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A QUASILINEAR PARABOLIC MODEL FOR POPULATION EVOLUTION

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Dedicated to Professor Jesús Ildefonso Díaz
on the occasion of his 60th birthday

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Abstract. A quasilinear parabolic problem is investigated. It models the evolution of a single population species with a nonlinear diffusion and a logistic reaction function. We present a new treatment combining standard theory of monotone operators in $L^2(\Omega)$ with some order-preserving properties of the evolutionary equation. The advantage of our approach is that we are able to obtain the *existence* and *long-time asymptotic behavior* of a weak solution almost simultaneously. We do not employ any uniqueness results; we rely on the uniqueness of the minimal and maximal solutions instead. At last, we answer the question of (long-time) survival of the population in terms of a critical value of a spectral parameter.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $\lambda > 0$ a real parameter, $q > 0$ a fixed number, $u_0 \in L^2(\Omega)$, and $m \in L^\infty(\Omega)$ a possibly sign-changing function. Consider the quasilinear parabolic problem

$$\begin{cases} \partial_t u - \Delta_p u = \lambda |u|^{p-2} u \left(m(x) - |u|^q\right) \text{ in } \Omega \times (0,\infty), \\ \qquad \partial_\nu u = 0 \text{ on } \partial\Omega \times (0,\infty), \\ \qquad u(0) = u_0 \text{ in } \Omega, \end{cases} \tag{1}$$

where Δ_p is the p-Laplace operator, defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$, and ∂_ν stands for the derivative in the direction of the exterior unit normal to the boundary $\partial\Omega$. The purpose of this paper is to investigate the existence and the asymptotic behavior as t goes to infinity of *nonnegative* solutions of (1).

The equation in (1) is actually a nonlinear generalization of the well-known logistic equation which is used in biology to model a population of density u at time t (see for instance [3, 7, 18]). From this point of view, the existence of a nonnegative solution not tending to zero as t goes to infinity means that the population survives after infinite

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time. Our aim is thus to find conditions on the intrinsic growth rate λ and the weight function m that guarantee (long-time) survival of the species.

To emphasize the main features of problem (1), we formulate a part of our results in a more general framework. More precisely, we replace the equation in (1) by

$$\partial_t u + Au = g(x,u) \quad \text{in } \Omega \times (0,\infty), \tag{2}$$

where A is a second order differential operator of Leray-Lions type, and g an increasing function in u. Assuming the presence of a pair of ordered sub- and supersolutions (α_0, β_0), and under some conditions on A and g, we show the existence of minimal and maximal solutions in the interval $[\alpha_0, \beta_0]$. This result is more or less classical (see [5, 11] and related papers), but here we present a proof from which information on the asymptotic behavior of solutions can be deduced. Our approach consists in combining the theory of *monotone operators* from [19] (see also [6]) with the method of *monotone iterations* from [23]. The minimal and maximal solutions being uniquely determined, we are able to allow for nonuniqueness of weak solutions of problem (2).

The technique of monotone iterations (also called the Chaplygin method) was first introduced in [23] in the context of semilinear *parabolic* PDEs. In addition, we point out that the use of iterative schemes in the case of general *nonlinear* elliptic operators of Leray-Lions type first appeared in Díaz' famous book [12]. A nice review on monotone methods can be found in the work [8] where the one-dimensional φ-Laplacian is considered. In the literature, the study of quasilinear parabolic problems similar to (1) has mostly been carried out in the case of Dirichlet boundary conditions; see [13, 17] and references therein. We also refer to [14, 21] for existence and uniqueness results on the *stationary* problem associated to (1). Although both works [8, 12], and [14] as well, treat only problems with nonlinear *elliptic* operators, their approach is based on a number of very general properties shared by both, elliptic and parabolic operators, such as the weak comparison principle and regularity of solutions, for instance. Therefore, in the work reported here we have been able to adopt some of their techniques and apply them to problems with nonlinear *parabolic* operators; cf. [13, 19, 21].

In the present paper, one of our contributions is the proof of the continuity of the solution operator associated to (2) (see Proposition 2.1). Since our problem is nonlinear, we also rewrite the arguments from [23] by using a parabolic comparison principle (see Proposition 2.2) instead of the standard linear parabolic maximum principle. Finally, in the study of the asymptotic behavior of solutions, we handle the possible nonuniqueness of solutions by proving a comparison principle for minimal and maximal solutions (see Lemma 3.3).

This paper is organized as follows. In Section 2, we deal with the existence of solutions of an abstract parabolic problem involving a strongly monotone operator. The asymptotic behavior of solutions of this problem is studied in Section 3. At last, in Section 4 we apply our results to problem (1). In the case when m changes sign and $\int_\Omega m < 0$, it is established that if $\lambda > \lambda_1(m)$ then any positive solution u of (P_{u_0}) converges to the unique positive stationary solution $\bar{u}$ of (1) as $t \to \infty$, whereas u converges to zero when $\lambda \leqslant \lambda_1(m)$. In other words, the principal eigenvalue $\lambda_1(m)$ appears as the threshold value of λ under which the species does not survive.

2. Existence via monotone iterations

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ (with $N \geqslant 1$) and $T > 0$ be given. In this section, we are concerned with the existence of solutions $u = u(x,t)$ of the problem

$$\begin{cases} \partial_t u + Au = g(x,u) \text{ in } \Omega \times (0,T), \\ \partial_\nu u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(0) = u_0 \text{ in } \Omega. \end{cases} \tag{3}$$

Here $u_0 \in L^2(\Omega)$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is increasing in its second argument, and A is a second order differential operator of Leray-Lions type [19]

$$Au(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(x, u(x,t), \nabla u(x,t)\big), \tag{4}$$

satisfying properties (A_1)-(A_3) below.

More precisely, our aim is to show the existence of at least one (bounded) solution of (3) in the presence of a pair of well-ordered upper and lower solutions. As mentioned before, this result has already been established by several authors. However, in order to deduce some information on the asymptotic behavior of solutions, we give here a proof based on the method of monotone iterations from [23].

