

FINITE EXTINCTION AND CONTROL IN SOME DELAY MODELS

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*Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday*

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Abstract. For a controllable linear time-invariant system $x' = Ax + bu(t)$ in \mathbb{R}^n a general delayed feedback action $u(t) = -k(t)u(t - \tau)$ is proposed so that the solutions of the closed-loop system $x' = Ax - bk(t)x(t - \tau)$ are driven to zero in finite time. Optimality with respect to some integral performance indices is also analyzed.

1. Introduction

It is an honor and a pleasure for us to contribute to the celebration of J.I. Díaz's 60th birthday. Since 1975 we have kept a fruitful contact with him, first and above all as a close friend, as well as by the privilege of collaborating with such a first-class scientist.

In previous papers ([1]-[8]), Dr. J.I. Díaz and the authors have considered different aspects of a general delay-differential equation

$$x' = Ax - M(t)x(t - \tau), \quad t \geq 0, \quad (1)$$

where $\tau > 0$ is a given delay, A is the infinitesimal generator of a C^0 -semigroup in some Banach space X and $M(t)$ is a linear operator which guarantees the existence and uniqueness of solutions of (1) in some appropriate function space and is supposed to vanish outside some finite time interval.

The present work will only deal with the finite-dimensional situation, so X will be \mathbb{R}^n and A and $M(t)$ will be $n \times n$ matrices, M with continuous (or at least locally integrable) entries. Our basic hypothesis is

$$M(t) = 0 \quad \text{for every } t \notin [\tau, \tau'], \quad (2)$$

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where numbers τ and τ' satisfy $0 < \tau < \tau' \leq 2\tau$, so that (1) reduces to the linear ODE $x' = Ax$ except on the time interval $[\tau, \tau']$. Other choices for the time interval are possible but this is the simplest one.

The structure of this equation and the hypothesis (2) on $M(t)$ guarantee that any solution $x(t)$ which vanishes at some $t^* \geq \tau'$ will automatically vanish for all future times $t \geq t^*$ (this property it never holds for time-independent differential delay equations like $x' = Ax - Mx(t - \tau)$ for constant matrices A and M). We thus have *extinction in finite time* t^* or, more to the point of this paper, *exact null controllability* in the prescribed time interval $[\tau, \tau']$.

In [6] the following result was proven:

PROPOSITION 1. *Assume that $M(t) = 0$ for $t \notin [\tau, \tau']$, $M(t)$ commutes with e^{At} and $\int_{\tau}^{\tau'} M(t)dt = e^{A\tau}$. Then all solutions of (1) vanish for $t \geq \tau'$. In particular, if $m : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous scalar function which vanishes outside $[\tau, \tau']$ and satisfies $\int_{\tau}^{\tau'} m(t)dt = 1$, then all solutions of*

$$x'(t) = Ax(t) - m(t)e^{A\tau}x(t - \tau)$$

vanish for $t \geq \tau'$.

We now propose here a useful adaptation of this result to a general controllable system

$$x' = Ax + Bu \tag{3}$$

which will produce a *delayed feedback controller* of the form

$$u = K(t)x(t - \tau) \tag{4}$$

which will settle the system at the zero state after time $t^* = \tau'$, *independently of the initial conditions*. On the one hand, it is clear that the *closed loop system*

$$x' = Ax - BK(t)x(t - \tau) \tag{5}$$

has the structure (1) with matrix $M(t) = BK(t)$. Yet, the problem is that B and K will not be square in general, the product BK will not be invertible, and it is not clear how one can modify the previous choice $M(t) = m(t)e^{A\tau}$ in order to meet our needs. This is precisely what we propose in this paper for the “most extreme” case in which B has only one column and A has n different real eigenvalues.

The final section deals with some optimality questions. The usual optimal controls associated to quadratic integral criteria do not meet our continuity considerations and must be modified by including time-derivative penalty terms in the performance index. The resulting optimal controller gain matrix $K(t)$ turns out to be the solutions of second-order boundary value problems whose solutions can be explicitly computed by standard ODE methods, giving expressions of $k_{ij}(t)$ involving linear combinations of the functions $\exp((-\lambda_i + \lambda_j)t)$, where λ_i are the eigenvalues of A .

Section 2 contains the statements of the main results. The proofs and some extensions are given in Section 3.

2. Statement of results and remarks

Let us begin by extending the proposition mentioned above.

THEOREM 1. 1. *Let A be a real $n \times n$ matrix and let $M : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ be a continuous matrix-valued function vanishing outside the interval $[\tau, \tau']$, where $0 < \tau < \tau' \leq 2\tau$. Then all the solutions of the delay system*

$$x'(t) = Ax(t) - M(t)x(t - \tau), \quad t \geq 0 \quad (6)$$

vanish for $t \geq \tau'$ if and only if

$$\int_{\tau}^{\tau'} e^{-As} M(s) e^{As} ds = e^{A\tau}. \quad (7)$$

2. *If matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and associated modal matrix P such that $P^{-1}AP = D \stackrel{\text{def}}{=} \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, then (7) has some special solutions of the form*

$$M(t) = m(t)M,$$

where $m : [0, \infty) \rightarrow \mathbb{R}$ is a scalar continuous function vanishing outside $[\tau, \tau']$ with $\int_{\tau}^{\tau'} m(t) dt = 1$ and M is any constant matrix which can be written as

$$M = e^{A\tau} + PNP^{-1},$$

where N is an $n \times n$ matrix with zero diagonal entries.

The proof will be given in the next section.

