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EXISTENCE AND UNIQUENESS FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH NONSTANDARD GROWTH CONDITIONS

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Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

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Abstract. We study the homogeneous Dirichlet problem for the equation

$$u_t = \sum_{i=1}^{n} D_i\left(a_i|D_i(|u|^{m(x)-1}u)|^{p_i(x,t)-2}D_i(|u|^{m(x)-1}u)\right) + b|u|^{\sigma(x,t)-2}u$$

with given exponents $m(x)$, $p_i(x,t)$ and $\sigma(x,t)$. It is proved that the problem has a solution in a suitable variable exponent Sobolev space. In dependence on the properties of the coefficient b and the exponents of nonlinearity, the solution exists globally or locally in time. The comparison principle and uniqueness are proved under additional restrictions on the data.

1. Introduction

The paper addresses the questions of existence and uniqueness of solutions of the Dirichlet problem for the doubly nonlinear anisotropic parabolic equation with variable nonlinearity:

$$\begin{aligned} u_t = \sum_{i=1}^{n} D_i\Big(a_i(z)|D_i(|u|^{m(x)-1}u)|^{p_i(z)-2}D_i(|u|^{m(x)-1}u)\Big) \\ + b(z)|u|^{\sigma(z)-2}u, \quad z=(x,t)\in Q_T, \\ u(x,0)=u_0(x) \text{ in } \Omega, \quad u=0 \text{ on } \Gamma_T, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the Lipschitz-continuous boundary $\partial\Omega$, $Q_T = \Omega\times(0,T)$ and $\Gamma_T = \partial\Omega\times(0,T)$. The exponents of nonlinearity $m(x)$, $p_i(z)$ and $\sigma(z)$ are given functions of their arguments.

The nonlinear equations with variable nonlinearity are usually termed equations with nonstandard growth conditions. In the last decades, the theory of such equations

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has been developing developing very rapidly and already accounts for numerous results concerning the issues we discuss in the present work. Equations of the type (1.1) with constant exponents m and p_i arise in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology - see, e.g., [2, 16, 17]. The questions of existence and uniqueness of solutions to equations of the types

$$\begin{aligned} &u_t = \operatorname{div}\left(a\,|u|^{\alpha}|\nabla u|^{p-2}\nabla u\right) + f(x,t,u), \\ &(|u|^{\beta-1}u)_t = \operatorname{div}\left(a\,|\nabla u|^{p-2}\nabla u\right) + g(x,t,u) \end{aligned} \tag{1.2}$$

with constant exponents of nonlinearity were studied by many authors - see [11, 12, 13, 16, 22, 25, 27]. Existence, uniqueness, and qualitative properties of solutions for parabolic equations with variable nonlinearity, including doubly nonlinear equations (1.2) with variable p and α, were studied in [1, 3, 4, 5, 6, 7, 8, 9], see also [10] for a study of elliptic equations with triple variable nonlinearity.

The Cauchy problem for doubly nonlinear parabolic equations with constant exponents of nonlinearity was studied in [14, 15, 20].

We prove that the Dirichlet problem for equation (1.1), rewritten in the formally equivalent form for the new unknown $v = |u|^{m(x)-1}u$, has a weak energy solution in a suitable Sobolev-Orlicz space prompted by the equation. The existence result is established under very weak restrictions on the low-order term, which entails the possibility that the solutions exist locally in time and may blow-up in a finite time. The comparison principle and uniqueness are established under stronger assumptions on the data: the proof is given for the case when coefficient $b(z)$ is nonpositive which means, in particular, that the solutions exist globally in time.

The paper is organized as follows. In Section 2 we define the variable exponent Sobolev spaces and collect some known facts from the theory of these spaces used in the further proceeding. The rigorous assumptions on the problem data and the main existence result are given in Section 3. In Sections 4, 5, 6 we construct a sequence of solutions to the regularized problem and show that the limit of this sequence is a solution of the problem under study. In Section 7-8 we show that under certain restriction on the data the solutions possess higher regularity, and then use this fact to establish the comparison principle and uniqueness. These results are confined to the solutions which exist globally in time, the latter property is provided by suitable restrictions on the coefficient $b(z)$ and the exponents of nonlinearity $m(x)$ and $\sigma(z)$.

2. The function spaces

In this section we collect some known facts from the theory of the Sobolev spaces with variable exponent. A rigorous and detailed exposition of this theory, as well as the exhaustive review of the existing bibliographic sources, can be found in the monograph [18].

2.1. Orlicz-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$: definitions and basic properties

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with Lipschitz–continuous boundary $\partial\Omega$. Let $p(x) : \Omega \mapsto [p^-, p^+] \subset (1,\infty)$ be a continuous function with the logarithmic module of continuity:

$$\forall\, z, \zeta \in \Omega,\ |z-\zeta| < 1, \qquad \sum_i |p_i(z) - p_i(\zeta)| \leqslant \omega(|z-\zeta|), \tag{2.1}$$

where

$$\overline{\lim_{\tau\to 0^+}}\, \omega(\tau) \ln \frac{1}{\tau} = C < +\infty.$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm (the Luxemburg norm)

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 :\ A_{p(\cdot)}\big(f/\lambda\big) \leqslant 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,p(\cdot)}(\Omega)$ with $p(x) \in [p^-, p^+] \subset (1,\infty)$ is defined by

$$\begin{cases} W_0^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) :\ |\nabla u|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega}. \end{cases} \tag{2.2}$$

A thorough discussion of the variable exponent Lebesgue and Sobolev spaces can be found in the monograph [18]. We limit ourselves by mentioning the basic properties of the spaces $W_0^{1,p(\cdot)}(\Omega)$ used in the rest of this paper.

- The space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive, provided that $p(x) \in C^0(\overline{\Omega})$.

- If condition (2.1) is fulfilled, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$, which can be defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm (2.2). The density of smooth functions in the space $W_0^{1,p(\cdot)}(\Omega)$ is crucial for the further proceeding. The condition of log-continuity of $p(x)$ is the best known and the most frequently used sufficient condition for the density of C_0^∞ in $W_0^{1,p(x)}(\Omega)$ - [18, 24, 28]). Although this condition is not necessary and can be substituted by other conditions - see [18, 19, 21, 28] - we keep it throughout the paper for the sake of simplicity of presentation.

- It follows directly from the definition of the norm that

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \leqslant A_{p(\cdot)}(f) \leqslant \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right). \tag{2.3}$$

• Hölder's inequality. For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1,\infty), \quad p'(x) = \frac{p(x)}{p(x)-1},$$

the following inequality holds:

$$\int_\Omega |f\, g|\, dx \leqslant \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \leqslant 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}. \tag{2.4}$$

2.2. Spaces $L^{p(\cdot,\cdot)}(Q_T)$ and anisotropic space $\mathbf{W}(Q_T)$

Let $m(x) > 0$ and $p_i(x,t) > 1$, $i = 1,\dots,n$, be given functions. We assume that $m(x) \in C^0(\overline{\Omega})$ and $p_i(x,t)$ satisfy the log-continuity condition in Q_T:

$$\forall\ (x,t), (y,\tau) \in Q_T, \ \text{such that}\ |(x,t)-(y,\tau)| = \sqrt{|x-y|^2 + (t-\tau)^2} < 1$$

it holds

$$\sum_i |p_i(x,t) - p_i(y,\tau)| \leqslant \omega(|(x,t)-(y,\tau)|), \qquad \overline{\lim_{\tau \to 0^+}}\, \omega(\tau) \ln \frac{1}{\tau} = C. \tag{2.5}$$

For every fixed $t \in [0,T]$ we introduce the Banach space

$$\begin{aligned}
&\mathbf{V}_t(\Omega) = \left\{ u(x) \in L^{\frac{m(\cdot)+1}{m(\cdot)}}(\Omega) \cap W_0^{1,1}(\Omega) :\ |D_i u(x)|^{p_i(x,t)} \in L^1(\Omega) \right\},\\
&\|u\|_{\mathbf{V}_t(\Omega)} = \|u\|_{\frac{m(\cdot)+1}{m(\cdot)},\Omega} + \sum_i \|D_i u\|_{p_i(\cdot,t),\Omega},
\end{aligned}$$

and denote by $\mathbf{V}_t'(\Omega)$ its dual. By $\mathbf{W}(Q)$ we denote the Banach space

$$\mathbf{W}(Q_T) = \left\{ u \in L^{\frac{m(\cdot)+1}{m(\cdot)}}(Q_T) \,\middle|\, \begin{array}{l} |D_i u|^{p_i(x,t)} \in L^1(Q_T), \\ u(\cdot,t) \text{ in } \mathbf{V}_t(\Omega) \text{ for a.e. } t \in (0,T) \end{array} \right\},$$
$$\|u\|_{\mathbf{W}(Q_T)} = \sum_i \|D_i u\|_{p_i(\cdot),Q_T} + \|u\|_{\frac{m(\cdot)+1}{m(\cdot)},Q_T}.$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$w \in \mathbf{W}'(Q_T) \ \Leftrightarrow \begin{cases} \exists w = (w_0, w_1, \dots, w_n) \text{ such that} \\ w_0 \in L^{m(\cdot)+1}(Q_T), \quad w_i \in L^{p_i'(\cdot)}(Q_T), \\ \forall \phi \in \mathbf{W}(Q_T) \quad \langle\langle w, \phi \rangle\rangle = \displaystyle\int_{Q_T} \Big(w_0 \phi + \sum_{i=1}^n w_i D_i \phi \Big) dz. \end{cases}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup \left\{ \langle\langle v, \phi \rangle\rangle \,|\, \phi \in \mathbf{W}(Q_T), \|\phi\|_{\mathbf{W}(Q_T)} \leqslant 1 \right\}.$$

The possibility to approximate a function $u \in \mathbf{W}(Q_T)$ is crucial for the further proceeding. Let ρ be the Friedrichs' mollifying kernel

$$\rho(s) = \begin{cases} \kappa \exp\left(-\frac{1}{1-|s|^2}\right) & \text{if } |s| < 1, \\ 0 & \text{if } |s| > 1, \end{cases} \qquad \kappa = \text{ const}, \qquad \int_{\mathbb{R}^{n+1}} \rho(z)\,dz = 1.$$

Given a function $v \in L^1(Q_T)$, we extend it to the whole $\mathbb{R}^{n+1}$ by a function with compact support (keeping the same notation for the continued function) and then define

$$v_h(z) = \int_{\mathbb{R}^{n+1}} v(s)\rho_h(z-s)\,ds \ \text{ with } \ \rho_h(s) = \frac{1}{h^{n+1}}\rho\left(\frac{s}{h}\right),\ h > 0.$$

The following assertion are known.

PROPOSITION 2.1. *If $u \in \mathbf{W}(Q_T)$ with the exponents $p_i(z)$ satisfying* (2.1) *in Q_T, then*

$$\|u_h\|_{\mathbf{W}(Q_T)} \leqslant C\left(1 + \|u\|_{\mathbf{W}(Q_T)}\right) \ \text{ and } \ \|u_h - u\|_{\mathbf{W}(Q_T)} \to 0 \ \text{ as } \ h \to 0.$$

Let us denote

$$p^+ = \sup_{Q_T} p_i(z), \quad m^- = \inf_{\Omega} m(x)$$

and set

$$\mathbf{V}_+(\Omega) = \left\{u(x) |\ u \in L^{1+\frac{1}{m^-}}(\Omega) \cap W_0^{1,1}(\Omega),\ |\nabla u| \in L^{p^+}(\Omega)\right\}.$$

Since $\mathbf{V}_+(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\psi_k\} \subset \mathbf{V}_+(\Omega)$.