First of all, we introduce a few assumptions and definitions. Let $1 < p < \infty$ and let V be a separable reflexive Banach space which is continuously embedded in $W^{1,p}(\Omega) \cap L^2(\Omega)$, and dense in $L^2(\Omega)$. Then we denote by $L^p(0,T;V)$ the Banach space of all strongly measurable (i.e. Bochner-measurable) functions $u : (0,T) \to V$ such that

$$\|u\|_{L^p(0,T;V)} := \Big(\int_0^T \|u\|_V^p \,\mathrm{d}t\Big)^{1/p} < \infty.$$

It is well-known that $L^p(0,T;V)$ is reflexive whenever V is reflexive; its dual space coincides with $L^{p'}(0,T;V^*)$. The duality pairing between V^* and V will be denoted by $\langle\ ,\ \rangle$. In the sequel, we also set

$$W^p(0,T;V) := \{u \in L^p(0,T;V) : \partial_t u \in L^{p'}(0,T;V^*)\},$$

where $\partial_t u$ is the distributional derivative of u. The space $W^p(0,T;V)$ is a Banach space for the norm

$$\|u\|_{W^p} := \Big(\int_0^T \|u\|_V^p \,\mathrm{d}t\Big)^{1/p} + \Big(\int_0^T \|\partial_t u\|_{V^*}^{p'} \,\mathrm{d}t\Big)^{1/p'}.$$

The reader is referred to [19, 25] for more details about the spaces $L^p(0,T;V)$, and $W^p(0,T;V)$.

The operator A is given by (4), and satisfies the following assumptions.

(A_1) $A \colon V \to V^*$ is continuous;

(A_2) there exists $C > 0$ such that $\|Au\|_{V^*} \leqslant C\|u\|_V^{p-1}$ for all $u \in V$;

(A_3) A is *strongly monotone* in the following interpolation sense: there exist $\theta \in (0,1]$, and $c > 0$ such that for all $u, v \in V$,

$$c\|u-v\|_V \leqslant \langle Au - Av, u-v\rangle^{\theta} \left(\|u\|_V + \|v\|_V\right)^{1-\theta}.$$

Let us point out that for $A = -\Delta_p$, we have $\theta = \min\{1, p/2\}$ (see Section (17)). Furthermore, the continuity of the embedding $V \subset W^{1,p}(\Omega) \cap L^2(\Omega)$ implies that A obeys also the following *coercivity condition* from [19, Chapter 2]:

(A_4) there exists $\alpha > 0$ such that $\langle Au, u\rangle \geqslant \alpha \|\nabla u\|_{L^p}^p$ for all $u \in V$, where $\|\nabla u\|_{L^p}^p$ is a seminorm on $W^{1,p}(\Omega)$, and hence on V.

With the above notations, a function u is called a *solution* of (3) if $u \in W^p(0,T;V)$, $g(x,u(t)) \in L^{p'}(0,T;V^*)$, $u(0) = u_0$ in Ω, and u satisfies

$$\langle \partial_t u(t), v\rangle + \langle Au(t), v\rangle = \langle g(x,u(t)), v\rangle, \tag{5}$$

for all $v \in V$, and for almost all $t \in (0,T)$. Observe that, by the continuous embedding $W^p(0,T;V) \subset C([0,T];L^2(\Omega))$, the condition $u(0) = u_0$ makes sense (see [25, Proposition 23.23 (ii)]).

To obtain the existence of at least one solution of (3), two preliminary propositions are needed.

PROPOSITION 2.1. *Let $f \in L^{p'}(0,T;V^*)$ and $u_0 \in L^2(\Omega)$ be given. Under assumptions (A_1)-(A_3) above, the problem*

$$\begin{cases} \partial_t u + Au = f \text{ in } \Omega \times (0,T), \\ \partial_\nu u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(0) = u_0 \text{ in } \Omega, \end{cases} \tag{6}$$

has a unique solution $u \in W^p(0,T;V)$. Moreover, if $f_n, f \in L^{p'}(0,T;V^*)$ and $u_{0,n}, u_0 \in L^2(\Omega)$ are such that $f_n \to f$ in $L^{p'}(0,T;V^*)$ and $u_{0,n} \to u_0$ in $L^2(\Omega)$ then

$$u_n \to u \text{ in } W^p(0,T;V) \text{ as } n \to \infty, \tag{7}$$

where $u_n, u \in W^p(0,T;V)$ are the solutions of (6) with f_n, f as right-hand sides and $u_{0,n}, u_0$ as initial conditions, respectively.

Proof. The existence and uniqueness of a solution of (6) has been obtained in [19, Chapter 2] and [25, Chapter 30] by means of the Faedo-Galerkin method. Another approach involving the semigroup theory can be found in [2, Chapter 3] and [6, Chapter 3]. Arguments to establish (7) are in fact contained in these references, even though not stated explicitly. Here we give the main ideas of the proof of (7), and refer to [9] for more details.

Firstly, using the boundedness of f_n and coercivity condition (A_4), one shows that the sequence (u_n) is bounded in $L^p(0,T;V) \cap L^\infty(0,T;L^2(\Omega))$. Therefore, passing to subsequences if necessary, we have

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)),\\ u_n &\rightharpoonup u \quad \text{weakly in } L^p(0,T;V),\\ u_n(T) &\rightharpoonup \xi \quad \text{weakly in } L^2(\Omega). \end{aligned}$$

Next, assumption (A_2) implies that (Au_n) is bounded in $L^{p'}(0,T;V^*)$, and thus $Au_n \rightharpoonup \chi$ in $L^{p'}(0,T;V^*)$ up to a subsequence, where $\chi \in L^{p'}(0,T;V^*)$ satisfies

$$\partial_t u + \chi = f$$

in the weak sense. Finally, a classical argument from the theory of monotone operators shows that u solves (6) with f as right-hand side (see [19, Chapter 2]).

To get to the strong convergence $u_n \to u$ in $W^p(0,T;V)$, it suffices to take $u_n - u$ as test function in the equations of u_n, u, and to substract:

$$\begin{aligned} 0 \leqslant \frac{1}{2}\|(u_n-u)(T)\|_{L^2}^2 + \int_0^T \langle Au_n - Au, u_n - u\rangle \,\mathrm{d}t \\ = \int_0^T \langle f_n - f, u_n - u\rangle \,\mathrm{d}t + \frac{1}{2}\|u_{n,0}-u_0\|_{L^2}^2. \end{aligned}$$

Since the right-hand side tends to zero as n goes to infinity, it follows that

$$\|u_n(T)-u(T)\|_{L^2}^2 \to 0 \ \text{ and } \int_0^T \langle Au_n - Au, u_n - u\rangle \,\mathrm{d}t \to 0.$$

Hence, by assumption (A_3), $\|u_n - u\|_{L^p(0,T;V)} \to 0$ as $n \to \infty$. Recalling the equations satisfied by u_n, u, and using continuity assumption (A_1), one concludes that $u_n \to u$ in $W^p(0,T;V)$. □

The next proposition is a weak comparison principle, and will be crucial in the construction of our monotone iterative scheme. In the case when A is linear, this result is an easy consequence of the parabolic weak maximum principle from [20].