In order to adapt this result to the control problem stated in the introduction we need some preliminary comments on controllability in its easiest form.

DEFINITION 1. *Given an $n \times n$ matrix A and an $n \times m$ matrix B , the linear, time invariant open-loop dynamical system*

$$x' = Ax + Bu \quad (8)$$

is said to be controllable if for every two states x^1 and x^2 and every time interval $[t_1, t_2]$ there exists at least one integrable control function $u : [t_1, t_2] \rightarrow \mathbb{R}^m$ such that the (unique) solution $x(t)$ to

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in [t_1, t_2] \\ x(t_1) = x^1 \end{cases}$$

satisfies $x(t_2) = x^2$.

PROPOSITION 2. (Kalman) *The necessary and sufficient condition for controllability is that the so-called "controllability matrix"*

$$\mathcal{C} \stackrel{\text{def}}{=} (B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B)$$

have full rank.

We may now state our main result in this section:

THEOREM 2. *Let A be a real $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and $b = \text{col}(b_1, \dots, b_n)$ a real column vector such that the pair (A, b) is controllable. Then there exists a vector continuous function $k(t) = \text{col}(k_1(t), \dots, k_n(t))$ vanishing outside $[\tau, \tau']$ such that all the solutions of the closed-loop delayed feedback system*

$$x'(t) = Ax(t) - bk(t)^T x(t - \tau) \quad (9)$$

vanish for $t \geq \tau'$. In fact,

$$k(t) = P^T \tilde{k}(t) \quad \text{or} \quad k(t)^T = \tilde{k}(t)^T P,$$

where P is the modal matrix associated to A (that is, the j -th column of P is an eigenvector of A associated to eigenvalue λ_j), and the functions $\tilde{k}_j(t)$ satisfy:

1. $\int_{\tau}^{\tau'} \tilde{k}_j(s) e^{(-\lambda_i + \lambda_j)s} ds = 0$ for $i \neq j$, $i, j = 1, 2, \dots, n$.
2. $\tilde{b}_i \int_{\tau}^{\tau'} \tilde{k}_i(s) ds = e^{\lambda_i \tau}$ for $i = 1, 2, \dots, n$, where $\tilde{b} = P^{-1}b$.

In particular, if m is any continuous scalar function vanishing outside $[\tau, \tau']$ and satisfying

$$\int_{\tau}^{\tau'} m(t) dt \stackrel{\text{def}}{=} \bar{m} \neq 0, \quad \int_{\tau}^{\tau'} m(t) e^{(\lambda_j - \lambda_i)t} dt = 0 \quad \text{for } i \neq j, \quad i, j \in \{1, 2, \dots, n\}. \quad (10)$$

then $k(t)$ may be chosen as $k(t) = m(t)k$, where $k = P^T \tilde{k}$ and

$$\tilde{k}_i = \frac{e^{\lambda_i \tau}}{\bar{m} \tilde{b}_i}, \quad i = 1, 2, \dots, n.$$

REMARKS 1. 1. The necessity of the controllability hypothesis is obvious: the standard (linear, finite-dimensional) theory proves that if we manage to steer any initial state $x(0)$ to zero, then the system is necessarily controllable.

2. The hypothesis that all eigenvalues be different is also necessary if A is diagonalizable and the control matrix B is a column (denoted b). Indeed, for a diagonal $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ the controllability matrix is

$$\mathcal{C} = \begin{pmatrix} b_1 & \lambda_1 b_1 & \dots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \dots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \lambda_n b_n & \dots & \lambda_n^{n-1} b_n \end{pmatrix}$$

which has two equal rows if (and only if) two eigenvalues coincide. Also, all entries b_i must be nonzero for controllability.

3. If the matrix A is not diagonalizable, the final conclusion of Theorem 2 may not hold: If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then (A, b) is controllable, but

$$e^{-At}bk(t)^T e^{At} = \begin{pmatrix} -tk_1(t) & -t(tk_1(t) + k_2(t)) \\ k_1(t) & tk_1(t) + k_2(t) \end{pmatrix}$$

and for the condition $\int_{\tau}^{\tau'} e^{-At}bk(t)^T e^{At} dt = e^{A\tau}$ to hold it is necessary that

$$\int_{\tau}^{\tau'} k_1(t)dt = 0, \int_{\tau}^{\tau'} tk_1(t) dt = -1, \int_{\tau}^{\tau'} k_2(t) dt = 2$$

These conditions are easily seen to be incompatible if we require $(k_1(t), k_2(t))$ to be a scalar multiple $m(t)$ of a constant vector (k_1, k_2) .

Optimality considerations

The functions m and k_i in Theorems 1 and 2 can be chosen in a number of meaningful optimal ways. The most immediate of them is to find the control m^* which minimizes the quadratic performance index or “energy” $\int_{\tau}^{\tau'} m(t)^2 dt$ on $[\tau, \tau']$ and the extend it by zero outside that interval.

However, since we are interested in having m continuous on $[0, \infty)$, we need to impose the “boundary conditions” $m(\tau) = m(\tau') = 0$, and this means that an additional derivative “penalty” term $\int_{\tau}^{\tau'} m'(t)^2 dt$ will need to be added to the performance index.

We will state here the simplest result. More general comments appear along with the proof in Section 3.