PROPOSITION 2.2. *Let $p_i(z)$ satisfy condition* (2.1) *in Q_T. Then the set $\{\psi_k\}$ is dense in $\mathbf{V}_t(\Omega)$ for every $t \in [0,T]$.*

PROPOSITION 2.3. *For every $u \in \mathbf{W}(Q_T)$ there is a sequence $\{d_k(t)\}$, $d_k(t) \in C^1[0,T]$, such that*

$$\left\| u - \sum_{k=1}^{s} d_k(t)\psi_k(x) \right\|_{\mathbf{W}(Q_T)} \to 0 \ \text{ as } \ s \to \infty.$$

PROPOSITION 2.4. *Let in the conditions of Proposition* 2.1 *$u_t \in \mathbf{W}'(Q_T)$. Then $(u_h)_t \in \mathbf{W}'(Q_T)$, and for every $\psi \in \mathbf{W}(Q_T)$ $\langle\langle (u_h)_t, \psi \rangle\rangle \to \langle\langle u_t, \psi \rangle\rangle$ as $h \to 0$.*

3. Assumptions and results

It is convenient to reformulate problem (1.1) introducing the new unknown function v and it's inverse u by the formulas

$$v = |u|^{m(x)-1}u, \quad \Phi(v,x) = \frac{1}{m(x)} \int_0^v |s|^{\frac{1-m(x)}{m(x)}}\,ds \equiv |v|^{\frac{1}{m(x)}} \operatorname{sign} v.$$

The function v solves the problem

$$\begin{cases} \partial_t \Phi(v,x) = \sum_{i=1}^{n} D_i\Big(a_i(x)|D_i v|^{p_i(x)-2} D_i v\Big) + f(z,v) \ \text{ in } \ Q_T, \\ v(x,0) = v_0(x) \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \Gamma_T \end{cases} \tag{3.1}$$

with the right-hand side

$$f(z,v) = b(z)|v|^{\frac{\sigma(z)-1}{m(z)}-1} v.$$

Throughout the paper assume that the coefficients and the exponents of nonlinearity satisfy the following conditions:

$$\begin{cases} \text{there exist finite positive constants } a^{\pm}, p^{\pm}, m^{\pm}, \sigma^{\pm} \text{ such that} \\ 0 < a^{-} \leqslant a_i(z) \leqslant a^{+}, \quad 1 < p^{-} \leqslant p_i(z) \leqslant p^{+}, \\ 0 < m^{-} \leqslant m(x) \leqslant m^{+}, \quad 1 < \sigma^{-} \leqslant \sigma(z) \leqslant \sigma^{+}. \end{cases} \tag{3.2}$$

The solution of problem (3.1) is understood in the following way.

DEFINITION 3.1. *A function $v(z)$ is called* energy solution *of problem* (3.1) *if*

(1) $v \in L^{\infty}(Q_T) \cap \mathbf{W}(Q_T)$ *and* $\partial_t \Phi(v,x) \in \mathbf{W}'(Q_T)$,

(2) *for every test-function* $\phi \in \mathbf{W}(Q_T)$, $\partial_t \phi \in \mathbf{W}'(Q_T)$,

$$\int_{Q_T} \left[\phi\, \partial_t \Phi(v,x) + \sum_{i=1}^{n} a_i\, |D_i v|^{p_i(z)-2} D_i v \cdot D_i \phi\right] dz = \int_{Q_T} f(z,v)\, \phi\, dz, \tag{3.3}$$

(3) *for every* $\psi(x) \in C_0^{\infty}(\Omega)$

$$\int_{\Omega} \psi(x)(\Phi(v(x,t),x) - \Phi(v_0(x),x))\, dx \to 0 \ \text{ as } \ t \to 0.$$

The main result of the paper is given in the following theorem.

THEOREM 3.1. *Let $m(x) \in C^0(\Omega)$, $p_i(x,t)$ satisfy the Log-continuity condition* (2.5) *in Q_T, and let conditions* (3.2) *be fulfilled. Let us assume that*

$$\nabla m(x) \in L^{\beta}(\Omega) \ \text{ with some } \ \beta > 1,$$

and that the exponents m, p_i satisfy one of the following conditions:

(1) $p_i > 1$ *are independent of t,* $m(x) > 0$ *in* Ω,

(2) $p_i(z) > 1$, $m(x) \in (0,1]$ *in* Ω, $\nabla m \in L^{p_i(z)}(Q_T)$ *for all* $i = 1, \dots, n$,

(3) $p_i(z) > 1$, $m(x) > 0$, $\nabla m \in L^{p_i(z)}(Q_T)$ *for all* $i = 1, \dots, n$, *and*

$$1 > \frac{1}{p_i(z)} + \frac{1}{m(x)} \ \text{ in } \ Q_T.$$

Then for every $v_0 \in L^\infty(\Omega)$ *problem* (3.1) *has at least one solution in a cylinder* Q_{T^*} *with*

$$T^* = \sup\{\theta : \ \|v(t)\|_{\infty,\Omega} < \infty \ \forall t \in (0,\theta)\}.$$

Moreover, for small τ *the solution satisfies the estimate*

$$\|v(t)\|_{\infty,\Omega} \leqslant \|v_0\|_{\infty,\Omega} \mathrm{e}^{At}, \quad t \in [0,\tau],$$

with a constant A *depending only on the data. The solution is nonnegative if* $v_0 \geqslant 0$ *a.e. in* Ω.

In the special case when the exponents of nonlinearity are constant, the assumptions of Theorem 3.1 reduce to the conditions $m > 0$, $p_i > 1$.

The solution of problem (3.1) is constructed as the limit of the sequence of solutions of the regularized problem with three regularization parameters. The solution of this problem is obtained as the limit of the sequence of finite-dimensional Galerkin's approximations. We impose no restrictions on the growth of the term f, which leads to the fact that the solution need not exist for all times. If $b(z) \leqslant 0$ in Q_T, the term f does not influence the a priori estimates for the solutions of the regularized problems and the solution exists for all times. The same happens if $\sigma^+ \leqslant 2$, although $b(z)$ is allowed to take positive values on a part of Q_T. The solution of problem (3.1) is constructed on a time interval $[0,T_*]$, with T_* depending on $\|v_0\|_{\infty,\Omega}$ in such a way that $T_* \to 0$ as $\|v_0\|_{\infty,\Omega} \to \infty$. Proceeding in small steps in time we continue the solution up to the moment T^* where either $T^* = T$, or $\|v(x,T^*)\|_{\infty,\Omega}$ becomes unbounded. If $T^* < \infty$, we obtain a local in time solution.

In Section 7 we derive stronger estimates on the constructed weak solutions. This is done under additional restrictions on the regularity of the initial function and properties of the coefficients a_i, b and the variable exponents of nonlinearity $p_i(z)$, $\sigma(z)$, $m(x)$. It is shown in Theorem 7.1 that under these restrictions on the data the weak solutions of problem (3.1) satisfy the estimate $\||v|^{\frac{1-m}{2m}} v_t\|_{2,Q_T} < \infty$. This estimate is used in Section 8 in the proof of the comparison principle for such solutions. The results of these sections hold for the solutions of equation (3.1) with $b(z) \leqslant 0$ and $\sigma^+ \leqslant 2$, which means that the solutions exist globally in time. Besides, to prove the comparison principle we claim that $m(x) \in (0,1]$. The proof of the comparison principle follows [1, 16, 17].

4. Regularization

A solution of problem (3.1) is constructed as the limit of the sequence of solutions of the regularized problems

$$\begin{cases} \partial_t \Phi_\varepsilon(v,x) = \operatorname{div}\big(\mathscr{F}_\delta(z,\nabla v)\big) + f_K(v,z) \ \text{ in } \ Q_T, \\ v(x,0) = v_0(x) \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \Gamma_T, \end{cases} \tag{4.1}$$

where

$$\mathscr{F}_\delta(\nabla v,z) = \{\mathscr{F}_{\delta,1},\ldots,\mathscr{F}_{\delta,n}\}, \quad \mathscr{F}_{\delta,i} = \delta\,|\nabla v|^{q-2} D_i v + a_i(x,t)|D_i v|^{p_i(z)-2} D_i v,$$

with constant $q > \max\{n, p^+\}$, given parameters ε, $\delta > 0$, and

$$\Phi_\varepsilon(v,x) = \int_0^{v(z)} A_\varepsilon(s,x)\,ds, \quad A_\varepsilon(s,x) = \varepsilon + \frac{1}{m}|s|^{\frac{1}{m(x)}-1} \geqslant \varepsilon > 0.$$

For $K > 1$ the function f_K is defined by the equality

$$f_K(w,z) = b(z)\begin{cases} \left(\min\{w^2,K^2\}\right)^{\frac{\sigma-2}{2m}} |w|^{\frac{1}{m}-1}w & \text{if } \sigma^+ > 2, \\ |w|^{\frac{\sigma-1}{m}-1}w & \text{if either } b(z) \leqslant 0 \text{ in } Q_T, \text{ or } \sigma^+ \leqslant 2. \end{cases} \tag{4.2}$$

Let us fix some $s \in \mathbb{N}$,

$$\frac{s-1}{n} \geqslant \frac{1}{2} - \frac{1}{\min\{p^+,q\}},$$

and denote $r = \max\left\{p^+, q, 1 + \dfrac{1}{m^-}\right\}$.

THEOREM 4.1. *Let $v_0 \in L^2(\Omega) \cap L^{\frac{m(\cdot)+1}{m(\cdot)}}(\Omega)$. For every $\delta > 0$, $\varepsilon > 0$, $K > 1$ problem* (4.1) *has at least one solution $v \in \mathbf{W}(Q_T) \cap L^q(0,T;W_0^{1,q}(\Omega))$ such that $\partial_t \Phi_\varepsilon(v,x) \in L^{r'}(0,T;H^{-s}(\Omega))$, for every test-function $\phi \in L^r(0,T;H_0^s(\Omega))$,*

$$\int_{Q_T} \left[\phi\, \partial_t \Phi_\varepsilon(v,x) + \sum_{i=1}^n \mathscr{F}_{\delta,i}(z,v) \cdot D_i \phi\right] dz = \int_{Q_T} f_K(v,z)\,\phi\,dz$$

and for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega \phi(x)(\Phi_\varepsilon(v(x,t),x) - \Phi_\varepsilon(v_0(x),x))\,dx \to 0 \ \text{ as } \ t \to 0.$$

4.1. Galerkin's approximations

Let us fix some $\delta > 0$, $\varepsilon > 0$ and $K > 1$. A solution of the regularized problem (4.1) is constructed as the limit of the sequence of finite-dimensional Galerkin's approximations

$$v^{(k)} = \sum_{i=1}^k c_i(t)\psi_i(x),$$

where $\{\psi_i\}$ is the orthonormal basis of $L^2(\Omega)$, composed of the eigenfunctions of the operator

$$(\psi_i, w)_{H_0^s(\Omega)} = \lambda_i(\psi_i, w)_{L^2(\Omega)} \quad \forall w \in H_0^s(\Omega).$$

The coefficients $c_i(t)$ are defined from the system of equations

$$\begin{aligned} \int_\Omega \partial_t \Phi_\varepsilon(v^{(k)},x)\,\psi_j\,dx + \delta \int_\Omega |\nabla v^{(k)}|^{q-2}\nabla v^{(k)} \cdot \nabla \psi_j\,dx \\ + \sum_{i=1}^n \int_\Omega a_i |D_i v^{(k)}|^{p_i(z)-2} D_i v^{(k)} \cdot D_i \psi_j\,dx \\ = \int_\Omega f_K(z, v^{(k)})\psi_j\,dx, \end{aligned} \tag{4.3}$$