PROPOSITION 2.2. *Let $T>0$ and let $u_1, u_2 \in W^p(0,T;V)$ satisfy*

$$\langle \partial_t u_1 + Au_1, v\rangle \leqslant \langle \partial_t u_2 + Au_2, v\rangle, \tag{8}$$

for all $v \in V$, $v \geqslant 0$, and almost all $t \in (0,T)$. If $u_1(0) \leqslant u_2(0)$ a.e. in Ω then $u_1 \leqslant u_2$ a.e. in $\Omega \times (0,T)$.

Proof. Let us take $v = \max\{u_1 - u_2, 0\} \in W^p(0,T;V)$ as a test function in (8) and integrate over $(0,t)$ for some $t \in (0,T)$. Then

$$\int_0^t \langle \partial_t(u_1-u_2), v\rangle + \int_0^t \langle Au_1 - Au_2, v\rangle \leqslant 0. \tag{9}$$

Then, by the integration-by-parts formula (see [25, Proposition 23.23 (iv)]), the divergence form of A, and assumption (A_3), we deduce

$$\frac{1}{2}\|v(t)\|_{L^2}^2 - \frac{1}{2}\|v(0)\|_{L^2}^2 \leqslant 0.$$

Since $v(0)=0$ a.e. in Ω, we conclude that $v(t)=0$ a.e. in Ω, and so $u_1 \leqslant u_2$ a.e. in $\Omega \times (0,T)$. □

We say that a function $\alpha\colon \Omega\times(0,T)\to\mathbb{R}$ is a *subsolution* for problem (3) if it satisfies the following conditions:

(i) $\alpha \in W^p(0,T;V)\cap L^\infty(\Omega\times(0,T))$;

(ii) the function

$$(x,t)\mapsto g(x,\alpha(x,t))\colon \Omega\times(0,T)\to\mathbb{R}$$

belongs to $L^{p'}(0,T;V^*)$;

(iii) $\alpha(x,0)\leqslant u_0(x)$ holds for a.e. $x\in\Omega$; and

(iv) the inequality

$$\langle \partial_t\alpha(t),v\rangle + \langle A\alpha(t),v\rangle \leqslant \langle g(x,\alpha(x,t)),v\rangle$$

holds for all $v\in V$, $v\geqslant 0$, and for almost every $t\in(0,T)$.

A *supersolution* for problem (3) is defined in the same way by reversing the corresponding inequalities.

Let us now assume that α_0 is a subsolution of (3), that is, α_0 satisfies (i)-(iv) above, in place of α. In addition, let $\{\alpha_{0,n}\}_{n=0}^\infty$ be a monotone increasing (i.e., nondecreasing) sequence of functions in $L^2(\Omega)$ with $\alpha_{0,0}=\alpha_0(\cdot,0)$ in Ω for $n=0$ and $t=0$. Then we can introduce the following monotone iteration scheme: For each $n\in\mathbb{N}$, assuming that $\alpha_{n-1}\colon \Omega\times(0,T)\to\mathbb{R}$ is known, such that α_{n-1} satisfies both conditions (i) and (ii) in place of α, we denote by $\alpha_n\colon \Omega\times(0,T)\to\mathbb{R}$ recursively the unique weak solution of the following initial-boundary value problem:

$$\begin{cases} \partial_t\alpha_n + A\alpha_n = g(x,\alpha_{n-1}(x,t)) \text{ in } \Omega\times(0,T),\\ \quad \partial_\nu\alpha_n(x,t)=0 \text{ on } \partial\Omega\times(0,T),\\ \qquad \alpha_n(x,0)=\alpha_{0,n}(x) \text{ in } \Omega. \end{cases} \tag{10}$$

In view of Proposition 2.1, the sequence $\{\alpha_n\}_{n=0}^\infty$ exists and is uniquely defined on the whole of $(0,T)$. Similarly, if β_0 is a supersolution of (3) and $\{\beta_{0,n}\}_{n=0}^\infty$ is a monotone decreasing (i.e., nonincreasing) sequence of functions in $L^2(\Omega)$ with $\beta_{0,0}=\beta_0(\cdot,0)$ in Ω, we can construct recursively the sequence $\{\beta_n\}_{n=0}^\infty$ of weak solutions of the initial--boundary value problem (10) with α's replaced by the corresponding β's. It is easy to see that each α_n (β_n, respectively) is a subsolution (supersolution) to problem (3). From Proposition 2.2, the definition of sub- and supersolution, and the monotone increasing property of g, we derive that $\alpha_n\geqslant\alpha_{n-1}$ and $\beta_n\leqslant\beta_{n-1}$ a.e. in $\Omega\times(0,T)$ first for $n = 1$ and subsequently, by induction, for all $n \in \mathbb{N}$. If, in addition, $\alpha_0 \leqslant \beta_0$ and $\alpha_{0,n} \leqslant \beta_{0,n}$ for all $n \in \mathbb{N}$ then Proposition 2.2 yields $\alpha_n \leqslant \beta_n$ for all $n \in \mathbb{N}$, as well, and hence

$$\alpha_0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_n \leqslant \ldots \leqslant \beta_n \leqslant \ldots \leqslant \beta_1 \leqslant \beta_0 \quad \text{a.e. in } \Omega \times (0,T). \tag{11}$$

Using the construction described above, we prove the following existence result.

THEOREM 2.3. *Let $A : V \to V^*$ be given by* (4)*, and satisfy* (A_1)-(A_3) *and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, increasing in its second argument. Suppose there exist a subsolution α_0 and a supersolution β_0 of* (3) *such that $\alpha_0 \leqslant \beta_0$ a.e. in $\Omega \times (0,T)$. Then there exists a minimal solution $u_{\min}$ and a maximal solution $u_{\max}$ of* (3) *in the interval $[\alpha_0, \beta_0]$ such that*

$$\alpha_0 \leqslant u_{\min} \leqslant u_{\max} \leqslant \beta_0 \;\; \textit{a.e. in } \Omega \times (0,T). \tag{12}$$

In other words, $u_{\min}$ and $u_{\max}$ solve (3) *together with* (12)*, and for any other solution w of* (3) *such that $\alpha_0 \leqslant w \leqslant \beta_0$, one has $u_{\min} \leqslant w \leqslant u_{\max}$ a.e. in $\Omega \times (0,T)$.*