THEOREM 3. *There exists a unique function $m : [\tau, \tau'] \rightarrow \mathbb{R}$ satisfying $m(\tau) = m(\tau') = 0$ and minimizing*

$$J(m) \stackrel{\text{def}}{=} \int_{\tau}^{\tau'} (m(t)^2 + \alpha^2 m'(t)^2) dt$$

subject to the constraints

$$\int_{\tau}^{\tau'} m(t)dt = 1, \int_{\tau}^{\tau'} m(t)e^{(-\lambda_i + \lambda_j)t} dt = 0 \text{ for } i, j = 1, 2, \dots, n, i \neq j.$$

Such function solves the boundary value problem

$$\begin{cases} m - \alpha^2 m'' = \mu_0 + \sum_{i \neq j} \mu_{ij} e^{(-\lambda_i + \lambda_j)t}, \\ m(\tau) = m(\tau') = 0, \end{cases}$$

for some appropriate “Lagrange multipliers” μ_{ij} , μ_0 and therefore has the form

$$m(t) = c_1 e^{-t/\alpha} + c_2 e^{t/\alpha} + c_0 + \sum_{i \neq j} c_{ij} e^{(-\lambda_i + \lambda_j)t},$$

where the constants c_1, c_2, c_0 and c_{ij} ($i \neq j$) are found by solving the $(n(n-1) + 3) \times (n(n-1) + 3)$ determinate system of linear equations obtained by imposing the boundary conditions and the integral constraints.

If only the condition $\int_{\tau}^{\tau'} m(t) dt = 1$ is imposed (as in Theorem 1), the solution is simply

$$m(t) = \frac{1}{\tau' - \tau} \left[1 - \frac{1}{\cosh(1/\alpha)} \cosh \frac{2t - \tau - \tau'}{\alpha(\tau' - \tau)} \right]$$

whose graph is a catenary.

3. Proofs

3.1. Proof of Theorem 1

Let us recall our basic delay system (1):

$$x' = Ax - M(t)x(t - \tau), \quad t \geq 0. \quad (11)$$

STEP 1. *The integral equation for $M(t)$.* Let us apply the variation of constants formula on $[\tau, \tau']$:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}M(s)x(s - \tau) ds \quad \text{for } t \geq 0,$$

where $x(\theta)$ (the “initial function”) is given on $[-\tau, 0]$. We are interested in finding conditions on $\tilde{M}(t)$ that will ensure that $x(\tau') = 0$.

The fact that $M(t) = 0$ for $t \leq \tau$ implies two things: first, that for $t \in [\tau, \tau']$ the endpoints of integral on the right-hand side may be taken as τ and t ; and second, that when $\sigma \in [0, \tau]$ we have $x(\sigma) = e^{A\sigma}x(0)$. Hence, denoting $\xi \stackrel{\text{def}}{=} x(0)$ we may write:

$$\begin{aligned} x(t) &= e^{At}\xi - \int_0^t e^{A(t-s)}M(s)e^{A(s-\tau)}\xi ds \\ &= e^{At}\xi - e^{At} \left(\int_0^t e^{-As}M(s)e^{As} ds \right) e^{-A\tau}\xi \\ &= e^{At} \left[I - \left(\int_{\tau}^t e^{-As}M(s)e^{As} ds \right) e^{-A\tau} \right] \xi \quad \text{for } t \in [\tau, \tau']. \end{aligned}$$

Therefore, $x(\tau')$ will vanish for every $\xi \in \mathbb{R}^n$ if and only if

$$I - \left(\int_{\tau}^{\tau'} e^{-As}M(s)e^{As} ds \right) e^{-A\tau} = 0$$

or, equivalently

$$\int_{\tau}^{\tau'} e^{-As}M(s)e^{As} ds = e^{A\tau} \quad (12)$$

as was to be proven.

STEP 2. *Reduction to diagonal or "modal" form.* We now assume that

$$P^{-1}AP = D \stackrel{\text{def}}{=} \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

where the real numbers λ_i are all different.

We may write the integral equation 12 as

$$\begin{aligned} \int_{\tau}^{\tau'} e^{-As} M(s) e^{As} ds &= \int_{\tau}^{\tau'} P e^{-Ds} P^{-1} M(s) P e^{Ds} P^{-1} ds \\ &= P \left(\int_{\tau}^{\tau'} e^{-Ds} P^{-1} M(s) P e^{Ds} ds \right) P^{-1} \\ &= e^{A\tau} = P e^{D\tau} P^{-1}. \end{aligned}$$

Hence, by denoting $\tilde{M}(t) \stackrel{\text{def}}{=} P^{-1}M(t)P$, we have a similar integral equation with a diagonal D :

$$\int_{\tau}^{\tau'} e^{-Ds} \tilde{M}(s) e^{Ds} ds = e^{D\tau}. \quad (13)$$

The advantage of this change of variables is that, for a diagonal D , the analysis of the integral becomes feasible because of the easy identity

$$[\text{diag}(\alpha_1, \dots, \alpha_n) (q_{ij}) \text{diag}(\beta_1, \dots, \beta_n)]_{ij} = \alpha_i q_{ij} \beta_j \quad \text{for } i, j = 1, \dots, n.$$

In our case,

$$e^{-Ds} \tilde{M}(s) e^{Ds} = \begin{pmatrix} \tilde{m}_{11}(s) & \tilde{m}_{12}(s) e^{(-\lambda_1 + \lambda_2)s} & \dots & \tilde{m}_{1n}(s) e^{(-\lambda_1 + \lambda_n)s} \\ \tilde{m}_{21}(s) e^{(-\lambda_2 + \lambda_1)s} & \tilde{m}_{22}(s) & \dots & \tilde{m}_{2n}(s) e^{(-\lambda_2 + \lambda_2)s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{m}_{n1}(s) e^{(-\lambda_n + \lambda_1)s} & \tilde{m}_{n2}(s) e^{(-\lambda_n + \lambda_2)s} & \dots & \tilde{m}_{nn}(s) \end{pmatrix}.$$

