EXISTENCE OF SOLUTIONS FOR SOME NONCOERCIVE ELLIPTIC PROBLEMS INVOLVING DERIVATIVES OF NONLINEAR TERMS

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Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

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Abstract. We study a nonlinear equation with an elliptic operator having degenerate coercivity. We prove the existence of a $W_0^{1,1}(\Omega)$ solution which is distributional or entropic, according to the growth assumptions on a lower order term in divergence form.

To Ildefonso:
"But of all these friends and lovers
There is no one compares with you."– see [9]

1. Introduction and statements of the results

In a joint paper with Jesús Ildefonso Díaz, the authors of [5] studied boundary value problems of the type

$$\begin{cases} A(u) = f - \operatorname{div}(\Phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

$$\Omega \text{ is a bounded, open subset of } \mathbb{R}^N, \text{ with } N > 2, \tag{2}$$

$$A \text{ is a coercive nonlinear differential operator,} \tag{3}$$

acting on $W_0^{1,p}(\Omega)$, $1 < p < \infty$, defined by $A(v) = -\operatorname{div}(a(x, v, \nabla v))$, which satisfies the classical Leray-Lions assumptions,

$$f \in W^{-1,p'}(\Omega),$$

$$\Phi \text{ belongs to } C^0(\mathbb{R}, \mathbb{R}^N). \tag{4}$$

The main feature of problem (1) is that no growth assumption was assumed on Φ.

Despite that, the authors proved the existence of a solution, in the following sense. Let $h \in C_c^1(\mathbb{R})$, then u is a renormalized solution to problem (1) if

$$\int_\Omega [a(x, u, \nabla u) - \Phi(u)] \cdot \nabla[h(u)\phi] = \int_\Omega f[h(u)\phi], \quad \forall \, \phi \in \mathscr{D}(\Omega). \tag{5}$$

In [2] the above problem is studied under the weaker assumption that $f \in L^1(\Omega)$, proving the existence of a solution in a slightly different sense. For $k \geqslant 0$ and $s \in \mathbb{R}$, let $T_k(s) = \max\{-k, \min\{s, k\}\}$. Then u is an entropy solution to (1) if $T_k(u)$ belongs to $H_0^1(\Omega)$ for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega [a(x, u, \nabla u) - \Phi(u)] \cdot \nabla T_k(u - \varphi) \leqslant \int_\Omega f T_k(u - \varphi). \tag{6}$$

In this note we will use the latter approach to prove the existence of a $W_0^{1,1}(\Omega)$ solution to the following degenerate elliptic problem:

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x)\nabla u}{(1 + b(x)|u|)^2}\right) + u = f - \mathrm{div}(\Phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Here $a(x), b(x)$ are measurable functions such that

$$0 < \alpha \leqslant a(x) \leqslant \beta, \quad 0 \leqslant b(x) \leqslant B, \tag{8}$$

with $\alpha, \beta \in \mathbb{R}^+$, $B \in \mathbb{R}$ and

$$f(x) \text{ belongs to } L^2(\Omega). \tag{9}$$

We point out that the main difference between the boundary value problems (1) and (7) is that the coercivity assumption (3) is not satisfied by the differential operator in (7).

We are going to prove that problem (7) has a solution u belonging to the non-reflexive Sobolev space $W_0^{1,1}(\Omega)$. We point out that this is quite unusual for an elliptic problem. According to the growth of Φ, u will be either a distributional or an entropy solution.

We recall that the problems

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x)\nabla u}{(1 + |u|)^\theta}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x)\nabla u}{(1 + |u|)^2}\right) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

have been studied in [6], [3], [7] and [4] proving existence results. In this note we prove that the same results hold even in the presence of a term in divergence form, that is, for problem (7).

We are going to prove the following theorems, according to the growth of Φ.

THEOREM 1. *Assume* (2), (4), (8) *and* (9) *and that there exists a positive C such that*

$$|\Phi(t)| \leqslant C|t|^2, \quad \forall\, t \in \mathbb{R}. \tag{12}$$

Then there exists a distributional solution $u \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$ *to problem* (7), *in the sense that*

$$\int_\Omega \frac{a(x)\,\nabla u \cdot \nabla \varphi}{(1 + b(x)|u|)^2} + \int_\Omega u\,\varphi = \int_\Omega f\,\varphi + \int_\Omega \Phi(u)\cdot\nabla\varphi\,,$$

for all $\varphi \in W_0^{1,\infty}(\Omega)$.

In the case where assumption (12) is not satisfied, one can prove the existence of more general solutions, that is, renormalized solutions as in [5], or entropy solutions as in [2]. Since the proof of existence of entropy solutions is easier (due to the fact that the weak convergence proved in Lemma 5 is enough), we will only prove the second result. Note however that the two concepts of solutions are equivalent (at least in the framework of Lebesgue data, see [8]) so that one can recover the existence of a renormalized solution from the existence of an entropy one.