Proof. Apart from Propositions 2.1 and 2.2, the proof follows similar lines as in [23]. Consider the sequences (α_n), $(\alpha_{0,n})$, (β_n) and $(\beta_{0,n})$ introduced above, with $\alpha_{0,n} \nearrow u_0$ and $\beta_{0,n} \searrow u_0$ in $L^2(\Omega)$. From (11), we deduce that the pointwise limits

$$\tilde{\alpha}(x,t) := \lim_{n\to\infty} \alpha_n(x,t) \;\text{ and }\; \tilde{\beta}(x,t) := \lim_{n\to\infty} \beta_n(x,t)$$

exist for almost all $(x,t) \in \Omega \times (0,T)$. By the Lebesgue dominated convergence theorem, it follows that $\alpha_n \to \tilde{\alpha}$ in $L^p(\Omega \times (0,T))$ as $n \to \infty$, and so $g(x,\alpha_n) \to g(x,\tilde{\alpha})$ in $L^{p'}(0,T;V^*)$ as $n \to \infty$ (see [25, Example 23.4]). The same holds, of course, for (β_n). Therefore, recalling the continuity result from Proposition 2.1, one gets

$$\alpha_n \to \tilde{\alpha} \;\text{ and }\; \beta_n \to \tilde{\beta} \;\text{ in } W^p(0,T;V).$$

Hence, passing to the limit in (10), we obtain that $\tilde{\alpha}$ is a weak solution of problem (3). In the same way, one proves that $\tilde{\beta}$ solves (3). Also, it follows from (11) that $\alpha_0 \leqslant \tilde{\alpha} \leqslant \tilde{\beta} \leqslant \beta_0$ a.e. in $\Omega \times (0,T)$.

To finish, if w is any solution of (3) satisfying $\alpha_0 \leqslant w \leqslant \beta_0$ then w is a subsolution of (3) with $w \leqslant \beta_0$. Thus one may consider the sequence of monotone iterations (α_n) defined by (10) with $\alpha_0 = w$ and $\alpha_{0,n} = u_0$ for all n. Obviously, $\alpha_n = w$ for all n, and Proposition 2.2 yields $w = \alpha_n \leqslant \beta_n$ for all n. Passing to the limit for $n \to \infty$, we obtain $w \leqslant \tilde{\beta}$. Inequality $\tilde{\alpha} \leqslant w$ is similar. The functions $u_{\min} := \tilde{\alpha}$ and $u_{\max} := \tilde{\beta}$ are therefore minimal and maximal solutions of (3), respectively. □

3. Asymptotic behavior of solutions

This section is devoted to the asymptotic behavior of solutions of the problem

$$(P_{u_0}) \begin{cases} \partial_t u + Au = g(x,u) \text{ in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 \text{ on } \partial\Omega \times (0,\infty), \\ u(0) = u_0 \text{ in } \Omega, \end{cases}$$

where the differential operator A and the nonlinearity g are as in Theorem 2.3. In the sequel, we use the notations $L^p_{\mathrm{loc}}([0,\infty);V)$ ($W^p_{\mathrm{loc}}([0,\infty);V)$, resp.) for functions belonging to $L^p(0,T;V)$ ($W^p_{\mathrm{loc}}(0,T;V)$, resp.) for any $T>0$. We say that u is a *solution* of (P_{u_0}) if $u\in W^p_{\mathrm{loc}}(0,\infty;V)$, $g(x,u(t))\in L^{p'}_{\mathrm{loc}}(0,\infty)$, $u(0)=u_0$ in Ω, and u satisfies (5) for all $v\in V$ and for almost all $t\in(0,\infty)$.

As we have already announced, our study relies strongly on the previous monotone iterative procedure used in the proof of Theorem 2.3. We start with a few auxiliary results in which some more properties of the sequences (α_n) and (β_n) defined in (10) are given.

Our first lemma is concerned with the monotonicity of solutions with respect to the time variable t. In the case when A is linear, this was established in [23] by differentiating the equation with respect to t, and using a maximum principle. Here an important role is played by the comparison principle from Proposition 2.2 whose proof involves the monotonicity of A.

LEMMA 3.1. *Let $\alpha_0\leqslant\beta_0$ be a sub- and a supersolution of problem $(P_{\alpha_0(0)})$. If α_0 is nondecreasing in t then the minimal solution $\alpha_{\min}$ of $(P_{\alpha_0(0)})$ in $[\alpha_0,\beta_0]$ is nondecreasing in t for almost all $x\in\Omega$. Similarly, if β_0 is nonincreasing in t, the maximal solution $\beta_{\max}$ of $(P_{\beta_0(0)})$ in $[\alpha_0,\beta_0]$ is nonincreasing in t.*

Proof. We prove the result for $\alpha_{\min}$. The same kind of argument applies to $\beta_{\max}$. Let us consider the sequence (α_n) of monotone iterations defined in (10) with $\alpha_{0,n}:=\alpha_0(0)$ for all n. By Theorem 2.3, one has

$$\alpha_n\to\alpha_{\min}\text{ in }W^p_{\mathrm{loc}}(0,\infty;V)\text{ as }n\to\infty.$$

Moreover, α_n is nondecreasing in t for all $n\geqslant 0$. Indeed, suppose by induction that α_{n-1} is nondecreasing. Fix $T>0$ and define for any $h\geqslant 0$ the translation $\tau_h\alpha_n$ by

$$(\tau_h\alpha_n)(x,t):=\alpha_n(x,t+h),$$

for all $t\in[0,T]$ and almost all $x\in\Omega$. One has

$$\begin{aligned}\partial_t\alpha_n(t)+A\alpha_n(t)&=g(x,\alpha_{n-1}(t))\\&\leqslant g(x,\alpha_{n-1}(t+h))\\&=\partial_t(\tau_h\alpha_n)(t)+A(\tau_h\alpha_n)(t),\end{aligned}$$

in the weak sense. Since the sequence (α_n) is monotone in n and since α_{n-1} is nondecreasing in t, $\tau_h\alpha_n(x,0)\geqslant\alpha_{n-1}(x,h)\geqslant\alpha_{n-1}(x,0)=\alpha_n(x,0)$ for all $n\geqslant 0$. Thus, Proposition 2.2 applies and we obtain $\alpha_n(x,t)\leqslant\tau_h\alpha_n(x,t)=\alpha_n(x,t+h)$ for almost all $(x,t)\in\Omega\times(0,T)$ and all $h\geqslant 0$. This shows that α_n is nondecreasing in t for all $n\geqslant 0$. Passing to the limit for $n\to\infty$, we conclude that $\alpha_{\min}$ is nondecreasing in t. □

Let us now assume that α_0 and β_0 are, respectively, sub- and supersolutions of the *stationary problem*

$$(P)\begin{cases}Au=g(x,u)\text{ in }\Omega,\\ \partial_\nu u=0\text{ on }\partial\Omega.\end{cases}$$

Then the minimal solution $\alpha_{\min}$ of (P_{α_0}) and the maximal solution $\beta_{\max}$ of (P_{β_0}) are not only monotone in t, but also converge to a solution of (P). This is the purpose of the following lemma. Again, this result is proved in [23] in the linear case, where the author uses the self-adjointness of the studied operator. Below we rely on the strong monotonicity of A instead.