Thus, for our integral equation to be satisfied it is necessary and sufficient that

1. $\int_{\tau}^{\tau'} \tilde{m}_{ij}(s) e^{(-\lambda_i + \lambda_j)s} ds = 0 \quad \text{for } i \neq j, i, j = 1, 2, \dots, n,$
2. $\int_{\tau}^{\tau'} \tilde{m}_{ii}(s) ds = e^{\lambda_i \tau} \quad \text{for } i = 1, 2, \dots, n,$

or just

$$\int_{\tau}^{\tau'} \tilde{m}_{ij}(s) e^{(-\lambda_i + \lambda_j)s} ds = \delta_{ij} e^{\lambda_i \tau} \quad \text{for } i, j = 1, \dots, n.$$

Once these functions have been chosen according to some specific criterion, the original matrix $M(t)$ is recovered as

$$M(t) = P \tilde{M}(t) P^{-1}.$$

STEP 3. *The case $M(t) = m(t)M$ for fixed M and $m(t) = 0$ outside $[\tau, \tau']$.* Under the hypothesis $M(t) = m(t)M$ (or $\tilde{M}(t) = m(t)\tilde{M}$), conditions (1) and (2) above translate into

$$1'. \quad \tilde{m}_{ij} \int_{\tau}^{\tau'} m(s) e^{-(\lambda_i + \lambda_j)s} ds = 0 \text{ for } i \neq j, i, j = 1, 2, \dots, n,$$

$$2'. \quad \tilde{m}_{ii} \int_{\tau}^{\tau'} m(s) ds = e^{\lambda_i \tau} \text{ for } i = 1, 2, \dots, n.$$

If $\int_{\tau}^{\tau'} m(s) ds = \bar{m} \neq 0$ then any $\tilde{M} = (1/\bar{m})e^{D\tau} + N$, where N has zero diagonal entries, will satisfy (1') and (2') without any further assumptions on $m(t)$. In terms of the original matrix M ,

$$M = P\tilde{M}P^{-1} = P((1/\bar{m})e^{D\tau} + N)P^{-1} = \frac{1}{\bar{m}}e^{A\tau} + PNP^{-1}$$

is a parametrization of the matrices M which solve our problem.

3.2. Proof of Theorem 2

The problem of going from a general square $M(t)$ to a control-theoretic factored matrix $bk(t)^T$ is no easy matter, not even with the parametrizations obtained before. This is because the non-vanishing integral condition imposed on $m(t)$ is no longer enough to solve our problem.

Continuing with the notations introduced in the previous proof and assuming that the pair (A, b) is controllable, let us see how the expression $M(t) = bk(t)^T$ changes after diagonalization, that is, after introducing matrix P and its associated $\tilde{M}(t) = P^{-1}M(t)P$.

If we perform the usual change of variables $y(t) = P^{-1}x(t)$, system $x' = Ax + bu$ becomes

$$y' = P^{-1}AP + P^{-1}b = Dy + P^{-1}bu.$$

It thus suffices to define the associated vectors

$$\tilde{b} \stackrel{\text{def}}{=} P^{-1}b, \quad \tilde{k}(t) \stackrel{\text{def}}{=} P^T k(t)$$

so that

$$bk(t)^T = (P\tilde{b})(P^{-T}\tilde{k}(t))^T = P(\tilde{b}\tilde{k}(t)^T)P^{-1},$$

where P^{-T} stands for $(P^{-1})^T = (P^T)^{-1}$. Therefore

$$M(t) = bk(t)^T \iff P\tilde{M}(t)P^{-1} = P(\tilde{b}\tilde{k}(t)^T)P^{-1}$$

is then equivalent to

$$\tilde{M}(t) = \tilde{b}\tilde{k}(t)^T$$

that is,

$$\tilde{m}_{ij}(t) = \tilde{b}_i \tilde{k}_j(t), \quad i, j = 1, 2, \dots, n$$

and the conditions given in the previous theorem now correspond to:

$$1. \quad \int_{\tau}^{\tau'} \tilde{m}_{ij}(s) e^{(-\lambda_i + \lambda_j)s} ds = \tilde{b}_i \int_{\tau}^{\tau'} \tilde{k}_j(s) e^{(-\lambda_i + \lambda_j)s} ds = 0 \text{ for } i \neq j, i, j = 1, 2, \dots, n,$$

$$2. \quad \int_{\tau}^{\tau'} \tilde{m}_{ij}(s) ds = \tilde{b}_i \int_{\tau}^{\tau'} \tilde{k}_i(s) ds = e^{\lambda_i \tau} \text{ for } i = 1, 2, \dots, n.$$

The hypothesis of controllability comes into play at this point. This property is invariant under linear changes of variables, so it is enough to check the controllability matrix $\tilde{\mathcal{C}}$ associated to the transformed diagonal system $y' = Dy + \tilde{b}u$, which is

$$\tilde{\mathcal{C}} = \begin{pmatrix} \tilde{b}_1 & \lambda_1 \tilde{b}_1 & \cdots & \lambda_1^{n-1} \tilde{b}_1 \\ \tilde{b}_2 & \lambda_2 \tilde{b}_2 & \cdots & \lambda_2^{n-1} \tilde{b}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_n & \lambda_n \tilde{b}_n & \cdots & \lambda_n^{n-1} \tilde{b}_n \end{pmatrix}.$$

Therefore, $\tilde{b}_1 \neq 0$, $\tilde{b}_2 \neq 0$, \dots , $\tilde{b}_n \neq 0$ is both necessary and sufficient for controllability. And this is precisely what was needed for the diagonal conditions (2) to hold. It is then clear that functions $\tilde{k}_i(s)$ may be chosen so that (1) also hold, and finally $k(t)$ be recovered as $P^{-T}\tilde{k}(t)$.