$j=1,...,k$. System (4.3) can be written in the form

$$\begin{cases} \sum_{i=1}^{k} B_{ij}c_i'(t) = F_j(c_1(t),\dots,c_k(t)), \\ c_j(0) = \int_\Omega v_0(x)\psi_j(x)\,dx, \quad j=1,\dots,k, \end{cases} \tag{4.4}$$

with continuous functions F_j and the matrix B with the entries

$$B_{ij}(t) = \int_\Omega A_\varepsilon(v^{(k)},x)\,\psi_i(x)\psi_j(x)\,dx, \quad i,j=1,\dots,k.$$

Since ψ_j are linearly independent, so is the system of functions

$$\Lambda_k = \left\{ \sqrt{A_\varepsilon(v^{(k)},x)}\,\psi_i(x) \right\}_{i=1}^{k}.$$

The determinant of B is the Gram determinant of the system Λ_k. Since $A_\varepsilon(s,x) \geqslant \varepsilon > 0$, system (4.4) can be solved with respect to the derivatives $c_j'(t)$ and written in the normal form. By Peano's theorem for every $k \in \mathbb{N}$ there exists at least one solution of system (4.4) on an in interval $(0,T_k)$. The solution $\{c_i(t),\dots,c_k(t)\}$ of system (4.4) defines the functions

$$u^{(k)} = \Phi_{\varepsilon,r}(v^{(k)},x) \equiv \sum_{j=1}^{k} Z_{k,j}(t)\psi_j(x) \tag{4.5}$$

with the coefficients $Z_{k,j}(t)$ given by the formulas

$$Z_{k,j}(t) = Z_{k,j}(0) + \int_0^t \sum_{i,j=1}^{k} B_{ij}(\tau)c_i'(\tau)\,d\tau, \quad j=1,\dots,k.$$

5. A priori estimates

Let us introduce the function

$$\Psi_\varepsilon(v^{(k)},x) = \int_0^{v^{(k)}} sA_\varepsilon(s,x)\,ds = \frac{\varepsilon}{2}(v^{(k)})^2 + \frac{1}{m}|v^{(k)}|^{\frac{m+1}{m}}.$$

Multiplying each of the equations in (4.3) by $c_j(t)$, integrating over the interval $(0,T_k)$ and summing up in $j=\overline{1,k}$, we obtain the inequality

$$\begin{aligned} \int_\Omega \Psi_\varepsilon(v^{(k)}(x,T_k),x)\,dx + \delta\int_{Q_{T_k}} |\nabla v^{(k)}|^q\,dz + a^- \sum_{i=1}^{n} \int_{Q_{T_k}} |D_i v^{(k)}|^{p_i(x)}\,dz \\ \leqslant \int_{Q_{T_k}} |v^{(k)}|\,|f_K(v^{(k)},z)|\,dz + \int_\Omega \Psi_\varepsilon(v_0^{(k)},x)\,dx, \ \ Q_{T_k} = \Omega\times(0,T_k). \end{aligned}$$

Due to the definition of f_K, this estimate can be continued as follows:

$$\begin{aligned} \int_\Omega \Psi_\varepsilon(v^{(k)}(x,t),x)\,dx\Big|_{t=0}^{t=T_k} + \delta\|\nabla v^{(k)}\|_{q,Q_T}^q \\ + \frac{a^-}{2}\sum_{i=1}^{n}\int_{Q_{T_k}} |D_i v^{(k)}|^{p_i(x)}\,dz \leqslant J, \end{aligned} \tag{5.1}$$

where

$$J = \begin{cases} \int_{Q_{T_k}} b(z)|v^{(k)}|^{\frac{\sigma-1}{m}+1}\,dz, & \text{if either } b(z)\leqslant 0 \text{ in } Q_T, \text{ or } \sigma^+\leqslant 2,\\ b^+\big(\sup_{Q_T} K^{\frac{\sigma-2}{m}}\big)\int_{Q_{T_k}} |v^{(k)}|^{\frac{1}{m}+1}\,dz, & \text{if } \sigma^+>2. \end{cases}$$

To estimate J we consider the following two possibilities: if $b(z)\leqslant 0$, then J is merely nonpositive, otherwise by Young's inequality

$$J\leqslant b^+\big(C+Y(T_k)\big),\quad Y(t)=\int_{Q_T}|v^{(k)}|^{1+\frac{1}{m}}\,dz,$$

and (5.1) provides Gronwall's inequality for $Y(t)$.

LEMMA 5.1. *Let $v_0^{(k)}\in L^2(\Omega)\cap L^{\frac{m(\cdot)+1}{m(\cdot)}}(\Omega)$. Then each of the functions $v^{(k)}$ can be continued from the cylinder Q_{T_k} to Q_T. The continued functions satisfy the uniform with respect to k, ε and δ estimates*

$$\sup_{(0,T)}\int_\Omega \Psi_\varepsilon(v^{(k)}(x,t),x)\,dx+\delta\|\nabla v^{(k)}\|^q_{q,Q_T}+\sum_{i=1}^n\int_{Q_T}|D_iv^{(k)}|^{p_i(x)}\,dz\leqslant M. \tag{5.2}$$

Proof. The possibility of continuation of $v^{(k)}$ to the same interval $[0,T]$ follows from (5.1) because the function $v^{(k)}(x,T_k)$ possesses the same properties that $v_0^{(k)}$. □

LEMMA 5.2. *For every fixed $j\in\mathbb{N}$ the sequence $\{Z_{k,j}(t)\}_{k=1}^\infty$, defined in* (4.5)*, contains a subsequence which converges to a function $Z_j(t)\in C^0[0,T]$.*

Proof. The assertion will follow from the Ascoli-Arzela theorem if we prove that for every fixed $j\in\mathbb{N}$ the sequence $\{Z_{k,j}(t)\}_{k=1}^\infty$ is equicontinuous and uniformly bounded in $[0,T]$. Let us accept the notation $Q_t^{t+h}=\Omega\times(t,t+h)$. By virtue of (4.3), for every $t,t+h\in[0,T]$,

$$\begin{aligned} Z_{k,j}(t+h)-Z_{k,j}(t) &= -\delta\int_{Q_t^{t+h}}|\nabla v^{(k)}|^{q-2}\nabla v^{(k)}\cdot\nabla\psi_j\,dz\\ &\quad -\sum_{i=1}^n\int_{Q_t^{t+h}}\big(a_i|D_iv^{(k)}|^{p_i(z)-2}D_iv^{(k)}\big)\cdot D_i\psi_j\,dz+\int_{Q_t^{t+h}}f_K(v^{(k)},z)\,\psi_j\,dz, \end{aligned}$$

whence

$$\begin{aligned} &|Z_{k,j}(t+h)-Z_{k,j}(t)|\\ &\qquad\leqslant 2a_1\sum_{i=1}^n\|D_iv^{(k)}\|_{p_i(\cdot),Q_t^{t+h}}\|D_i\psi_j\|_{p_i(\cdot),Q_t^{t+h}}\\ &\qquad\quad+\delta\|\nabla v^{(k)}\|_{q,Q_t^{t+h}}\|\nabla\psi_j\|_{q,Q_t^{t+h}}+\|f_K(v^{(k)},z)\|_{2,Q_t^{t+h}}\|\psi_j\|_{2,\Omega}h^{\frac12}\\ &\qquad\leqslant C\sum_{i=1}^n\max\left\{\Big(\int_{Q_t^{t+h}}|D_iv^{(k)}|^{p_i(z)}\,dz\Big)^{\frac{1}{p_i^+}},\Big(\int_{Q_t^{t+h}}|D_iv^{(k)}|^{p_i(z)}\,dz\Big)^{\frac{1}{p_i^-}}\right\} \end{aligned}$$

$$\times \max\left\{\left(\int_{Q_t^{t+h}} |D_i\psi_j|^{p_i(z)}\,dz\right)^{\frac{1}{p_i^+}}, \left(\int_{Q_t^{t+h}} |D_i\psi_j|^{p_i(z)}\,dz\right)^{\frac{1}{p_i^-}}\right\}$$
$$+\|f_K(v^{(k)},z)\|_{2,Q_t^{t+h}}\|\psi_j\|_{2,\Omega}h^{\frac{1}{2}}+\delta\|\nabla v^{(k)}\|_{q,Q_t^{t+h}}\|\nabla\psi_j\|_{q,Q_t^{t+h}}.$$

In our choice of the basis $\{\psi_j\}$

$$\begin{aligned}\int_{Q_t^{t+h}}|D_i\psi_j|^{p_i(z)}\,dz &\leqslant C\,\||D_i\psi_j|^{p_i}\|_{\frac{p^+}{p_i(\cdot)},Q_t^{t+h}}\|1\|_{\left(\frac{p^+}{p_i(\cdot)}\right)',Q_t^{t+h}}\\ &\leqslant C(n,T,|\Omega|,p^\pm)\max\left\{\left(h\int_\Omega|\nabla\psi_j|^{p^+}\,dz\right)^{\frac{p^-}{p^+}}, h\int_\Omega|\nabla\psi_j|^{p^+}\,dz\right\}\\ &\leqslant C'(s,n,T,|\Omega|,p^\pm)\max\left\{h,\,h^{\frac{p^-}{p^+}}\right\}\left(\|\psi_j\|^{p^-}_{H_0^s(\Omega)}+\|\psi_j\|^{p^+}_{H_0^s(\Omega)}\right)\end{aligned}$$

with constants C, C' independent of j and h. Since $\|\psi_j\|_{2,\Omega}\leqslant C''\|\psi_j\|_{H_0^s(\Omega)}$, estimate (5.2) yields the inequality

$$|Z_{k,j}(t+h)-Z_{k,j}(t)|\leqslant C''\left(\|\psi_j\|^{p^-}_{H_0^s(\Omega)}+\|\psi_j\|^{p^+}_{H_0^s(\Omega)}+\delta^{1-\frac{1}{q}}\|\psi_j\|^{q}_{H_0^s(\Omega)}\right)\gamma(h),$$

with $\gamma(h)=\max\left\{\sqrt{h},\,h,\,h^{\frac{p^-}{p^+}},h^{q'}\right\}$ and a constant $C''=C''(T,|\Omega|,p^\pm,K)$ independent of k and j. This means equicontinuity of the sequence $\{Z_{k,j}(t)\}_{k=1}^\infty$. Uniform boundedness of $Z_{k,j}(t)$ follows from the last estimate with $t=0$. Namely, $\forall h\in[0,T]$

$$|Z_{k,j}(h)|\leqslant|Z_{k,j}(0)|+C\left(\|\psi_j\|^{p^-}_{H_0^s(\Omega)}+\|\psi_j\|^{p^+}_{H_0^s(\Omega)}+\|\psi_j\|^{q}_{H_0^s(\Omega)}\right), \tag{5.3}$$

what gives the proof. □

Using the diagonal procedure we extract from $\{Z_{k,j}(t)\}$ a subsequence which converges as $k\to\infty$ to $Z_j(t)\in C^0[0,T]$ for every fixed $j\in\mathbb{N}$. By agreement, for this subsequence we will use the same notation. Let us introduce the functions

$$u=\sum_{j=1}^\infty\psi_j(x)Z_j(t).$$

LEMMA 5.3. *For every $\phi(x)\in C_0^\infty(\Omega)$*

$$\sup_{(0,T)}\int_\Omega\phi(x)\left(u(z)-\Phi_\varepsilon(v^{(k)}(z),z)\right)dx\to 0 \ \text{ as } \ k\to\infty.$$

Proof. It follows from (5.3) and the Parseval equality that

$$\begin{aligned}&\sum_{j=s+1}^\infty\left|(\phi,\psi_j)_{2,\Omega}\left(u^{(k)}-u,\psi_j\right)_{2,\Omega}\right|\\ &\qquad\leqslant\sup_{(0,T)}\|u^{(k)}-u\|_{2,\Omega}\Big(\sum_{j=s+1}^\infty(\phi,\psi_j)^2_{2,\Omega}\Big)^{\frac{1}{2}}\to 0 \text{ as } s\to\infty.\end{aligned}$$

At the same time, Lemma 5.2 yields

$$\left|\sum_{j=1}^{s}(\phi,\psi_j)_{2,\Omega}\big(u^{(k)}-u,\psi_j\big)_{2,\Omega}\right| \leqslant \sum_{j=i}^{s}\sup_{(0,T)}|Z_{k,j}(t)-Z_j(t)|\|\phi\|_{2,\Omega}\to 0 \ \text{ as } \ k\to\infty.$$

LEMMA 5.4. *Let*

$$r=\max\Big\{q,p^+,1+\frac{1}{m^-}\Big\}.$$

Then $\partial_t\Phi_\varepsilon(v^{(k)},x)$ *are bounded in* $L^{r'}(0,T;H^{-s}(\Omega))$ *uniformly with respect to* k, δ, ε.