LEMMA 3.2. *Suppose that α_0 is a subsolution and β_0 a supersolution of the stationary problem (P), with $\alpha_0 \leqslant \beta_0$. Let $\alpha_{\min}$ be the minimal solution of (P_{α_0}) and let $\beta_{\max}$ be the maximal solution of (P_{β_0}) in $[\alpha_0, \beta_0]$. Then, for all $1 \leqslant r < \infty$,*

$$\alpha_{\min}(t) \nearrow \bar{\alpha} \text{ and } \beta_{\max}(t) \searrow \bar{\beta} \text{ in } L^r(\Omega) \text{ as } t \to \infty,$$

where $\bar{\alpha}$ and $\bar{\beta}$ are solutions of (P), and $\alpha_0 \leqslant \bar{\alpha} \leqslant \bar{\beta} \leqslant \beta_0$ a.e. in Ω.

Proof. By Lemma 3.1, we know that $\alpha_{\min}$ and $\beta_{\max}$ are respectively nondecreasing and nonincreasing in t and, by construction, we have $\alpha_{\min}, \beta_{\min} \in [\alpha_0, \beta_0]$. Therefore, the pointwise limits

$$\bar{\alpha}(x) := \lim_{t\to\infty} \alpha_{\min}(x,t) \text{ and } \bar{\beta}(x) := \lim_{t\to\infty} \beta_{\max}(x,t)$$

exist for almost all $x \in \Omega$. By the Lebesgue dominated convergence theorem, this convergence also holds in $L^r(\Omega)$ for all $1 \leqslant r < \infty$.

Our aim is to show that $\bar{\alpha}$ is a solution of (P). Let us fix $T > 0$. For $h \geqslant 0$, define as before $(\tau_h \alpha_{\min})(x,t) := \alpha_{\min}(x, t+h)$ for all $t \in [0,T]$ and for almost all $x \in \Omega$. Then we have

$$\tau_h \alpha_{\min}(t) \to \bar{\alpha} \text{ in } L^r(\Omega) \text{ as } h \to \infty, \tag{13}$$

for all $t \in [0,T]$ and all $1 \leqslant r < \infty$.

In fact, $(\tau_h \alpha_{\min})$ is a Cauchy sequence in the space $L^p(0,T;V)$. Indeed, for all $h \geqslant 0$, $\tau_h \alpha_{\min}$ solves problem $(P_{\alpha_{\min}(h)})$ and hence

$$\begin{aligned} &\frac{1}{2}\|(\tau_{h_1}\alpha_{\min} - \tau_{h_2}\alpha_{\min})(T)\|_{L^2}^2 \\ &\qquad + \int_0^T \big\langle A(\tau_{h_1}\alpha_{\min}) - A(\tau_{h_2}\alpha_{\min}), \tau_{h_1}\alpha_{\min} - \tau_{h_2}\alpha_{\min}\big\rangle \,\mathrm{d}t \\ &= \int_0^T \big\langle g(\tau_{h_1}\alpha_{\min}) - g(\tau_{h_2}\alpha_{\min}), \tau_{h_1}\alpha_{\min} - \tau_{h_2}\alpha_{\min}\big\rangle \,\mathrm{d}t \\ &\qquad + \frac{1}{2}\|(\tau_{h_1}\alpha_{\min} - \tau_{h_2}\alpha_{\min})(0)\|_{L^2}^2, \end{aligned}$$

for all $h_1, h_2 \geqslant 0$. In view of (13), the right-hand side goes to zero as h_1 and h_2 tend to infinity. It follows therefore from monotonicity assumption (A_3) that

$$\|\tau_{h_1}\alpha_{\min} - \tau_{h_2}\alpha_{\min}\|_{L^p(0,T;V)} \to 0 \quad \text{as } h_1, h_2 \to \infty.$$

Thus, there exists $\tilde{\alpha} \in L^p(0,T;V)$ such that

$$\tau_h \alpha_{\min} \to \tilde{\alpha} \quad \text{in } L^p(0,T;V) \text{ as } h \to \infty.$$

Moreover, using the equation satisfied by $\tau_h \alpha_{\min}$, one easily verifies that

$$\partial_t(\tau_h \alpha_{\min}) \to \partial_t \tilde{\alpha} \quad \text{in } L^{p'}(0,T;V^*),$$

and

$$\partial_t \tilde{\alpha} + A\tilde{\alpha} = g(x,\tilde{\alpha}). \tag{14}$$

Finally, the convergence $\tau_h \alpha_{\min} \to \tilde{\alpha}$ in $W^p(0,T;V)$ and the continuous embedding $W^p(0,T;V) \subset C([0,T];L^2(\Omega))$ yield

$$(\tau_h \alpha_{\min})(0) \to \tilde{\alpha}(0) \text{ and } (\tau_h \alpha_{\min})(T) \to \tilde{\alpha}(T) \text{ in } L^2(\Omega) \text{ as } h \to \infty.$$

Hence (13) implies that $\tilde{\alpha}(T) = \bar{\alpha}$ for all $T \geqslant 0$, and $\partial_t \tilde{\alpha} \equiv 0$ in (14). This shows that $\bar{\alpha}$ solves (P). The same reasoning applies to $\bar{\beta}$. □

The last important lemma for the asymptotic behavior of solutions of (P_{u_0}) is the following weak comparison principle. This lemma involves minimal and maximal solutions, and is particularly interesting because solutions of (P_{u_0}) are not necessarily unique.

LEMMA 3.3. *Let $u_0, v_0 \in L^2(\Omega)$ and let α_0 and β_0 be a sub- and a supersolution of both problems (P_{u_0}) and (P_{v_0}) with $\alpha_0 \leqslant \beta_0$ a.e. in $\Omega \times (0,\infty)$. Suppose $u_{\min}$ and $v_{\min}$ are the minimal solutions of (P_{u_0}) and (P_{v_0}) in $[\alpha_0, \beta_0]$, respectively. If $u_0 \leqslant v_0$ a.e. in Ω then*

$$u_{\min} \leqslant v_{\min} \text{ a.e. in } \Omega \times (0,\infty).$$

A similar result holds for maximal solutions: $u_{\max} \leqslant v_{\max}$ a.e. in $\Omega \times (0,\infty)$.