However, if we want to factor out

$$\tilde{k}(t) = \tilde{m}(t)\tilde{k}$$

for a scalar $\tilde{m}(t)$ and a *constant* vector \tilde{k} , we run into some difficulties not encountered in the previous theorem. In fact, the first condition cannot be done away with by merely imposing $\tilde{m}_{ij} = 0$ like before, because now \tilde{b}_i is not zero (recall that it is *essential* that it be nonzero for controllability). Hence we need to impose the “orthogonality conditions”

$$\int_{\tau}^{\tau'} \tilde{m}(t) e^{(-\lambda_i + \lambda_j)t} dt = 0, \quad i \neq j, \quad i, j = 1, 2, \dots, n,$$

as *extra hypotheses* for (2) to hold true.

REMARKS 2. If matrix A is not diagonalizable, the final conclusion of Theorem 2 may not hold: If

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then the pair (A, b) is controllable, but

$$e^{-At}bk(t)^T e^{At} = \begin{pmatrix} -tk_1(t) & -t(tk_1(t) + k_2(t)) \\ k_1(t) & tk_1(t) + k_2(t) \end{pmatrix}$$

and for the condition $\int_{\tau}^{\tau'} e^{-At}bk(t)^T e^{At} dt = e^{A\tau}$ to hold it is necessary and sufficient that

$$\int_{\tau}^{\tau'} k_1(t) dt = 0, \quad \int_{\tau}^{\tau'} tk_1(t) dt = -1, \quad \int_{\tau}^{\tau'} k_2(t) dt = 2.$$

These conditions are easily seen to be *incompatible* if we require $(k_1(t), k_2(t))$ to be a *scalar multiple* $m(t)$ of a constant vector (k_1, k_2) .

3.3. Optimal delayed feedback controls

Since the problem we are dealing with is basically concentrated on the interval $[\tau, \tau']$, it is natural to compare the feedback controllers we have produced

$$u(t) = k(t)^T x(t - \tau)$$

with the *optimal control problem* of the non-delayed initial problem

$$\begin{cases} \dot{x}' = Ax + bu(t), \\ x(\tau) = \eta, x(\tau') = 0, \end{cases}$$

where η ($= e^{A\tau}\xi$ in our previous notation) is arbitrary and “optimal” refers to some standard performance index like

$$J(u) = \int_{\tau}^{\tau'} |u(t)|^2 dt = \int_{\tau}^{\tau'} u(t)^T u(t) dt.$$

Unfortunately, it is easy to show that this optimal control $\bar{u}(t)$, given by

$$\bar{u}(t) = b^T e^{-A^T t} \left(\int_{\tau}^{\tau'} e^{-As} b b^T e^{A^T s} ds \right)^{-1} e^{-A\tau} \eta$$

(see Sontag ([10])) does not vanish on the endpoints τ, τ' , so it will not fit into the present discussion.

In order to force \bar{u} to vanish at these endpoints, the simplest device is to add a derivative penalty term, so we will consider instead

$$J(u) \stackrel{\text{def}}{=} \int_{\tau}^{\tau'} \left(|u(t)|^2 + \alpha^2 |u'(t)|^2 \right) dt = \int_{\tau}^{\tau'} \left(u(t)^T u(t) + \alpha^2 u'(t)^T u'(t) \right) dt,$$

where α is a nonzero positive number to be chosen according to the importance accorded to the penalty term.

Going back to our delayed problem, we thus assume that the control actions previously considered have an associated cost functional of the type

$$J(k) = \int_{\tau}^{\tau'} \left(|k(t)|^2 + \alpha^2 |k'(t)|^2 \right) dt = \int_{\tau}^{\tau'} \left(k(t)^T k(t) + \alpha^2 k'(t)^T k'(t) \right) dt.$$

Performing the standard change of variables to diagonal form

$$y(t) = P^{-1}x(t), \quad \dot{x}(t) = Py(t)$$

we obtain again to the associated diagonal delay system

$$y'(t) = Dt - P^{-1}bk(t)^T Py(t - \tau) \stackrel{\text{def}}{=} Dt - \tilde{b}\tilde{k}(t)^T y(t - \tau)$$

with $\tilde{b} = P^{-1}b$ and $\tilde{k}^T(t) = k(t)^T P$ as before. The functional J is now expressed as

$$J(\tilde{k}) = \int_{\tau}^{\tau'} \left(\tilde{k}(t)^T P^{-1}P^{-T}\tilde{k}(t) + \alpha^2 k'(t)^T P^{-1}P^{-T}k'(t) \right) dt$$

$$= \int_{\tau}^{\tau'} (\tilde{k}(t))^T \tilde{W} \tilde{k}(t) + \alpha^2 \tilde{k}'(t)^T \tilde{W} \tilde{k}'(t) dt,$$

where $\tilde{W} = P^{-1}P^{-T}$ is symmetric and positive definite. The integral constraints on \tilde{k} are given by

$$\int_{\tau}^{\tau'} e^{-Dt} \tilde{b} \tilde{k}(t) e^{Dt} dt = e^{D\tau}$$

or, equivalently,

$$\int_{\tau}^{\tau'} e^{(-\lambda_i + \lambda_j)t} \tilde{b}_i \tilde{k}_j(t) dt = \delta_{ij} e^{\lambda_i \tau}, \quad i, j = 1, 2, \dots, n.$$