Proof. It suffices to show that

$$\left|\int_{Q_T}\phi\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz\right|\leqslant C,\ \ \forall\,\phi\in L^r(0,T;H^s_0(\Omega)),\ \ \|\phi\|_{L^r(0,T;H^s_0(\Omega))}\leqslant 1,$$

with a constant C, independent of k, δ, ε and ϕ. By the definition

$$\partial_t\Phi_\varepsilon(v^{(k)},x)=\sum_{j=1}^{k}Z'_{k,j}(t)\psi_j(x).$$

Writing $\phi^{(k)}=\sum_{j=1}^{k}(\phi,\,\psi_j)_{2,\Omega}\,\psi_j$ and using orthogonality of $\{\psi_j\}$, we have

$$\begin{aligned}\int_{Q_T}\phi\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz&=\int_{Q_T}\Big(\sum_{j=1}^{\infty}(\phi,\,\psi_j)_{2,\Omega}\psi_j\Big)\Big(\sum_{s=1}^{k}Z'_{k,s}(t)\psi_s\Big)\,dz\\&=\int_0^T\sum_{j,s=1}^{k}Z'_{k,s}(t)(\phi,\,\psi_j)_{2,\Omega}\Big(\psi_s,\,\psi_j\Big)_{2,\Omega}\,dt\\&=\int_{Q_T}\Big(\sum_{j=1}^{k}(\phi,\,\psi_j)_{2,\Omega}\psi_j\Big)\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz\\&=\int_{Q_T}\phi^{(k)}\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz.\end{aligned}$$

Using (4.3) and the uniform estimates (5.2) we obtain

$$\begin{aligned}&\left|\int_{Q_T}\phi\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz\right|=\left|\int_{Q_T}\phi^{(k)}\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz\right|\\&\leqslant a^+\sum_{i=1}^{n}\||D_iv^{(k)}|^{p_i(z)-1}\|_{\frac{p^+}{p^+-1},Q_T}\|D_i\phi^{(k)}\|_{p^+,Q_T}\\&\quad+\delta^{\frac{1}{q'}}\Big(\delta\,\|\nabla v^{(k)}\|^q_{q,Q_T}\Big)^{\frac1q}\|\nabla\phi^{(k)}\|_{q',Q_T}+\|\phi^{(k)}\|_{\frac{m+1}{m},Q_T}\|f_K(v^{(k)},z)\|_{m+1,Q_T}\\&\leqslant C\Big(\sum_{i=1}^{n}(\int_{Q_T}|D_iv^{(k)}|^{(p^+)'(p_i(z)-1)}\,dz)^{\frac{1}{(p^+)'}}+\|f_K\|_{m+1,Q_T}+\delta^{\frac1q}\,\|\nabla v^{(k)}\|_{q,Q_T}\Big)\\&\quad\times\Big(\|\nabla\phi^{(k)}\|_{p^+,Q_T}+\|\phi^{(k)}\|_{\frac{m+1}{m},Q_T}+\delta^{\frac{1}{q'}}\|\nabla\phi^{(k)}\|_{q',Q_T}\Big)\end{aligned}$$

with a constant C independent of k,δ and ε. Since $(p^+)'\leqslant (p_i(z))'$, by Young's inequality

$$|D_i v^{(k)}|^{p^+\frac{p_i(z)-1}{p^+-1}}\leqslant C+|D_i v^{(k)}|^{p_i(z)}$$

which, by virtue of (5.2), provides the estimate

$$\Big|\int_{Q_T}\phi\,\partial_t\Phi_\varepsilon(v^{(k)},x)\,dz\Big|\leqslant C\left(\|\nabla\phi^{(k)}\|_{p^+,Q_T}+\|\phi^{(k)}\|_{\frac{m+1}{m},Q_T}+\delta^{\frac{1}{q'}}\|\nabla\phi^{(k)}\|_{q',Q_T}\right).$$

By the choice of the basis $\{\psi_j\}$,

$$\begin{aligned}\|\nabla\phi^{(k)}\|_{p^+,Q_T}^{p^+}&=\int_0^T\|\nabla\phi^{(k)}\|_{p^+,\Omega}^{p^+}dt\leqslant\int_0^T\|\phi^{(k)}\|_{H_0^s(\Omega)}^{p^+}dt\\&=\int_0^T\left\|\sum_{j=1}^k\psi_j(\phi,\,\psi_j)_{2,\Omega}\right\|_{H_0^s(\Omega)}^{p^+}dt\\&=\int_0^T\left(\sum_{j=1}^k(\phi,\,\psi_j)_{2,\Omega}^2\|\psi_j\|_{H_0^s(\Omega)}^2\right)^{\frac{p^+}{2}}dt\\&\leqslant\int_0^T\left(\sum_{j=1}^\infty(\phi,\,\psi_j)_{2,\Omega}^2\|\psi_j\|_{H_0^s(\Omega)}^2\right)^{\frac{p^+}{2}}dt\\&=\|\phi\|_{L^{p^+}(0,T;H_0^s(\Omega))}^{p^+}\end{aligned}$$

and

$$\|\nabla\phi^{(k)}\|_{q,Q_T}^q\leqslant C\|\phi\|_{L^q(0,T;H_0^s(\Omega))}^q,\quad\|\phi^{(k)}\|_{\frac{m+1}{m},Q_T}\leqslant C\|\phi\|_{L^{\frac{m^-+1}{m^-}}(0,T;H_0^s(\Omega))}.$$

Let us introduce the difference operator $\Delta_{(h)}$ as follows

$$\Delta_{(h)}V(x,t):=V(x,t+h)-V(x,t).$$

LEMMA 5.5. *Let*

$$w^{(k)}=|v^{(k)}|^{\frac{1}{m(x)}}\operatorname{sign}v^{(k)}.$$

For every $t,t+h\in[0,T]$ the functions $v^{(k)}$, $w^{(k)}$ satisfy the inequalities

$$\begin{aligned}\varepsilon\|\Delta_{(h)}v^{(k)}(x,t)\|_{2,Q_{T-h}}^2&+\int_0^{T-h}\int_{\Omega\cap\{m<1\}}\left|\Delta_{(h)}v^{(k)}(x,t)\right|^{\frac{m+1}{m}}dz\\&+\int_0^{T-h}\int_{\Omega\cap\{m\geqslant1\}}\left|\Delta_{(h)}w^{(k)}(x,t)\right|^{m+1}dz\leqslant C|h|\end{aligned}\tag{5.4}$$

with a constant C independent of k and ε. If the exponents p_i do not depend on t, the constant C is also independent of δ.

Proof. For every $t, t+h \in [0,T]$

$$
\begin{aligned}
I &:= \int_{Q_{T-h}} \Delta_{(h)} v^{(k)}(x,t)\, \Delta_{(h)} \Phi_\varepsilon(v^{(k)}(x,t),x)\, dz \\
&= \int_{Q_{T-h}} \Delta_{(h)} v^{(k)}(x,t) \Big(\int_t^{t+h} \partial_\tau \Phi_\varepsilon(v^{(k)}(x,\tau),x)\, d\tau \Big)\, dz \\
&= \int_0^h \Big(\int_{Q_{T-h}} \Delta_{(h)} v^{(k)}(x,t)\, \partial_t \Phi_\varepsilon(v^{(k)}(x,t+\tau),x)\, dz \Big)\, d\tau.
\end{aligned}
$$

By virtue of (4.3) and (5.1),

$$
\begin{aligned}
|I| \leqslant & \int_0^h \int_{Q_{T-h}} \sum_i a_i |D_i v^{(k)}(x,t+\tau)|^{p_i-1} |D_i v^{(k)}(x,t+h)|\, dz d\tau \\
& + \int_0^h \int_{Q_{T-h}} \sum_i a_i |D_i v^{(k)}(x,t+\tau)|^{p_i-1} |D_i v^{(k)}(x,t)|\, dz d\tau \\
& + \int_0^h \int_{Q_{T-h}} |\Delta_{(h)} v^{(k)}(x,t)|\, |f_K(v^{(k)}(x,t+\tau),(x,t+\tau))|\, dz d\tau \\
& + \delta \int_0^h \int_{Q_{T-h}} |\nabla v^{(k)}(x,t+\tau)|^{q-1} |\nabla(\Delta_{(h)} v^{(k)}(x,t))|\, dz d\tau \\
\equiv & \, I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

If p_i are independent of t, by Young's inequality

$$
I_1 + I_2 \leqslant C a_1 \int_0^h \Big(\sum_i \int_{Q_T} |D_i v^{(k)}(x,t)|^{p_i(x)}\, dz \Big)\, d\tau \leqslant C|h|
$$

with an independent of δ constant C. If $p_i(z) \equiv p_i(x)$, in the same way we obtain the estimate

$$
\begin{aligned}
I_1 + I_2 &\leqslant C a^+ \int_0^h \left(\sum_i \left(\int_{Q_T} |D_i v^{(k)}(x,t)|^{(p_i-1)q'}\, dz \right)^{\frac{1}{q'}} \delta^{-\frac{1}{q}} \big(\delta^{\frac{1}{q}} \|v^{(k)}\|_{q,Q_T} \big) \right) d\tau \\
&\leqslant C a^+ |h| \left(1 + \frac{1}{\delta} \big(\delta \|\nabla v^{(k)}\|^q_{q,Q_T} \big) \right) \delta^{-\frac{1}{q}} \big(\delta^{\frac{1}{q}} \|v^{(k)}\|_{q,Q_T} \big).
\end{aligned}
$$

Further,

$$
I_3 \leqslant C|h|\, \|v^{(k)}\|_{\frac{m+1}{m},Q_T} \|f_K\|_{m+1,Q_T} \leqslant C|h|,
$$

and, finally,

$$
I_4 \leqslant C|h| \big(\delta^{\frac{q'}{q}} \|\nabla v^{(k)}\|_{q,Q_T} \big)^{\frac{1}{q'}} \big(\delta^{\frac{1}{q}} \|\nabla v^{(k)}\|_{q,Q_T} \big).
$$

Finally, $\forall \mu \geqslant 2$, $\xi, \zeta \in \mathbb{R}$ by applying the inequality

$$
(|\xi|^{\mu-2}\xi - |\zeta|^{\mu-2}\zeta)(\xi - \zeta) \geqslant 2^{-\mu} |\xi - \zeta|^\mu
$$

we arrive at (5.4).