THEOREM 2. *Assume* (2), (4), (8) *and* (9). *Then there exists an entropy solution* $u \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$ *to problem* (7), *in the sense that* $T_k(u)$ *belongs to* $H_0^1(\Omega)$ *for every* $k > 0$ *and*

$$\int_\Omega \frac{a(x)\,\nabla u \cdot \nabla T_k(u - \varphi)}{(1 + b(x)|u|)^2} + \int_\Omega u\,T_k(u - \varphi)$$

$$\leqslant \int_\Omega f\,T_k(u - \varphi) + \int_\Omega \Phi(u)\cdot\nabla T_k(u - \varphi),\ \ \forall\,\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)\,.$$

2. Proofs of the results

To prove our existence results, we begin by approximating the boundary value problem (7). Let $\{f_n\}$ be a sequence of $L^\infty(\Omega)$ functions such that f_n strongly converges to f in $L^2(\Omega)$, and $|f_n| \leqslant |f|$ for every n in $\mathbb{N}$.

LEMMA 3. *There exists a solution* u_n *in* $H_0^1(\Omega) \cap L^\infty(\Omega)$ *of*

$$\begin{cases} -\mathrm{div}\left(\dfrac{a(x)\,\nabla u_n}{(1 + b(x)|u_n|)^2}\right) + u_n = f_n - \mathrm{div}(\Phi(u_n)) & \text{in } \Omega, \\[4mm] \qquad\qquad\qquad u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Proof. Let $M_n = \|f_n\|_{L^\infty(\Omega)} + 1$, and consider the problem

$$
\begin{cases}
-\text{div}\left(\dfrac{a(x)\nabla w}{(1 + b(x)|T_{M_n}(w)|)^2}\right) + w = f_n - \text{div}(\Phi(T_{M_n}(w))) & \text{in } \Omega, \\
\qquad\qquad\qquad\qquad w = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{14}
$$

The existence of a $H_0^1(\Omega)$ weak solution w to (14) follows from Schauder's Theorem. Choosing

$$(|w| - \|f_n\|_{L^\infty(\Omega)})_+ \text{sgn}(w)$$

as a test function we obtain, dropping the nonnegative first term, and using the divergence Theorem on the last one, that

$$|w| \leqslant \|f_n\|_{L^\infty(\Omega)} < M_n.$$

Therefore, $T_{M_n}(w) = w$, and w is a bounded weak solution of (13). $\square$

In the following result we are going to prove some a priori estimates on the solutions u_n to problems (13).

LEMMA 4. *Let u_n be the sequence of solutions to (13). Then for every $k \geqslant 0$,*

$$\int\limits_{\{|u_n| \geqslant k\}} |u_n|^2 \leqslant \int\limits_{\{|u_n| \geqslant k\}} |f|^2; \tag{15}$$

$$\lim_{k \to +\infty} \text{meas}(\{|u_n| \geqslant k\}) = 0 \text{ uniformly with respect to } n; \tag{16}$$

$$\alpha \int\limits_\Omega \frac{|\nabla u_n|^2}{(1 + B|u_n|)^2} \leqslant \int\limits_\Omega |f|^2; \tag{17}$$

$$\|\nabla T_k(u_n)\|_{L^2(\Omega)}^2 \leqslant \frac{\|f\|_{L^1(\Omega)}}{\alpha} k(1 + Bk)^2. \tag{18}$$

Proof. Let $k \geqslant 0$, $i > 0$, and let $\psi_{i,k}(s)$ be the function defined by

$$
\psi_{i,k}(s) = \begin{cases}
\qquad 0 & \text{if } 0 \leqslant s \leqslant k, \\
\quad i(s - k) & \text{if } k < s \leqslant k + \frac{1}{i}, \\
\qquad 1 & \text{if } s > k + \frac{1}{i}, \\
\psi_{i,k}(s) = -\psi_{i,k}(-s) & \text{if } s < 0.
\end{cases}
$$

The choice of $|u_n|\,\psi_{i,k}(u_n)$ as a test function in (13) yields

$$\int\limits_\Omega \frac{a(x)|\nabla u_n|^2}{(1 + b(x)|u_n|)^2}|\psi_{i,k}(u_n)| + \int\limits_\Omega \frac{a(x)|\nabla u_n|^2}{(1 + b(x)|u_n|)^2}\psi'_{i,k}(u_n)|u_n|$$

$$+ \int_\Omega u_n |u_n| \psi_{i,k}(u_n)$$

$$= \int_\Omega f_n |u_n| \psi_{i,k}(u_n) + \int_\Omega \Phi(u_n) \cdot \nabla(|u_n| \psi_{i,k}(u_n)).$$

By the divergence Theorem the last term is zero. Since $\psi'_{i,k}(s) \geqslant 0$, we can drop the second term of the left hand side. By (8) one gets

$$\alpha \int_\Omega \frac{|\nabla u_n|^2}{(1 + b(x)|u_n|)^2} |\psi_{i,k}(u_n)| + \int_\Omega u_n |u_n| \psi_{i,k}(u_n) \leqslant \int_\Omega |f| |u_n| |\psi_{i,k}(u_n)|.$$

We infer (15) from this estimate as in [4], letting $i \to \infty$. One can prove (16) and (17) with the same arguments as in [4].