Proof. Let us denote by (α_n^u) and (α_n^v), respectively, the monotone sequences from Theorem 2.3 that converge to u and v almost everywhere. Without loss of generality, we may choose the initial data $\alpha_{0,n}^u$ and $\alpha_{0,n}^v$ such that $\alpha_{0,n}^u \leqslant \alpha_{0,n}^v$ for all n. Then, by induction and by Proposition 2.2, one easily shows that $\alpha_n^u \leqslant \alpha_n^v$ for all $n \in \mathbb{N}$. Our statement follows by taking the limit for $n \to \infty$. □

We are now in a position to conclude the following result.

THEOREM 3.4. *Let α_0 be a subsolution and β_0 a supersolution of (P), and let $u_0 \in L^2(\Omega)$ satisfy $\alpha_0 \leqslant u_0 \leqslant \beta_0$ a.e. in Ω. If u is any solution of (P_{u_0}) such that $\alpha_0 \leqslant u \leqslant \beta_0$ then*

$$\bar{\alpha} \leqslant \liminf_{t\to\infty} u(x,t) \leqslant \limsup_{t\to\infty} u(x,t) \leqslant \bar{\beta} \text{ a.e. in } \Omega, \tag{15}$$

where $\bar{\alpha}$ and $\bar{\beta}$ solve the stationary problem (P). In particular, if (P) admits a unique solution $\bar{u}$ in $[\alpha_0, \beta_0]$ then, for all $1 \leqslant r < \infty$,

$$u(t) \to \bar{u} \text{ in } L^r(\Omega) \text{ as } t \to \infty. \tag{16}$$

Proof. Consider the minimal solution $\alpha_{\min}$ of (P_{α_0}), and the maximal solution $\beta_{\max}$ of (P_{β_0}) in $[\alpha_0, \beta_0]$, whose existence has been established in Theorem 2.3. In view of Lemma 3.3, one has $\alpha_{\min} \leqslant u_{\min}$ and $u_{\max} \leqslant \beta_{\max}$ a.e. in $\Omega \times (0,\infty)$. As a result, $\alpha_{\min} \leqslant u \leqslant \beta_{\max}$ a.e. in $\Omega \times (0,\infty)$, and (15) follows from Lemma 3.2. In the case when (P) has a unique solution $\bar{u}$ in $[\alpha_0, \beta_0]$, one has $\bar{\alpha} \equiv \bar{\beta} \equiv \bar{u}$ in (15). Hence, u converges pointwise to $\bar{u}$ as $t \to \infty$, and it suffices to apply the Lebesgue dominated convergence theorem to get (16). □

4. Application to a model for population evolution

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $\lambda > 0$ a parameter, $m \in L^\infty(\Omega)$ a possibly sign-changing function, $p \in (1,\infty)$, $q > 0$, and $u_0 \in L^2(\Omega)$. In this last section, we build upon our previous investigations to study the quasilinear parabolic problem

$$\begin{cases} \partial_t u - \Delta_p u = \lambda |u|^{p-2} u \left(m(x) - |u|^q\right) \text{ in } \Omega \times (0,\infty), \\ \qquad \partial_\nu u = 0 \text{ on } \partial\Omega \times (0,\infty), \\ \quad u(0) = u_0 \text{ in } \Omega. \end{cases} \tag{17}$$

From a biological point of view, (17) describes the density u at time t of a population living in the domain Ω (see [7]). The p-Laplacian $-\Delta_p u$ of u represents the diffusion of the population, while its growth is given by $\lambda |u|^{p-2} u (m - |u|^q)$. Neumann boundary conditions mean that individuals may not cross the boundary of their habitat. In the linear case $p = 2$, this problem has been considered in [7, 18, 23], and more recently in [3] among many others.

To recover the framework of Sections 2 and 3, we rewrite the equation in (17) as follows

$$\partial_t u - \Delta_p u + \lambda m^- |u|^{p-2} u + \lambda |u|^{p+q-2} u = \lambda m^+ |u|^{p-2} u.$$

More precisely, we consider the Banach space

$$V := W^{1,p}(\Omega) \cap L^{p+q}(\Omega) \cap L^2(\Omega)$$

and define, for all $x \in \Omega$ and all $u \in V$,

$$\begin{aligned} Au &:= -\Delta_p u + \lambda m^-(x) |u|^{p-2} u + \lambda |u|^{p+q-2} u, \\ g(x,u) &:= \lambda m^+(x) |u|^{p-2} u. \end{aligned}$$

Then $V \subset W^{1,p}(\Omega)$ is separable, reflexive, and continuously and densely embedded in $L^2(\Omega)$. (Of course, if $p + q \geqslant 2$ or $p^* \geqslant 2$, the intersection with $L^2(\Omega)$ is not necessary.) It is well-known that the p-Laplacian, and hence the operator $A : V \to V^*$, satisfy properties (A_1)-(A_2). Condition (A_3) follows from the standard inequality

$$|\xi - \eta|^p \leqslant C_p \big((|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \big)^{s/2} \big(|\xi|^p + |\eta|^p \big)^{1-s/2}, \tag{18}$$

which holds for all $\xi, \eta \in \mathbb{R}^N$, and $p \in (1,\infty)$, and where $C_p > 0$ is a constant, $s = 2$ if $p \geqslant 2$, and $s = p$ if $1 < p < 2$ (see [24]). Indeed, by (18) and a direct application of the Hölder inequality, it can be derived that (A_3) holds with $\theta = 1$ if $p \geqslant 2$, and $\theta = p/2$ if $1 < p < 2$. One also verifies that the Carathéodory function $g = g(x,u) = \lambda m^+(x)|u|^{p-2}u$ is increasing with respect to the variable u.

The asymptotic behavior of positive solutions of (17) is described Theorems 4.1 and 4.3 below. Our results involve the one-parameter eigenvalue problem

$$\begin{cases} -\Delta_p u - \lambda m|u|^{p-2}u = \mu|u|^{p-2}u \text{ in } \Omega, \\ \partial_\nu u = 0 \text{ on } \partial\Omega. \end{cases} \tag{19}$$

We recall that, for any $\lambda \in \mathbb{R}$, there exists a unique *principal* eigenvalue $\mu_1 = \mu_1(\lambda, m)$ of (19) (see [4]). Moreover, the associated principal eigenvalue ϕ_1 satisfies $\phi_1 \in C^1(\overline{\Omega})$, and can be chosen such that $\phi_1 > 0$ in $\overline{\Omega}$.