We must therefore solve the following quadratic *isoperimetric variational problem* (where we have dropped the tilde symbols for simplicity):

$$(V) \begin{cases} \text{minimize } J(k) \stackrel{\text{def}}{=} \int_{\tau}^{\tau'} (k(t))^T W k(t) + \alpha^2 k'(t)^T W k'(t) dt \\ \quad = \int_{\tau}^{\tau'} \sum_{i,j=1}^n w_{ij} (k_i(t) k_j(t) + \alpha^2 k'_i(t) k'_j(t)) dt, \\ \text{subject to } \begin{cases} k_i(\tau) = 0, k_i(\tau') = 0, i = 1, 2, \dots, n, \\ \int_{\tau}^{\tau'} e^{(-\lambda_i + \lambda_j)t} b_i k_j(t) dt = e^{\lambda_i \tau} \delta_{ij}, i, j = 1, 2, \dots, n. \end{cases} \end{cases} \quad (14)$$

In the function space $H^1(\tau, \tau')$ of \mathbb{R}^n -valued functions with square-integrable derivatives this quadratic problem has a unique solution as long as the integral constraints are not incompatible and the set of admissible functions is not empty, and this is indeed the case since we have already produced some particular functions satisfying those constraints. Furthermore, since for each j the integrands

$$e^{(-\lambda_i + \lambda_j)t} b_i, \quad i = 1, 2, \dots, n$$

are linearly independent, the standard isoperimetric theory (see [9], 4.7) guarantees that if we define the Lagrangian functional in the usual way

$$\begin{aligned} \mathcal{L}(k, \mu) &= J(k) - \sum_{i,j=1}^n \mu_{ij} \left(\int_{\tau}^{\tau'} e^{(-\lambda_i + \lambda_j)t} b_i k_j(t) dt - e^{\lambda_i \tau} \delta_{ij} \right) \\ &= \int_{\tau}^{\tau'} \sum_{i,j=1}^n w_{ij} (k_i(t) k_j(t) + \alpha^2 k'_i(t) k'_j(t)) dt \\ &\quad - \sum_{i,j=1}^n \mu_{ij} \left(\int_{\tau}^{\tau'} e^{(-\lambda_i + \lambda_j)t} b_i k_j(t) dt - e^{\lambda_i \tau} \delta_{ij} \right) \\ &= \int_{\tau}^{\tau'} \mathcal{F}(t, k(t), k'(t)) dt, \end{aligned}$$

where the Lagrangian function $\mathcal{F} : \mathbb{R}^{1+2n} \rightarrow \mathbb{R}$ is given by

$$\mathcal{F}(t, k, k') \stackrel{\text{def}}{=} \sum_{i,j=1}^n \left[w_{ij} (k_i k_j + \alpha^2 k'_i k'_j) - \mu_{ij} e^{(-\lambda_i + \lambda_j)t} b_i k_j \right].$$

Then there exists a unique choice of the Lagrange multipliers μ_{ij} for which the variation $\delta\mathcal{L}$ vanishes at the (unique) optimum $\bar{k}(t)$ and therefore the Euler-Lagrange equations are satisfied there:

$$\frac{d}{dt} \left[\frac{\partial \mathcal{F}}{\partial k'_j}(t, \bar{k}(t), \bar{k}'(t)) \right] - \frac{\partial \mathcal{F}}{\partial k_j}(t, \bar{k}(t), \bar{k}'(t)) = 0, \quad j = 1, 2, \dots, n.$$

In our case we obtain

$$\frac{d}{dt} \sum_{i=1}^n 2\alpha^2 w_{ij} k'_i - \sum_{i=1}^n (2w_{ij} k_i - \mu_{ij} e^{(-\lambda_i + \lambda_j)t} b_i) = 0, \quad j = 1, 2, \dots, n$$

which, together with the required boundary data, can be written as

$$(BVP) \begin{cases} 2 \sum_{i=1}^n w_{ij} (k_i - \alpha^2 k'_i) = \sum_{i=1}^n \mu_{ij} e^{(-\lambda_i + \lambda_j)t} b_i, & j = 1, 2, \dots, n, \\ k_j(\tau) = 0, \quad k_j(\tau') = 0, & j = 1, 2, \dots, n, \\ \int_{\tau}^{\tau'} e^{-\lambda_i t} b_i k_j(t) e^{\lambda_j t} dt = \delta_{ij} e^{\lambda_i \tau}, & i, j = 1, 2, \dots, n, \end{cases}$$

which is the boundary value problem with integral constraints to be analyzed. This system may be expressed in matrix form as follows:

$$2(k - \alpha^2 k'')^T W = b^T e^{-Dt} \mu e^{Dt},$$

where μ represents the matrix (μ_{ij}) .

In order to obtain a simpler closed-form solution to this problem, let us assume that A is symmetric and P is orthogonal. This ensures that $W = P^{-1} P^{-T}$ is the identity matrix and thus the problem becomes uncoupled:

$$(BVP)'_j \begin{cases} 2(k_j - \alpha^2 k'_j) = \sum_{i=1}^n \mu_{ij} e^{(-\lambda_i + \lambda_j)t} b_i, \\ k_j(\tau) = 0, \quad k_j(\tau') = 0, \\ \int_{\tau}^{\tau'} e^{-\lambda_i t} b_i k_j(t) e^{\lambda_j t} dt = \delta_{ij} e^{\lambda_i \tau}, \quad i = 1, 2, \dots, n, \end{cases} \tag{15}$$

(where $j = 1, 2, \dots, n$).