LEMMA 5.6. *Let* $\|\nabla m(x)\|_{\alpha,\Omega}<\infty$ *with some* $\alpha>1$. *Then for every* $\delta>0$ *the sequence* $\{v^{(k)}\}$ *contains a subsequence which converges to a function* v *pointwise in* Q_T.

Proof. It suffices to show that the sequences

$$w^{(k)}=|v^{(k)}|^{\frac{1}{m}}\operatorname{sign}v^{(k)},\quad s^{(k)}=|v^{(k)}|^{\frac{m+1}{m}}\operatorname{sign}v^{(k)}$$

are precompact in $L^1(0,T;L^{\mu_2}(\Omega\cap\{0<m<1\}))$ and $L^1(0,T;L^{\mu_1}(\Omega\cap\{m\geqslant 1\}))$ with some $\mu_1,\mu_2>1$. On the set $Q_T\cap\{0<m<1\}$, the assertion of the lemma for $\{v^{(k)}\}$ follows immediately from (5.2), (5.4) and the results of [26].

Let us denote $\Omega^+=\Omega\cap\{m\geqslant 1\}$ and $Q_T^+=\Omega^+\times(0,T)$. For every $t,t+h\in[0,T]$ one has

$$\begin{aligned}
&\int_{\Omega^+}|\Delta_{(h)}s^{(k)}(x,t)|\,dx\\
&=\int_{\Omega^+}\left|\int_0^1\frac{d}{d\theta}(\theta w^{(k)}(x,t+h)+(1-\theta)w^{(k)}(x,t))^{m+1}\,d\theta\right|\,dx\\
&\leqslant\int_0^1\int_{\Omega^+}(m+1)\Big(\int_0^1|\theta w^{(k)}(x,t+h)+(1-\theta)w^{(k)}(x,t)|^m\,d\theta\Big)|\Delta_{(h)}w^{(k)}(x,t)|\,dx\\
&\leqslant C\left(\||w^{(k)}(\cdot,t+h)|^m\|_{\frac{m+1}{m},\Omega^+}+\|w^{(k)}(\cdot,t)|^m)\|_{\frac{m+1}{m},\Omega^+}\right)\|\Delta_{(h)}w^{(k)}(\cdot,t)\|_{m+1,\Omega^+}\\
&:=I.
\end{aligned}$$

By virtue of (5.2), the first factor of I is bounded uniformly with respect to $t\in[0,T]$. The estimate for the second one follows from Lemma 5.5:

$$\begin{aligned}
&\int_0^{T-h}\|\Delta_h w^{(k)}(\cdot,t)\|_{m+1,\Omega^+}\,dt\\
&\leqslant C\int_0^{T-h}\left[\left(\int_{\Omega^+}|\Delta_{(h)}w^{(k)}|^{m+1}\,dx\right)^{\frac{1}{m^+ +1}}+\left(\int_{\Omega^+}|\Delta_{(h)}w^{(k)}|^{m+1}\,dx\right)^{\frac{1}{m^- +1}}\right]dt\\
&\leqslant C\left(h^{\frac{1}{m^+ +1}}+h^{\frac{1}{m^- +1}}\right).
\end{aligned}$$

On the other hand, for every $r>1$,

$$\begin{aligned}
\int_{Q_T^+}|\nabla s^{(k)}|^r\,dz\leqslant C_1\int_{Q_T^+}|v^{(k)}|^{\frac{r}{m}}|\nabla v^{(k)}|^r\,dz&\\
+C_2\int_{Q_T^+}|v^{(k)}|^{r\left(1+\frac{1}{m}\right)}|\ln|v^{(k)}||^r|\nabla m|^r\,dz&:=J_1+J_2.
\end{aligned}$$

By Hölder's inequality

$$J_1\leqslant C\left(\delta^{\frac{r}{q}}\|\nabla v^{(k)}\|^r_{q,Q_T}\right)\delta^{-\frac{r}{q}}\|v^{(k)}\|^{1-\frac{r}{q}}_{\frac{rq}{q-r},Q_T}\leqslant C\delta^{-\frac{r}{q}},\tag{5.5}$$

provided that $q > 2r$. To estimate J_2 we claim that $|\nabla m| \in L^{\alpha}(\Omega)$ and $\alpha > r > 1$. Since $|\ln s|^r \leqslant C(\gamma)s^{-\gamma}$ for $s \in (0,1]$ and $|\ln s|^r \leqslant C(\gamma)s^{\gamma}$ for $s > 1$ and an arbitrary positive γ (small),

$$J_2 \leqslant C\int_0^T \Big(\|\nabla m\|^{\alpha}_{\alpha,\Omega} + \int_{\Omega^+}\Big(|v^{(k)}|^{\left(\frac{\alpha}{r}\right)'\left(r\left(1+\frac{1}{m}\right)+\gamma\right)} \\ + |v^{(k)}|^{\left(\frac{\alpha}{r}\right)'\left(r\left(1+\frac{1}{m}\right)-\gamma\right)}\,dx\Big)\Big)\,dt$$

and for the sufficiently big q we may estimate both integrals by means of the Poincaré inequality and (5.2):

$$J_2 \leqslant C\Big(1 + \delta^{-\kappa}\big(\delta\|\nabla v^{(k)}\|_{q,Q_T}\big)^{\kappa}\Big). \tag{5.6}$$

By the compactness results in [26] the sequence $\{s^{(k)}\}$ contains a subsequence which converges in the norm of $L^1(0,T;L^{\mu_2}(\Omega))$.

COROLLARY 5.1. *The assertion of Lemma* 5.6 *remains true if instead of the condition* $\delta > 0$ *we claim that* $\delta \geqslant 0$ *and* $\|v^{(k)}\|_{\infty,Q_T} \leqslant \lambda$ *uniformly with respect to* k. *In this case we may take* $r = p^-$, *whence* $J_1 \leqslant C(1+\lambda)^{\frac{p^-}{m^-}}M$ *with the constant* M *from* (5.2), *the estimate on* J_2 *is obvious.*

Gathering the above assertions, for fixed $\varepsilon > 0$, $\delta > 0$ and $K > 1$ we may extract from the sequence $\{v^{(k)}\}$ a subsequence (for the sake of simplicity we keep for it the same notation) such that

$$\begin{cases} v^{(k)} \to v & \text{a.e. in } Q_T, \\ \Phi_\varepsilon(v^{(k)},x) \to \Phi_\varepsilon(v,x) & \text{a.e. in } Q_T, \\ \mathscr{F}_{\delta,i}(v^{(k)},z) \to A_i & \text{weakly in } L^{p_i'(\cdot)}(Q_T),\ i = 1,\dots,n, \\ \partial_t\Phi_\varepsilon(v^{(k)},x) \to V & \text{weakly in } L^{r'}(0,T;H^{-s}(\Omega)), \\ \Phi_\varepsilon(v^{(k)},x) \to \Phi_\varepsilon(v,x) & \text{weakly in } L^2(\Omega) \text{ uniformly in } t \in [0,T]. \end{cases} \tag{5.7}$$

It is easy to see that $V = \partial_t\Phi_\varepsilon(v,x) \in L^{r'}(0,T;H^{-s}(\Omega))$. By the definition of $v^{(k)}$

$$\int_{Q_T}\Big[\phi^{(s)}\partial_t\Phi_\varepsilon(v^{(k)},x) + \sum_i \mathscr{F}_{\delta,i}(v^{(k)},z)\cdot D_i\phi^{(s)} - \phi^{(s)}f_K(v^{(k)},z)\Big]\,dz = 0 \tag{5.8}$$

for every $\phi^{(s)} \in \operatorname{span}\{\psi_1,\dots,\psi_s\}$, $s \leqslant k$. Letting first $k \to \infty$ and then $s \to \infty$, we find that for every test-function $\phi \in L^r(0,T;H^s_0(\Omega))$,

$$\int_{Q_T}\Big[\phi\,\partial_t\Phi_\varepsilon(v,x) + \sum_i A_i\cdot D_i\phi - \phi\,f_K(v,z)\Big]\,dz = 0. \tag{5.9}$$

Since smooth functions are dense in $\mathbf{W}(Q_T)$, we then conclude that (5.9) holds for every test-function $\phi \in \mathbf{W}(Q_T) \cap L^q(0,T;W^{1,q}_0(\Omega))$ and that for the limit function $\partial_t\Phi_\varepsilon(v,x) \in \mathbf{W}'(Q_T) \cap L^{q'}(0,T;W^{1,q'}(\Omega))$. To identify A_i we rely on the monotonicity of the flux functions $\mathscr{F}_{\delta,i}(v^{(k)},z)$ and follow the classical scheme described in [23, Ch.2, Sec.2] (see also [4]) and conclude that $A_i = \mathscr{F}_{\delta,i}(v,z)$ a.e. in Q_T. Taking in (5.8) the independent of t test-function ϕ, integrating by parts in t, and then letting $k,s\to\infty$, by Lemma 5.3 we have that

$$\int_\Omega \phi(x)(\Phi_\varepsilon(v(x,t),x)-\Phi_\varepsilon(v_0,x))\,dx$$
$$= -\int_0^t\int_\Omega\Big[\sum_{i=1}^n \mathscr{F}_{\delta,i}(v,z)\cdot D_i\phi - f_K(v,z)\,\phi\Big]\,dz \to 0 \text{ as } t\to 0.$$

This completes the proof of Theorem 4.1.

6. Bounded solutions. Proof of Theorem 3.1

LEMMA 6.1. *Let in the conditions of Theorem* 4.1 $v_0\in L^\infty(\Omega)$. *Then for every* $K>\max\{\|v_0\|_{\infty,\Omega},1\}$ *there is* T_K *such that the solutions of problem* (4.1) *satisfy the uniform in* δ *and* ε *estimate*

$$\|v(t)\|_{\infty,\Omega}\leqslant \|v_0\|_{\infty,\Omega}\,\mathrm{e}^{A(K)t} \text{ for } t\in[0,T_K] \tag{6.1}$$

with a constant A *independent of* ε *and* δ.

The proof relies on the following assertion.