THEOREM 4.1. *Let* $1 < p < 2$ *and* $u_0 \in L^\infty(\Omega)$ *be given. Let* $u = u(x,t)$ *be any bounded solution of* (17)*. In case* (i) *assume that* $u_0, u \geqslant \delta$ *a.e. for some* $\delta > 0$.
(i) *If* $\mu_1 < 0$ *then* $u(t) \to \bar{u}$ *in* $L^r(\Omega)$*, for all* $1 \leqslant r < \infty$*, as* $t \to \infty$*, where* $\bar{u}$ *is the unique positive solution of the stationary problem associated to* (17).
(ii) *If* $\mu_1 \geqslant 0$ *then* $u(t) \to 0$ *in* $L^r(\Omega)$*, for all* $1 \leqslant r < \infty$*, as* $t \to \infty$.

Proof. (i) Let $\phi_1 > 0$ be the principal eigenfunction of (19) normalized by $\|\phi_1\|_\infty = 1$. If $\varepsilon > 0$ is a real number such that $\varepsilon \leqslant (-\mu_1/\lambda)^{1/q}$, then $\varepsilon\phi_1$ solves

$$-\Delta_p(\varepsilon\phi_1) - \lambda m(\varepsilon\phi_1)^{p-1} = \mu_1(\varepsilon\phi_1)^{p-1} \leqslant -\lambda(\varepsilon\phi_1)^{p+q-1} \text{ in } \Omega.$$

Hence $\alpha_0 := \varepsilon\phi_1$ is a subsolution of the stationary problem associated to (17), and every constant $\beta_0 > (\|m\|_\infty)^{1/q}$ is a supersolution. Choosing ε, β_0 such that $\alpha_0 \leqslant u \leqslant \beta_0$ a.e. in $\Omega \times (0,\infty)$, we may thus apply Theorem 3.4. It can easily be established by using the Picone identity for the p-Laplacian from [1] that the stationary problem associated to (17) has at most one nontrivial nonnegative solution u, and therefore, the conclusion follows.

(ii) Let us take u as test function in (17) and integrate over (t_1, t_2) with $0 < t_1 < t_2$. We have

$$\frac{1}{2}\|u(t_2)\|_{L^2}^2 - \frac{1}{2}\|u(t_1)\|_{L^2}^2 + \int_{t_1}^{t_2}\int_\Omega |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t + \lambda\int_{t_1}^{t_2}\int_\Omega |u|^{p+q}\,\mathrm{d}x\,\mathrm{d}t = \lambda\int_{t_1}^{t_2}\int_\Omega m(x)|u|^p\,\mathrm{d}x\,\mathrm{d}t. \tag{20}$$

Recalling the definition of μ_1, we get

$$\frac{1}{2}\|u(t_2)\|_{L^2}^2 - \frac{1}{2}\|u(t_1)\|_{L^2}^2 + \mu_1\int_{t_1}^{t_2}\int_\Omega |u|^p\,\mathrm{d}x\,\mathrm{d}t + \lambda\int_{t_1}^{t_2}\int_\Omega |u|^{p+q}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{21}$$

for almost all $t_1 < t_2$. The last two terms in (21) being nonnegative, we deduce that the function $t \to \|u(t)\|_{L^2}^2$ is nonincreasing on $(0,\infty)$. Let η be such that $\|u(t)\|_{L^2}^2 \searrow \eta$ as $t \to \infty$. Assume by contradiction that $\eta > 0$ and fix $T > 0$. Then the functions

$$(\tau_h u)(x,t) := u(x, t+h), \;\; h \geqslant 0,$$

are such that for all $t \in [0,T]$

$$\|\tau_h u(t)\|_{L^2}^2 \searrow \eta \text{ as } h \to \infty,$$

and solve problem (17) with the initial condition $u(h)$. Since u is assumed to be bounded, the limit $\bar{u} := \lim_{t\to\infty} u(x,t)$ exists for almost all $x \in \Omega$. Moreover, by the Lebesgue dominated convergence theorem, we have

$$\tau_h u \to \bar{u} \text{ in } L^r(\Omega) \text{ as } h \to \infty,$$

for all $t \in [0,T]$ and all $1 \leqslant r < \infty$. As in the proof of Lemma 3.2, one shows that $\tau_h u \to \tilde{u}$ in $L^p(0,T;V)$ as $h \to \infty$ (up to a subsequence), where $\tilde{u}$ satisfies

$$\partial_t \tilde{u} - \Delta_p \tilde{u} = \lambda |\tilde{u}|^{p-2}\tilde{u}(m - |\tilde{u}|^q) \quad \text{and} \quad \tau_h u \to \tilde{u} \text{ in } C([0,T];L^2) \text{ as } h \to \infty.$$

Therefore

$$\|\tau_h u(T)\|_{L^2}^2 \to \|\tilde{u}(T)\|_{L^2}^2 \text{ as } h \to \infty \quad \text{and} \quad \|\tilde{u}(T)\|_{L^2}^2 = \eta > 0 \text{ for all } T > 0.$$

Hence, from (21) with u replaced by $\tilde{u}$, we deduce that $\tilde{u} \equiv 0$, a contradiction. □

Note that the requirement that $u \geqslant \delta$ in Theorem 4.1 is closely related to the question of uniqueness of solutions of (17). When $1 < p < 2$, the nonlinear term $|u|^{p-2}u$ is not Lipschitz continuous, but only Hölder continuous with respect to u. Therefore, we are in a situation comparable to that of the classical ODE $u' = u^{p-1}$, and we have to restrict our study to solutions that are bounded away from zero (see also [9] for more details).

When $p \geqslant 2$, the assumption $u \geqslant \delta$ can be removed. In that case, one has indeed the following comparison principle which implies that solutions of (17) are unique. Its proof relies on the Lipschitz continuity of the function $|u|^{p-2}u$ (see for example [15]).

LEMMA 4.2. *Let $p \geqslant 2$ and $T > 0$. Assume that $u_1, u_2 \in W^p(0,T;V)$ are such that $u_1(0) \leqslant u_2(0)$ a.e. in Ω, and*

$$\begin{aligned} &\left\langle \partial_t u_1 - \Delta_p u_1 - \lambda |u_1|^{p-2} u_1 \left(m - |u_1|^q\right), v \right\rangle \\ &\qquad \leqslant \left\langle \partial_t u_2 - \Delta_p u_2 - \lambda |u_2|^{p-2} u_2 \left(m - |u_2|^q\right), v \right\rangle, \quad (22) \end{aligned}$$

for all $0 \leqslant v \in V$ and almost all $t \in (0,T)$. Then $u_1 \leqslant u_2$ a.e. in $\Omega \times [0,T]$.