If α is such that $1 - \alpha^2(\lambda_j - \lambda_i)^2$ is never zero, the solutions of $(BVP)'$ can be explicitly obtained:

$$k_j(t) = \frac{1}{2} \sum_{i=1}^n \frac{\mu_{ij} b_i}{1 - \alpha^2(\lambda_j - \lambda_i)^2} e^{(-\lambda_i + \lambda_j)t} + c_{j1} e^{-t/\alpha} + c_{j2} e^{t/\alpha}, \quad j = 1, \dots, n. \tag{16}$$

Otherwise the solution contains some extra polynomial terms multiplying the exponentials, but the analysis remains the same.

We have thus $n^2 + 2n$ unknowns $(\mu_{ij}$ and $c_{j1}, c_{j2})$ together with $n^2 + 2n$ side conditions:

$$\begin{cases} k_j(\tau) = 0, \quad k_j(\tau') = 0, & j = 1, 2, \dots, n, \\ \int_{\tau}^{\tau'} e^{-\lambda_i t} b_i k_j(t) e^{\lambda_j t} dt = \delta_{ij} e^{\lambda_i \tau}, & i, j = 1, 2, \dots, n, \end{cases}$$

which can be uniquely solved since the variational problem does have a unique solution.

In the general case (that is, when $W \neq I$), the system is coupled but it can be solved in the same way. From

$$2(k - \alpha^2 k'')^T W = b^T e^{-Dt} \mu e^{Dt},$$

we conclude

$$2(k - \alpha^2 k'')^T = b^T e^{-Dt} \mu e^{Dt} W^{-1}$$

which corresponds to

$$(V) \begin{cases} 2(k_j - \alpha^2 k_j'') = \sum_{i=1}^n \sum_{r=1}^n e^{(-\lambda_i + \lambda_r)t} b_i \mu_{ir} w_{rj}^{-1}, & j = 1, 2, \dots, n, \\ k_j(\tau) = 0, \quad k_j(\tau') = 0, & j = 1, 2, \dots, n, \\ \int_{\tau}^{\tau'} e^{-\lambda_i t} b_i k_j(t) e^{\lambda_j t} dt = 0 & \text{for } i, j = 1, 2, \dots, n, \end{cases}$$

where w_{rj}^{-1} is the (r, j) entry of W^{-1} . The general solution of the differential equation is now

$$k_j(t) = \frac{1}{2} \sum_{i=1}^n \sum_{r=1}^n \frac{\mu_{ir} w_{rj}^{-1} b_i}{1 - \alpha^2 (\lambda_r - \lambda_i)^2} e^{(-\lambda_i + \lambda_r)t} + c_{j1} e^{-t/\alpha} + c_{j2} e^{t/\alpha}$$

and the constants are again determined by the side conditions.

Finally, the original $k(t)$ (before the diagonalization process) is recovered from $\tilde{k}(t)$ given above by means of the relation $k(t)^T P = \tilde{k}(t)^T$.

Let us finish by considering the simpler case

$$\tilde{k}_i(t) = m(t) \tilde{k}_i, \quad i = 1, 2, \dots, n$$

for a fixed function $m(t)$ and constant gains \tilde{k}_i whose values were obtained in Theorem 2. Then

$$J(m) = \int_{\tau}^{\tau'} (\tilde{k}(t)^T \tilde{W} \tilde{k}(t) + \alpha^2 \tilde{k}'(t)^T \tilde{W} \tilde{k}'(t)) dt = \tilde{k}^T \tilde{W} \tilde{k} \int_{\tau}^{\tau'} (m(t) + \alpha^2 m'(t)) dt$$

and the variational problem to be considered is

$$(V) \begin{cases} \text{minimize } J(m) \stackrel{\text{def}}{=} \int_{\tau}^{\tau'} (m(t) + \alpha^2 m'(t)) dt, \\ \text{subject to } \begin{cases} m(\tau) = 0, \quad m(\tau') = 0, & i = 1, 2, \dots, n, \\ \int_{\tau}^{\tau'} m(t) dt = 1, \\ \int_{\tau}^{\tau'} e^{(-\lambda_i + \lambda_j)t} m(t) dt = 0, & i, j = 1, 2, \dots, n, \quad i \neq j, \end{cases} \end{cases}$$

which has the same, but simpler, structure than the one just analyzed. By following exactly the same steps, one shows that this problem has a unique solution of the form

$$m(t) = c_1 e^{-t/\alpha} + c_2 e^{t/\alpha} + c_0 + \sum_{i \neq j} c_{ij} e^{(-\lambda_i + \lambda_j)t}.$$

3.4. An example

Let us consider the controlled pendulum equation

$$m\theta'' + l \sin \theta = u(t),$$

where $u(t)$ stands for the external forces. We want to settle the pendulum in the upright position. Linearizing around $\bar{\theta} = \pi$, and denoting $z(t) \stackrel{\text{def}}{=} \theta(t) - \pi$ we find

$$mz'' - lz = u(t)$$

or, in matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ l & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

Assume $m = 1$ and $l = 1$ so that the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$. A modal matrix is

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and thus,

$$\tilde{b} = P^{-1}b = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \tilde{k}^T = k^T P = (k_1, k_2) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = (k_1 - k_2, k_1 + k_2)$$

and the associated diagonal system is

$$\frac{d}{dt} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} (\tilde{k}_1, \tilde{k}_2) \begin{pmatrix} y_1(t - \tau) \\ y_2(t - \tau) \end{pmatrix}.$$