LEMMA 6.2. *Denote* $A_\varepsilon(\rho,x)=\varepsilon+\dfrac{1}{m}|\rho|^{\frac{1}{m}-1}$. *For every constant* $K>0$, $q\geqslant 1$ *and* $C_q>\dfrac{1}{m}+2q-1$,

$$G(v)\equiv C_q\int_0^{\max\{0,v\}}\rho A_\varepsilon(\rho,x)\min\{K^{2(q-1)},\rho^{2(q-1)}\}\,d\rho$$
$$-v\max\{0,v\}A_\varepsilon(v,x)\min\{K^{2(q-1)},v^{2(q-1)}\}\geqslant 0.$$

Proof. The assertion follows because $G(0)=0$ and the function $G(v)$ is nondecreasing:

$$G'(v)=\frac{1}{m}v^{\frac{1}{m}-1}\begin{cases}0, & \text{if } v\leqslant 0,\\ \big(C_q-\dfrac{1}{m}-(2q-1)\big)v^{2q-1} & \text{if } 0<v<K,\\ \big(C_q-\dfrac{1}{m}+1\big) & \text{if } v>K.\end{cases}$$

PROOF OF LEMMA 6.1 We consider in detail the more complicated case when $b(z)\geqslant 0$ and $\sigma^+>2$. Let us show the following estimate

$$\|\max\{0,v\}(t)\|_{\infty,\Omega}\leqslant\|\max\{0,v_0\}\|_{\infty,\Omega}\exp(At),$$

where A is a suitable constant. Let us fix some $K>1$ and test (5.9) with the function $\phi_q(v)=\max\{0,v\}\min\{K^{2(q-1)},v^{2(q-1)}\}$:

$$\begin{aligned}
&\frac{1}{h}\int_t^{t+h}\int_\Omega A_\varepsilon(v,x)\phi_k(v)v_t\,dx\\
&=\frac{1}{h}\int_t^{t+h}\frac{d}{dt}\Big(\int_\Omega\int_0^{\max\{0,v\}} sA_\varepsilon(s,x)\min\{K^{2(q-1)},s^{2(q-1)}\}\,ds\,dx\Big)dt\\
&\leqslant\frac{1}{h}\int_t^{t+h}\int_\Omega |b(z)|\max\{0,v\}\,|v|^{\frac{1-m}{m}}\big(\min\{v^2,K^2\}\big)^{\frac{\sigma-2}{2m}}\min\{K^{2(q-1)},v^{2(q-1)}\}\,dz\\
&\leqslant C_q\,b^+m^+\Big(\sup_{Q_t}\big(\min\{v^2,K^2\}\big)^{\frac{\sigma-2}{2m}}\Big)\\
&\qquad\times\frac{1}{h}\int_t^{t+h}\int_\Omega\int_0^{\max\{0,v\}} sA_\varepsilon(s,x)\min\{K^{2(q-1)},s^{2(q-1)}\}\,ds\,dz.
\end{aligned}$$

Let us denote

$$I=\int_0^{\max\{0,v\}} sA_\varepsilon(s,x)\min\{K^{2(q-1)},s^{2(q-1)}\}\,ds.$$

Letting $h\to 0$, we obtain the differential inequality for I:

$$\frac{dI(t)}{dt}\leqslant C_q b^+m^+\Big(\sup_{Q_T}\big(\min\{v^2,K^2\}\big)^{\frac{\sigma-2}{2m}}\Big)I(t),\qquad I(0)=I_0.$$

Integration of this inequality leads to the inequality $I^{\frac{1}{C_q}}(t)\leqslant I_0^{\frac{1}{C_q}}\mathrm{e}^{At}$ with the constants

$$C_q>2q-1,\quad A=b^+m^+\Big(\sup_{Q_T}\min\{v^2,K^2\}\Big)^{\frac{\sigma-2}{2m}}\leqslant b^+m^+K^\mu,\quad \mu=\sup_{Q_t}\frac{\sigma-2}{m}.$$

Letting $q\to\infty$ (i.e. $C_q\to\infty$), we arrive at (6.1) with A still depending on v. Let us claim that on the interval $(0,T_K)$ the constant A is independent of v. Let $K>\max\{\|v_0\|_{\infty,\Omega},1\}$. We want to choose it from the condition

$$\|v_0\|_{\infty,\Omega}\exp\big(b^+m^+\,t\,K^\mu\big)\leqslant K,$$

which can be written in the equivalent form:

$$t\leqslant\frac{1}{b^+m^+K^\mu}\ln\Big(\frac{K}{\|v_0\|_{\infty,\Omega}}\Big)=\frac{\|v_0\|^\mu_{\infty,\Omega}}{b^+m^+}\big(s^{-\mu}\ln s\big),\quad s=\frac{K}{\|v_0\|_{\infty,\Omega}}.$$

It remains to show that this inequality is indeed true for small t. Consider the function $g(s)=s^{-\mu}\ln s$, $s\geqslant 1$. It is nonnegative and attains its maximum when $s=\exp(1/\mu)$, which gives the value of T_K.

COROLLARY 6.1. *If $v_0\geqslant 0$ a.e. in Ω, then the solutions of problem* (4.1) *are nonnegative in Q_{T_K}.*

It follows that for $K > \max\{\|v_0\|_{\infty,\Omega}, 1\}$ the solutions of problem (4.1) are independently of δ and ε bounded in O_{T_K}, which means that there is an interval $[0, T_0]$ where $\|v(t)\|_{\infty,\Omega} < K$. Thus, $f_K(v,z) \equiv f(v,z)$ in Q_{T_0} and v is a bounded solution of the problem

$$\begin{cases} \partial_t \Phi_\varepsilon(v,x) = \operatorname{div}\Big(\mathscr{F}_\delta(z, \nabla v)\Big) + f(v,z) \text{ in } Q_{T_0}, \\ v(x,0) = v_0(x) \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_{T_0}, \end{cases} \tag{6.2}$$

which includes only two regularizations parameters ε and δ.

6.1. Passage to the limit as $\varepsilon \to 0$

To pass to the limit as $\varepsilon \to 0$ we notice that all the estimates for the solutions of the regularized problem (4.1) are independent of ε, and that the presence of ε was used only once in the proof of invertibility of the matrix B. Let $\{v_\varepsilon\}$ be the sequence of solutions of problem (6.2) with fixed $\delta > 0$. The functions v_ε satisfy the uniform in ε and δ a priori estimates of the previous section needed to pass to the limit as $\varepsilon \to 0$. According to (5.1) $\sup_{(0,T)} \|\sqrt{\varepsilon} v_\varepsilon\|_{2,\Omega}$ are bounded uniformly in ε. It follows that for every smooth test-function ϕ with $\phi_t \in C^\infty(0,T; C_0^\infty(\Omega))$

$$\begin{aligned} \varepsilon \left| \int_{Q_{T_0}} \phi \, \partial_t v_\varepsilon \, dz \right| &\leqslant \varepsilon \int_\Omega |\phi| \, |v_\varepsilon| \, dx + \varepsilon \int_\Omega |\phi(x,0)| \, |v_{0\varepsilon}| \, dx + \varepsilon \int_{Q_{T_0}} |v_\varepsilon| |\phi_t| \, dz \\ &\leqslant \sqrt{\varepsilon} \sup_{(0,T_0)} \|\sqrt{\varepsilon} v_\varepsilon\|_{2,\Omega} \sup_{(0,T_0)} \|\phi\|_{2,\Omega} + \varepsilon \|v_0\|_{2,\Omega} \|\phi(x,0)\|_{2,\Omega} \\ &\qquad + \sqrt{\varepsilon} \|v_\varepsilon\|_{2,Q_{T_0}} \|\phi_t|_{2,Q_{T_0}} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Since the smooth functions are dense in the set

$$\left\{\phi \,:\, \phi \in \mathbf{W}(Q_{T_0}), \, \partial_t \phi \in \mathbf{W}'(Q_{T_0})\right\},$$

the limit function $v \equiv v_\delta$ solves problem (6.2) with $\varepsilon = 0$.

6.2. Passage to the limit as $\delta \to 0$

Let us denote by $\{v_\delta\}$ the sequence of solutions of problem (4.1) with $\varepsilon = 0$ and $f_K \equiv f$. The functions v_δ satisfy all the estimates we need to extract a converging subsequence, except the estimate of Lemma 5.5 which is independent of δ only if $p_i \equiv p_i(x)$. Estimate (5.4) was used only in the proof of relative compactness of the sequences $\{v^{(k)}\}$ and $\{v_\varepsilon\}$. The uniform boundedness of solutions of problem (6.2) allows one to prove the relative compactness of the sequence $\{v_\delta\}$ with respect to δ in a different way.

LEMMA 6.3. *Let the conditions of Lemma* 6.1 *be fulfilled and let* $\{v_\delta\}$ *be a sequence of weak solutions of problem* 6.2 *with* $\varepsilon = 0$ *and* $\delta > 0$. *If* $\nabla m \in L^{p_i(z)}(Q_{T_0})$ *for all* $i = 1, \ldots, n$ *and either* $m(x) \in (0,1]$, *or*

$$1 > \frac{1}{p_i(z)} + \frac{1}{m(x)} \ \textit{in} \ Q_{T_0},$$

then the sequence $\{u_\delta\}$, $u_\delta \equiv \Phi_0^{-1}(v_\delta,x)$, *is relatively compact in* $L^\mu(Q_{T_0})$ *with some* $\mu>1$.

Proof. By virtue of (6.2), estimate (5.2) and Lemma 6.1 we have

$$\partial_t u_\delta \in L^{q'}(0,T_0;W^{-1,q}(\Omega)).$$

Testing equation (6.2) with the function $\phi = \displaystyle\int_0^{v_\delta} |s|^{\alpha-1}ds$, $\alpha\in(0,1)$, we obtain the inequality

$$\int_\Omega \frac{1}{1+m\alpha}|v_\delta|^{\frac{1}{m}+\alpha}\,dx\Big|_{t=0}^{t=T_0} + \delta\int_{Q_{T_0}} |v_\delta|^{\alpha-1}|\nabla v_\delta|^q\,dz \\ + a^-\sum_{i=1}^n \int_{Q_{T_0}} |v_\delta|^{\alpha-1}|D_i v_\delta|^{p_i(z)}\,dz \leqslant C$$

with an independent of δ constant C. It is easy to calculate that for bounded $|v_\delta|$

$$|D_i u_\delta|^{p_i(z)} \leqslant \frac{C_1}{m^{p_i}}|v_\delta|^{\left(\frac{1}{m}-1\right)p_i}|D_i v_\delta|^{p_i} + C_2\frac{|\nabla m|^{p_i}}{m^{2p_i}}\left||v_\delta|^{\frac{1}{m}}\ln|v_\delta|\right|^{p_i} \\ \leqslant C_3\left(|v_\delta|^{\left(\frac{1}{m}-1\right)p_i}|D_i v_\delta|^{p_i} + |\nabla m|^{p_i}\right).$$

If $m(x)\in(0,1]$, this inequality gives the uniform estimate on $\|\nabla u_\delta\|_{p^-,Q_{T0}}$. In the case $m\geqslant 1$, the same estimate follows if we claim that $(1/m-1)\,p_i(z)\leqslant \alpha-1$. The conclusion about relative compactness of the family $\{u_\delta\}$ in $L^\mu(Q_{T_0})$ with some $\mu>1$ follows from [29, Lemma 9.1]. □

We can now extract from $\{v_\delta\}$ a subsequence with the following properties:

$$\begin{cases} v_\delta \to v & \text{a.e. in } Q_{T_0}, \\ \Phi(v_\delta,x)\to\Phi(v,x) & \text{a.e. in } Q_{T_0}, \\ \Phi(v_\delta,x)\to\Phi(v,x) & \text{weakly in } L^2(\Omega) \text{ uniformly in } t\in[0,T_0], \\ \mathscr{F}_{\delta,i}(v_\delta,z)\to B_i & \text{weakly in } L^{p_i'(\cdot)}(Q_{T_0}), i=1,\dots,n, \\ \partial_t\Phi(v_\delta,x)\to W & \text{weakly in } L^{r'}(0,T_0;H^{-s}(\Omega)). \end{cases} \tag{6.3}$$

Moreover, the estimate $\delta\|\nabla v_\delta\|^q_{q,Q_T}\leqslant C$ yields

$$\delta\,|\nabla v_\delta|^{q-2}\nabla v_\delta \to 0 \ \text{ as } \ \delta\to 0 \ \text{ weakly in } L^q(Q_{T_0}).$$

Indeed: for every $\phi\in L^q(0,T;W_0^{1,q}(\Omega))$

$$\delta\left|\int_{Q_{T_0}} |\nabla v_\delta|^{q-2}\nabla v_\delta\cdot\nabla\phi\,dz\right| \leqslant \delta^{\frac{1}{q}}\left(\delta\int_{Q_{T_0}}|\nabla v_\delta|^q dz\right)^{1-\frac{1}{q}}\|\nabla\phi\|_{q,Q_{T_0}} \to 0$$

as $\delta\to 0$. The limit functions B_i are defined by monotonicity,

$$B_i = \mathscr{F}_{0,i}(\nabla v,z) = a_i(z)|D_i v|^{p_i(z)-2}D_i v.$$

These arguments are summarized in the following lemma.