The choice of $T_k(u_n)$ as a test function in (13) gives

$$\frac{\alpha}{(1 + Bk)^2} \|\nabla T_k(u_n)\|^2_{L^2(\Omega)} \leqslant k \|f\|_{L^1(\Omega)} + \int_\Omega \Phi(u_n) \cdot \nabla T_k(u_n)$$

by using (8) and dropping the positive term $\int_\Omega u_n T_k(u_n)$. By the divergence Theorem the last integral is zero. This implies (18). □

The estimates proved in Lemma 4 can be used as in [4] to prove the following result.

LEMMA 5. *Let u_n be the solutions to (13). Up to subsequences, the sequence $\{u_n\}$ converges to some function u strongly in $L^2(\Omega)$ and weakly in $W_0^{1,1}(\Omega)$.*

We are going to prove Theorem 1.

Proof. Let u_n and u be as in Lemma 5. We now pass to the limit in the approximate problems (13). The lower order term on the left hand side and the first term of right hand side easily pass to the limit, due to the $L^2(\Omega)$ convergence of u_n to u and of f_n to f. For the operator term one can pass to the limit as in [4].

For the last term, since u_n converges to u in $L^2(\Omega)$ and thus a.e. in Ω, and Φ is continuous, $\Phi(u_n) \to \Phi(u)$ a.e. in Ω. Moreover, if E is any measurable subset of Ω we have, by (12),

$$\int_E |\Phi(u_n)| \leqslant C \int_E |u_n|^2.$$

The last term tends to 0, as $\operatorname{meas}(E) \to 0$, uniformly with respect to n, by Vitali's Theorem. Again by Vitali's Theorem, we deduce that $\Phi(u_n) \to \Phi(u)$ in $(L^1(\Omega))^N$. This allows us to pass to the limit in the last term. □

We are now going to prove Theorem 2.

Proof. We consider $T_k(u_n - \varphi)$ as a test function in (13) and we pass to the limit as $n \to \infty$. We can write the operator term as

$$\int_\Omega \frac{a(x)}{(1+b(x)|u_n|)^2}|\nabla T_k(u_n - \varphi)|^2 + \int_\Omega \frac{a(x)}{(1+b(x)|u_n|)^2}\nabla\varphi \cdot \nabla T_k(u_n - \varphi).$$

Estimate (18) and the a.e. convergence of u_n to u imply that

$$T_k(u_n - \varphi) \to T_k(u - \varphi) \quad \text{weakly in } H_0^1(\Omega).$$

Since

$$\frac{a(x)}{(1+b(x)|u_n|)^2} \quad \text{is bounded in } \Omega,$$

we deduce that

$$\liminf_{n\to\infty} \int_\Omega \frac{a(x)|\nabla T_k(u_n - \varphi)|^2}{(1+b(x)|u_n|)^2} \geqslant \int_\Omega \frac{a(x)|\nabla T_k(u - \varphi)|^2}{(1+b(x)|u|)^2}.$$

For the second term one has

$$\int_\Omega \frac{a(x)}{(1+b(x)|u_n|)^2}\nabla\varphi \cdot \nabla T_k(u_n - \varphi) \to \int_\Omega \frac{a(x)}{(1+b(x)|u|)^2}\nabla\varphi \cdot \nabla T_k(u - \varphi)$$

since $T_k(u_n - \varphi) \to T_k(u - \varphi)$ weakly in $H_0^1(\Omega)$ and

$$\frac{a(x)}{(1+b(x)|u_n|)^2}\nabla\varphi \to \frac{a(x)}{(1+b(x)|u|)^2}\nabla\varphi \quad \text{in } (L^2(\Omega))^N$$

by Lebesgue's Theorem.

By the $L^2(\Omega)$ convergences of u_n to u and f_n to f we deduce that

$$\int_\Omega u_n T_k(u_n - \varphi) \to \int_\Omega u\, T_k(u - \varphi), \qquad \int_\Omega f_n T_k(u_n - \varphi) \to \int_\Omega f\, T_k(u - \varphi).$$

Let us now study the last term:

$$\int_\Omega \Phi(u_n) \cdot \nabla T_k(u_n - \varphi).$$

This is non zero only in $\{|u_n - \varphi| \leqslant k\}$. On this set $\Phi(u_n)$ is bounded, by the continuity of Φ. By the weak $H_0^1(\Omega)$ convergence of $T_k(u_n - \varphi)$ to $T_k(u_n - \varphi)$ we deduce that

$$\int_\Omega \Phi(u_n) \cdot \nabla T_k(u_n - \varphi) \to \int_\Omega \Phi(u) \cdot \nabla T_k(u - \varphi),$$

as desired. $\square$

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