The boundary-value problem (BVP) (15) becomes

$$\begin{cases} 2(\tilde{k}_1 - \alpha^2 \tilde{k}_1'') = -\frac{\mu_{11}}{2} + \frac{1}{2} \mu_{21} e^{-2t}, \\ 2(\tilde{k}_2 - \alpha^2 \tilde{k}_2'') = -\frac{\mu_{12}}{2} e^{2t} + \frac{1}{2} \mu_{22}, \end{cases}$$

whose general solution is given by (16):

$$\begin{cases} \tilde{k}_1(t) = -\frac{1}{4} \mu_{11} + \frac{1}{4} \frac{\mu_{21}}{1 - 4\alpha^2} e^{-2t} + c_{11} e^{-t/\alpha} + c_{12} e^{t/\alpha}, \\ \tilde{k}_2(t) = -\frac{1}{4} \frac{\mu_{12}}{1 - 4\alpha^2} e^{2t} + \frac{1}{4} \mu_{22} + c_{21} e^{-t/\alpha} + c_{22} e^{t/\alpha}, \end{cases}$$

together with the eight side conditions

$$\begin{cases} \tilde{k}_1(\tau) = 0, \quad \tilde{k}_1(\tau') = 0, \quad \int_{\tau}^{\tau'} \tilde{k}_1(t) dt = -2e^{-\tau}, \quad \int_{\tau}^{\tau'} e^{-2t} \tilde{k}_1(t) dt = 0, \\ \tilde{k}_2(\tau) = 0, \quad \tilde{k}_2(\tau') = 0, \quad \int_{\tau}^{\tau'} \tilde{k}_2(t) dt = 2e^{\tau}, \quad \int_{\tau}^{\tau'} e^{2t} \tilde{k}_2(t) dt = 0. \end{cases}$$

Let us take, for instance, $\alpha = 1$, $\tau = 1$ and $\tau' = 2$. Then the side conditions become

$$\begin{cases} 0.36788c_{11} - 0.011278\mu_{21} - 0.25\mu_{11} + 2.7183c_{12} = 0, \\ 0.13534c_{11} - 0.0015263\mu_{21} - 0.25\mu_{11} + 7.3891c_{12} = 0, \\ 0.23254c_{11} - 0.0048758\mu_{21} - 0.25\mu_{11} + 4.6708c_{12} + 0.73576 = 0, \\ 0.015769c_{11} - 0.00037459\mu_{21} - 0.014627\mu_{11} + 0.23254c_{12} = 0, \\ 0.61575\mu_{12} + 0.25\mu_{22} + 0.36788c_{21} + 2.7183c_{22} = 0, \\ 4.5498\mu_{12} + 0.25\mu_{22} + 0.13534c_{21} + 7.3891c_{22} = 0, \\ 1.967\mu_{12} + 0.25\mu_{22} + 0.23254c_{21} + 4.6708c_{22} - 5.4366 = 0, \\ 60.966\mu_{12} + 5.9011\mu_{22} + 4.6708c_{21} + 127.78c_{22} = 0, \end{cases}$$

whose unique solution is:

$$\begin{aligned} \mu_{11} &= 595.56, \mu_{21} = 10131.0, c_{11} = 637.19, c_{12} = 10.572, \\ \mu_{12} &= 186.4, \mu_{22} = 4420.5, c_{21} = -1576.1, c_{22} = -235.47. \end{aligned}$$

Therefore, the optimal solution is

$$\begin{cases} \tilde{k}_1(t) = -148.89 - 844.25e^{-2t} + 637.19e^{-t} + 10.572e^t, \\ \tilde{k}_2(t) = 1105.1 + 15.533e^{2t} - 235.47e^t - 1576.1e^{-t}. \end{cases}$$

The original $k(t)$ is recovered from $\tilde{k}(t)$ by means of:

$$k(t) = P^{-T}\tilde{k}(t) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{k}_1(t) \\ \tilde{k}_2(t) \end{pmatrix}.$$

4. Final remarks

1. The conditions (10) imply that no result of this type seems to be readily available for infinite-dimensional systems because of the infinitely many eigenvalues usually associated to their infinitesimal generators. This fact is also related to the fact that many infinite-dimensional control systems are only “approximately controllable”. The authors are working on this topic at the moment.

2. No substantial differences appear if vector b is allowed to depend on t . In that case the general solution (16) cannot be written explicitly and the best one can do is to use the Green’s function $G(t, s)$ associated to the Dirichlet problem

$$\begin{cases} v - \alpha^2 v'' = f(t) & \text{on } [\tau, \tau'], \\ v(\tau) = 0, v(\tau') = 0, \end{cases}$$

and obtain

$$k_j(t) = \frac{1}{2} \sum_{i=1}^n \mu_{ij} \int_{\tau}^{\tau'} G(t, s) e^{(-\lambda_i + \lambda_j)s} b_i(s) ds, \quad j = 1, \dots, n.$$

3. Similarly, nonautonomous systems like

$$x' = A(t)x + b(t)x(t - \tau)$$

can be considered by the usual change of variables $x = \Phi(t)y$, where $\Phi(t)$ is a fundamental matrix associated to $x' = A(t)$, obtaining

$$y' = \Phi(t)^{-1}b(t)\Phi(t - \tau)y(t - \tau)$$

and assuming that the gramian matrix

$$\int_{\tau}^t \Phi(t)^{-1}b(t)b(t)^T \Phi(t)^{-T} dt$$

is nonsingular. This is equivalent (see [11]).

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