LEMMA 6.4. *Let us assume that the conditions of Theorem* 4.1 *are fulfilled and, additionally,* $v_0 \in L^\infty(\Omega)$. *If either* p_i *are independent of* t, *or the conditions of Lemma* 6.3 *are fulfilled, then for every* $K > \max\{\|v_0\|_{\infty,\Omega}, 1\}$ *there exists* $T_0 > 0$ *such that problem* (6.2) *with* $\varepsilon = 0$ *has a solution* $v \in \mathbf{W}(Q_{T_0})$ *in the sense of Definition* 3.1. *Moreover,* $\|v(t)\|_{\infty,\Omega}$ *is bounded in* $[0,T_0]$ *and satisfies estimate* (6.1)*). The solution is nonnegative if* $v_0 \geqslant 0$ *in* Ω.

6.3. Continuation to the maximal existence interval

The constructed solution is defined on an interval $(0,T_0)$ with T_0 estimated in Lemma 6.1, the function $v(x,T_0)$ possesses the same properties that the initial function v_0. Taking now T_0 for the initial moment we may repeat all the above arguments to show that the solution of problem (3.1) can be continued to a time interval (T_0,T_1) with T_1 depending on $\|v(\cdot,T_0)\|_{\infty,\Omega}$. Continuing this process we obtain the sequence $\{T_k\}$ and a solution of problem (3.1) in the cylinders Q_{T_k}. If $T_k \to \infty$, the solution exists globally in time, otherwise $\lim T_k = T^* < \infty$ and problem (3.1) admits a local in time solution.

7. Strong energy solutions

In this section we improve estimates for the finite-dimensional Galerkin's approximations for the solutions of the regularized problem (4.1) under additional assumptions on the coefficients a_i, b and the nonlinearity exponents p_i, σ. These estimates remain true for the energy solution obtained as the limit of the sequence of the approximate solutions. For the sake of simplicity, we confine the derivation of these estimates to the simplest case when $b(z) \leqslant 0$ in Q_T and $\sigma^+ \leqslant 2$. Under these assumptions the solution constructed in the previous sections is global in time, i.e., can be continued to the cylinder Q_T with any $T > 0$. Let us introduce the energy function

$$E(t,v^{(k)}) = \int_\Omega \left(\frac{\delta}{q} |\nabla v^{(k)}|^q + \sum_{i=1}^n a_i \frac{|D_i v^{(k)}\ |^{p_i}}{p_i} + |b| \frac{|v^{(k)}|^{\frac{\sigma+m-1}{m}}}{\frac{\sigma+m-1}{m}} \right) dx. \tag{7.1}$$

We begin with deriving some energy relations. Multiplying each of the equations in (4.3) by $c'_j(t)$ and summing up in $j = \overline{1,k}$, we arrive at the equality

$$\begin{aligned} &\int_\Omega A_\varepsilon(v^{(k)},x)\left(v_t^{(k)}\right)^2 dx \\ &\qquad + \int_\Omega \left[\delta\, |\nabla v^{(k)}|^{q-2} \nabla v^{(k)} \nabla v_t^{(k)} + \sum_{i=1}^n \left(a_i |D_i v^{(k)}|^{p_i(z)-2} D_i v^{(k)} \right) D_i \left(v_t^{(k)}\right) \right] dx \\ &\qquad\qquad = \int_\Omega b |v^{(k)}|^{\frac{\sigma-m-1}{m}} v^{(k)} v_t^{(k)} dx. \end{aligned}$$

We will use the easily verified formulas

$$\delta\, |\nabla u|^{q-2} \nabla u \nabla u_t = \frac{\delta}{q} \frac{\partial}{\partial t} \left(|\nabla u|^q\right),$$

and

$$a_i|D_iu|^{p_i-2}D_iu\cdot D_iu_t=\frac{\partial}{\partial t}\Big(a_i\frac{|D_iu\ |^{p_i}}{p_i}\Big)$$
$$+a_i|D_iu\ |^{p_i}\left(\frac{1}{p_i^2}-\frac{\ln|D_iu\ |^{p_i}}{p_i}\right)p_{it}-a_{it}\frac{|D_iu\ |^{p_i}}{p_i},$$

$$b|u|^{\frac{\sigma-m-1}{m}}u\,u_t=\frac{\partial}{\partial t}\Big(b\frac{|u|^{\frac{\sigma+m-1}{m}}}{\frac{\sigma+m-1}{m}}\Big)$$
$$+b|u|^{\frac{\sigma+m-1}{m}}\left(\frac{1}{(\frac{\sigma+m-1}{m})^2}-\frac{\ln|u|}{\frac{\sigma+m-1}{m}}\right)\frac{\sigma_t}{m}-b_t\frac{|u|^{\frac{\sigma+m-1}{m}}}{\frac{\sigma+m-1}{m}}.$$

Then

$$\int_\Omega\delta\,|\nabla u|^{q-2}\nabla u\nabla u_t dx=\delta\frac{d}{dt}\int_\Omega|\nabla u(\cdot,t)|^q dx,$$

and

$$\int_\Omega\sum_{i=1}^n\big(a_i|D_iv^{(k)}|^{p_i(z)-2}D_iv^{(k)}\big)D_i(v_t^{(k)})dx$$
$$=\frac{d}{dt}\int_\Omega\Big(\sum_{i=1}^n a_i\frac{|D_iv^{(k)}\ |^{p_i}}{p_i}\Big)dx$$
$$+\int_\Omega\sum_{i=1}^n\Big(a_i|D_iv^{(k)}\ |^{p_i}\big(\frac{1}{p_i^2}-\frac{\ln|D_iv^{(k)}\ |^{p_i}}{p_i}\big)p_{it}-a_{it}\frac{|D_iv^{(k)}\ |^{p_i}}{p_i}\Big),$$

$$\int_\Omega b|v^{(k)}|^{\frac{\sigma-1}{m}-1}v^{(k)}\,v_t^{(k)}\,dx=\frac{d}{dt}\int_\Omega\left(b\frac{|v^{(k)}|^{\frac{\sigma+m-1}{m}}}{\frac{\sigma+m-1}{m}}\right)dx$$
$$+\int_\Omega\left(b|v^{(k)}|^{\frac{\sigma+m-1}{m}}\left(\frac{1}{(\frac{\sigma+m-1}{m})^2}-\frac{\ln|v^{(k)}|}{\frac{\sigma+m-1}{m}}\right)\frac{\sigma_t}{m}-b_t\frac{|v^{(k)}|^{\frac{\sigma+m-1}{m}}}{\frac{\sigma+m-1}{m}}\right)dx.$$

Gathering these formulas and taking into account the definition of the energy function, we arrive at the relation

$$\frac{dE(t,v^{(k)})}{dt}+\int_\Omega A_\varepsilon(v^{(k)})\big(v_t^{(k)}\big)^2\,dx=\Lambda_1+\Lambda_2,\tag{7.2}$$

where

$$\Lambda_1=\int_\Omega\sum_{i=1}^n\left(-a_i|D_iv^{(k)}\ |^{p_i}\left(\frac{1}{p_i^2}-\frac{\ln|D_iv^{(k)}\ |^{p_i}}{p_i}\right)p_{it}+a_{it}\frac{|D_iv^{(k)}\ |^{p_i}}{p_i}\right)dx,$$

$$\Lambda_2=\int_\Omega\left(b|v^{(k)}|^{\frac{\sigma+m-1}{m}}\left(\frac{1}{(\frac{\sigma+m-1}{m})^2}-\frac{\ln|v^{(k)}|}{\frac{\sigma+m-1}{m}}\right)\frac{\sigma_t}{m}-b_t\frac{|v^{(k)}|^{\frac{\sigma+m-1}{m}}}{\frac{\sigma+m-1}{m}}\right)dx.$$

LEMMA 7.1. *Let us assume that* $p_{it} = \sigma_t = 0$, *and that*

$$0 \leqslant -a_{it}(x,t) \leqslant C_a, \quad 0 \leqslant b_t(x,t) \leqslant C_b.$$

Then

$$E(t,v^{(k)}) + \int_0^t \int_\Omega A_\varepsilon(v^{(k)},x)\big(v_t^{(k)}\big)^2\,dxd\tau \leqslant E(0,v_0^{(k)}). \tag{7.3}$$

Inequality (7.3) *transforms into equality if* $a_{it} = b_t = 0$.

Proof. Under the assumptions of the lemma, relation (7.2) takes the form

$$\frac{d}{dt}E(t,v^{(k)}) + \int_\Omega A_\varepsilon(v^{(k)},x)\big(v_t^{(k)}\big)^2 dx$$
$$= -\int_\Omega\Big(\sum_{i=1}^n |a_{it}|\,\frac{|D_i v^{(k)}\ |^{p_i}}{p_i} + \frac{|b_t|\,m|u|^{\frac{\sigma+m-1}{m}}}{\sigma+m-1}\Big)dx \leqslant 0$$

and (7.3) upon integration in t.

LEMMA 7.2. *Let us assume that*

$$\begin{cases} 0 \leqslant -p_{it} \leqslant C_p, & 0 \leqslant \sigma_{it} \leqslant C_\sigma, \\ 0 \leqslant -a_{it}(x,t) \leqslant C_a, & 0 \leqslant b_t(x,t) \leqslant C_b \end{cases}$$

Then

$$E(t,v^{(k)}) + \int_0^t \int\limits_\Omega A_\varepsilon(v^{(k)})\big(v_t^{(k)}\big)^2\,dxd\tau \leqslant E(0,v_0^{(k)}) + Ct \tag{7.4}$$

with the constant $C = |\Omega|\mathrm{e}\left(a^+C_p + \frac{b^+}{m^-}C_\sigma\right)$.