Using the previous lemma, we deduce the following.

THEOREM 4.3. *Let $p \geqslant 2$ and $u_0 \in L^\infty(\Omega)$ be given. Let u be any bounded solution of* (17). *In case (i) below, assume that $u_0 \geqslant \delta$ a.e. for some $\delta > 0$.*

(i) If $\mu_1 < 0$ then $u(t) \to \bar{u}$ in $L^r(\Omega)$, for all $1 \leqslant r < \infty$, as $t \to \infty$, where $\bar{u}$ is the unique positive solution of the stationary problem associated to (17).

(ii) If $\mu_1 \geqslant 0$ then $u(t) \to 0$ in $L^r(\Omega)$, for all $1 \leqslant r < \infty$, as $t \to \infty$.

Proof. We start with (i), the proof of (ii) being exactly the same as in Theorem 4.1. Let $\phi_1 > 0$ be the principal eigenfunction of (19) normalized by $\|\phi_1\|_\infty = 1$. As $u_0 \geqslant \delta > 0$ a.e. in Ω, there exists some $\kappa > 0$ such that $\alpha_0 := \kappa\phi_1 \leqslant u_0$ a.e. in Ω. Since α_0 is a subsolution of the stationary problem associated to (17) for κ small enough, Lemma 3.1 implies that the unique solution α of

$$\begin{cases} \partial_t u - \Delta_p u = \lambda |u|^{p-2} u \left(m(x) - |u|^q\right) \text{ in } \Omega \times (0,\infty), \\ \qquad \partial_\nu u = 0 \text{ on } \partial\Omega \times (0,\infty), \\ \quad u(0) = \alpha_0 \text{ in } \Omega, \end{cases}$$

is nondecreasing in t. Moreover, from $\alpha(x,0) \leqslant u(x,0)$ a.e. in Ω, we deduce that $\alpha(x,t) \leqslant u(x,t)$ for almost all $(x,t) \in \Omega \times (0,\infty)$ by the weak comparison principle from Lemma 4.2. Hence, $u(x,t) \geqslant \alpha(x,t) \geqslant \alpha(x,0) = \alpha_0$ for almost all $(x,t) \in \Omega \times (0,\infty)$, and similar arguments as in the proof of Theorem 4.1 apply. □

Of course, in the linear case $p = 2$, the restriction $u_0 > \delta$ on the initial condition is not necessary either. This is a consequence of the parabolic strong maximum principle from [20].

At last, we mention that the conditions $\mu_1 < 0$ and $\mu_1 \geqslant 0$ may be expressed in terms of the parameter $\lambda > 0$ and the weight function $m \in L^\infty(\Omega)$ as in the corollary below. In this result, the principal eigenvalues of the problem

$$\begin{cases} -\Delta_p u = \lambda m |u|^{p-2} u \text{ in } \Omega, \\ \quad \partial_\nu u = 0 \text{ on } \partial\Omega, \end{cases} \tag{23}$$

play an important role. Indeed, a sign-changing weight m causes that problem (23) possesses *two distinct* principal eigenvalues, 0 and $\lambda_1(m)$.

In the case when $\int_\Omega m\,\mathrm{d}x < 0$ and m changes sign, $\lambda_1(m)$ is positive, and a necessary and sufficient condition for $\mu_1 = \mu_1(\lambda, m) < 0$ is that $\lambda > \lambda_1(m)$. If, on the contrary, $\int_\Omega m\,\mathrm{d}x > 0$ and m changes sign, then $\lambda_1(m)$ is negative, and $\mu_1(\lambda, m) \geqslant 0$ for all $\lambda > 0$. The reader is referred to [4] for more details on the shape of μ_1 as a function of λ and m. A thorough study of principal eigenvalues for problem (23) with $p = 2$ has been carried out in [16, 17] for both, Dirichlet or Neumann boundary conditions (homogeneous). In these works it was shown that a sign-changing (i.e., indefinite) weight m may cause that problem (23) possesses *two distinct* principal eigenvalues, i.e., real eigenvalues with strictly positive eigenfunctions.

COROLLARY 4.4. *Let $u_0 \in L^\infty(\Omega)$ be given and let u be any bounded solution of (P_{u_0}). In cases (i) and (iii) below, assume that $u_0, u \geqslant \delta > 0$ a.e. if $1 < p < 2$, and that $u_0 \geqslant \delta > 0$ a.e. if $p \geqslant 2$ for some $\delta > 0$.*

(i) If $\int_\Omega m\,\mathrm{d}x \geqslant 0$ and $m \not\equiv 0$ then $u(t) \to \bar{u}$ in $L^r(\Omega)$, for all $1 \leqslant r < \infty$, as $t \to \infty$, where $\bar{u}$ is the unique positive solution of the stationary problem associated to (17).

(ii) If $m \leqslant 0$ a.e. in Ω then $u(t) \to 0$ in $L^r(\Omega)$, for all $1 \leqslant r < \infty$, as $t \to \infty$.

Assume that m is sign-changing and $\int_\Omega m\,\mathrm{d}x < 0$. Let $\lambda_1(m)$ be the unique nonzero principal eigenvalue of (23).

(iii) If $\lambda > \lambda_1(m)$ *then* $u(t) \to \bar{u}$ *in* $L^r(\Omega)$, *for all* $1 \leqslant r < \infty$, *as* $t \to \infty$, *where* $\bar{u}$ *is the unique positive solution of the stationary problem* (P) *associated to* (17).

(iv) If $\lambda \leqslant \lambda_1(m)$ *then* $u(t) \to 0$ *in* $L^r(\Omega)$, *for all* $1 \leqslant r < \infty$, *as* $t \to \infty$.

In conclusion, we have proven that if $\int_\Omega m\,\mathrm{d}x \geqslant 0$ and $m \not\equiv 0$ then the population tends to survive, and if m is negative then it becomes extinct. If the weight m changes sign and $\int_\Omega m\,\mathrm{d}x < 0$ then $\lambda > \lambda_1(m)$ appears as a necessary and sufficient condition for the survival of a species modeled by (17). As pointed out in [3, 7] in the case $p = 2$, this is a rather elegant and simple criterion that can be verified numerically. In [10, 22], the authors show that the principal eigenvalue $\lambda_1(m)$ contains in fact much information on how to arrange the favorable region $\{x \in \Omega : m(x) \leqslant 0\}$ and unfavorable region $\{x \in \Omega : m(x) > 0\}$ in order to maximize the species survival.

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