Proof. In this case (7.2) takes on the form

$$\frac{d}{dt}E(t,v^{(k)}) + \int_\Omega A_\varepsilon(v^{(k)},x)\big(v_t^{(k)}\big)^2 dx$$
$$= \sum_{i=1}^n \int_\Omega\Big(-|a_{it}|\frac{|D_i v^{(k)}\ |^{p_i}}{p_i} + \frac{a_i|D_i v^{(k)}|^{p_i}}{p_i^2}\big(1 - p_i \ln|D_i v^{(k)}\ |\big)|p_{it}|\Big)\,dx$$
$$-\int_\Omega \frac{|b_t|\,m|v^{(k)}|^{\frac{\sigma+m-1}{m}}}{\sigma+m-1}dx$$
$$+\int_\Omega\Big(\frac{|\sigma_t|bm}{(\sigma+m-1)^2}|v^{(k)}|^{\frac{\sigma+m-1}{m}}\Big(1 - \frac{(\sigma+m-1)}{m}\ln|v^{(k)}|\Big)\Big)dx,$$

whence

$$\frac{dE(t,v^{(k)})}{dt} + \int_\Omega A_\varepsilon(v^{(k)},x)\Big(\frac{\partial v^{(k)}}{\partial t}\Big)^2 dx$$
$$\leqslant \sum_{i=1}^n \int_\Omega \frac{a_i|D_i v^{(k)}|^{p_i}}{p_i^2}\big(1 - p_i \ln|D_i v^{(k)}\ |\big)|p_{it}|dx$$

$$
\begin{aligned}
&+\int_\Omega \frac{|\sigma_t| bm}{(\sigma+m-1)^2}\left|v^{(k)}\right|^{\frac{\sigma+m-1}{m}}\left(1-\frac{(\sigma+m-1)}{m}\ln\left|v^{(k)}\right|\right)dx\\
&\equiv \Lambda_p+\Lambda_\sigma.
\end{aligned}
$$

We estimate Λ_p and Λ_σ in the following way:

$$
\begin{aligned}
\Lambda_p &\leqslant a^+ C_p \sum_{i=1}^n \int_{\Omega\cap\{p_i \ln|D_i v^{(k)}|\leqslant 1\}} \frac{|D_i v^{(k)}|^{p_i}}{p_i^2}dx\\
&\leqslant a^+ C_p\, n\,|\Omega|\left(\sup_{p^-\leqslant r\leqslant p^+}\left[\max_{0\leqslant \xi\leqslant \mathrm{e}^{\frac{1}{r}}}\frac{|\xi|^r}{r^2}\right]\right)\leqslant \mathrm{e}\, a^+ C_p\, n\,|\Omega| = C_1,
\end{aligned}
$$

$$
\Lambda_\sigma \leqslant \frac{b^+}{m^-}C_\sigma \int_{\Omega\cap\{(\sigma+m-1)\ln|v^{(k)}|\leqslant m\}} \frac{m^2\left|v^{(k)}\right|^{\frac{\sigma+m-1}{m}}}{(\sigma+m-1)^2}dx \leqslant \mathrm{e}\frac{b^+}{m^-}C_\sigma\,|\Omega| = C_2.
$$

Then

$$
\frac{d}{dt}E(t,v^{(k)})+\int_\Omega A_\varepsilon(v^{(k)},x)\left(\frac{\partial v^{(k)}}{\partial t}\right)^2 dx \leqslant C_1+C_2,
$$

and (7.4) follows upon integration in t. □

Lemma 7.1 and Lemma 7.2 allow one to derive stronger estimates for the energy solutions. Namely, under the conditions of Lemma 7.1,

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_\Omega\left(\delta|\nabla v^{(k)}|^q+\sum_{i=1}^n |D_i v^{(k)}|^{p_i}+|v^{(k)}|^{\frac{\sigma+m-1}{m}}\right)dx\\
&\qquad\qquad+\int_{Q_T} A_\varepsilon(v^{(k)},x)\left(v_t^{(k)}\right)^2 dz \leqslant CE(0,v_0^{(k)})
\end{aligned}
$$

with a constant C depending only on $a^\pm$, $b^\pm$, $p^\pm$, $\sigma^\pm$. If the conditions of Lemma 7.2 are fulfilled, then

$$
\begin{aligned}
&\sup_{t\in(0,T)}\int_\Omega\left(\delta|\nabla v^{(k)}|^q+\sum_{i=1}^n |D_i v^{(k)}|^{p_i}+|v^{(k)}|^{\frac{\sigma+m-1}{m}}\right)dx\\
&\qquad\qquad+\int_{Q_T} A_\varepsilon(v^{(k)},x)\left(v_t^{(k)}\right)^2 dz \leqslant E(0,v_0^{(k)})+CT
\end{aligned}
$$

with the constant $C=\mathrm{e}\,(a^+ nC_p+\frac{b^+}{m^-}C_\sigma)|\Omega|$.

These inequalities give the additional estimates for the energy solution obtained as the limit of the sequence of solutions of the regularized problem:

$$
\Psi(v)=\sup_{t\in(0,T)}\int_\Omega\left(\sum_{i=1}^n |D_i v|^{p_i}+|v|^{\frac{\sigma+m-1}{m}}\right)dx+\int_{Q_T}|v|^{\frac{1-m}{m}}|v_t|^2\,dz\leqslant C. \tag{7.5}
$$

The energy solutions of problem (3.1) satisfying (7.5) will be termed *Strong Energy Solutions*. Let us consider the function

$$
G(v^{(k)},x)=\int_0^{v^{(k)}}\sqrt{A_\varepsilon(s,x)}\,ds.
$$

Under the conditions of Lemmas 7.1, 7.2 it follows that $\|G_t(v^{(k)},x)\|_{2,Q_T}$ are uniformly bounded. Using the properties of convergence of the sequence of solutions of problem (4.1), we may extract a subsequence such that

$$
\begin{aligned}
&G(v^{(k)},x)=\int_0^{v^{(k)}}\sqrt{A_\varepsilon(s,x)}\,ds\to\frac{2\sqrt{m}}{1+m}|v|^{\frac{1+m}{2m}}v \text{ a.e. in } Q_T,\\
&G_t(v^{(k)},x)=\partial_t\Big(\int_0^{v^{(k)}}\sqrt{A_\varepsilon(s,x)}\,ds\Big)\to G^* \text{ weakly in } L^2(Q_T).
\end{aligned}
$$

It is easy to see that $G^*=G_t(v,x)$.

THEOREM 7.1. *Let in the conditions of Theorem* 3.1 $b(z)\leqslant b^-<0$ *in* Q_T *and* $\sigma^+\leqslant 2$. *Assume that either the conditions of Lemma* 7.1, *or the conditions of Lemma* 7.2 *are fulfilled. If* $E_0(0,v_0^{(k)})$ *are uniformly in* k *bounded, then the constructed solution of problem* (3.1)*) is a strong energy solution.*

The initial energy is bounded uniformly in k if we claim that v_0 possesses some extra regularity, say, $v_0\in H_0^s(\Omega)$ with a suitably big s (see the proof of Lemma 5.4). It is clear also that in this case the first term of $E(0,v_0^{(k)})$ given by (7.1) tends to zero as $\delta\to 0$.

8. Comparison principle and uniqueness of strong solutions

In this section we establish the comparison principle for strong energy solutions of the problem

$$
\begin{cases}
\partial_t\Big(|v|^{\frac{1}{m(x)}-1}v\Big)=\sum\limits_{i=1}^n D_i\Big(a_i|D_iv|^{p_i(z)-2}D_iv\Big)+b\,|v|^{\frac{\sigma(z)-1}{m(x)}-1}v \text{ in } Q_T,\\
v(x,0)=v_0(x) \text{ in } \Omega,\\
v=0 \text{ on } \Gamma_T.
\end{cases}
\tag{8.1}
$$

Given two solutions v_1, v_2 of problem (8.1), we introduce the functions

$$
w=|v_1|^{\frac{1}{m}-1}v_1-|v_2|^{\frac{1}{m}-1}v_2,\quad v=v_1-v_2.
$$

THEOREM 8.1. *Let* v_1, v_2 *be bounded strong energy solutions of problem* (8.1) *with the initial data* v_{01}, v_{02}. *Assume that*

$$
0<m^-\leqslant m(x)\leqslant m^+<1,\quad 2\leqslant\sigma^-. \tag{8.2}
$$

Then for all $t\in[0,T]$

$$
\|w(\cdot,t)\|_{L^1(\Omega)}\leqslant e^{Ct}\|w(\cdot,0)\|_{L^1(\Omega)},\quad C=\textit{ const.}>0.
$$

Proof. By virtue of (7.5) and (8.2),

$$\int_{Q_T}\left(\partial_t\left(|v|^{\frac{1}{m}-1}v\right)\right)^2 dz=\int_{Q_T}\frac{1}{m^2}|v|^{\frac{2(1-m)}{m}}|v_t|^2\,dz$$
$$\leqslant C\sup_{Q_T}\left(\frac{|v|^{\frac{1-m}{m}}}{m^2}\right)\int_{Q_T}|v|^{\frac{(1-m)}{m}}|v_t|^2\,dxdt$$
$$\leqslant C'.$$

Now we may follow the arguments of papers [1, 16, 17]. By the definition, for every $\phi\in\mathbf{W}(Q)$ with $\phi_t\in\mathbf{W}'(Q_T)$,

$$\int_Q\left[\phi\,\partial_t w+\sum_{i=1}^n a_i(z)\Big(|D_iv_1|^{p_i-2}D_iv_1-|D_iv_2|^{p_i-2}D_iv_2\Big)D_i\phi\right]dz$$
$$=\int_Q b(z)\Big(|v_1|^{\frac{\sigma-1}{m}-1}v_1-|v_2|^{\frac{\sigma-1}{m}-1}v_2\Big)\,\phi\,dz.$$

Let us introduce the function

$$T_\delta(s)=\frac{s}{\sqrt{\delta^2+s^2}},\quad \delta>0.$$

It is easy to check that

$$\int_0^w T_\delta(s)\,ds\to|w|,\quad T_\delta(s)\to\operatorname{sign}s\ \text{ as }\ \delta\to 0,$$

$$T'_\delta(s)=\frac{\delta^2}{(\delta^2+s^2)^{\frac{3}{2}}}>0,\quad |sT'_\delta(s)|\leqslant 1,\ s\in\mathbb{R}.$$

Now we will make use of the following assertion.

LEMMA 8.1. (Lemma 4.5, [1]) *Let* $v\in\mathbf{W}(Q_T)\cap L^\infty(Q_T)$, $\partial_t w\in\mathbf{W}'(Q_T)$. *If* $\partial_t w\in L^1(Q_T)$, *then*

$$\lim_{\delta\to 0}\int_{Q_t}T_\delta(v)\partial_t w\,dz=\int_\Omega|w(\cdot,s)|\,dx\Big|_{s=0}^{s=t}. \tag{8.3}$$

The proof is based on the observation that w and v have the same sign and, thus, if $T_\delta(v)\partial_t w$ has a limit as $\delta\to 0$, it must coincide with $\lim T_\delta(w)\partial_t w$ as $\delta\to 0$. Because of (7.5), in the case $m(x)\in(0,1]$ this lemma is applicable to the strong energy solutions. Testing (8.1) with $T_\delta(v)$, $\delta>0$, we get

$$\int_Q\left(T_\delta(v)\,\partial_t w+\sum_{i=1}^n a_i(z)\Big(|D_iv_1|^{p_i-2}D_iv_1-|D_iv_2|^{p_i-2}D_iv_2\Big)T'_\delta(v)D_iv\right)dz$$
$$=\int_Q b(z)\big(|v_1|^{\frac{\sigma-1}{m}-1}v_1-|v_2|^{\frac{\sigma-1}{m}-1}v_2\big)T_\delta(v)\,dz.$$

The second term on the left-hand side of this equality is nonnegative because of the monotonicity, while

$$\left|b(z)\left(|v_1|^{\frac{\sigma-1}{m}-1}v_1-|v_2|^{\frac{\sigma-1}{m}-1}v_2\right)\right|\leqslant C\left||v|^{\frac{1}{m}-1}v-|v|^{\frac{1}{m}-1}v\right|^{\sigma-2}\leqslant\widetilde{C}\,|w|\,.$$

Letting $\delta\to 0$, we arrive at the inequality

$$\int_{\Omega}|w(\cdot,s)|\,dx\Big|_{s=0}^{s=t}\leqslant C\int_{Q_t}|w|\,dz.$$

Writing this inequality in the form

$$Y'(t)\leqslant A+CY(t),\quad Y(t)=\int_0^t\int_{\Omega}|w|\,dz,\quad A=\int_{\Omega}|w(x,0)|\,dx,$$

and integrating, we obtain the estimate

$$Y(t)\leqslant\frac{A}{C}\left(\mathrm{e}^{Ct}-1\right),$$

whence

$$\int_{\Omega}|w(x,t)|\,dx=Y'(t)\leqslant A+CY(t)\leqslant\mathrm{e}^{Ct}\int_{\Omega}|w(x,0)|\,dx.$$

Under the conditions of Theorem 8.1 uniqueness of strong energy solutions is an immediate byproduct of the comparison principle.

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