Differential
Equations
& Applications
Volume 4, Number 1 (2012), 159–179

THE PROBABILISTIC BROSAMLER FORMULA FOR SOME NONLINEAR NEUMANN BOUNDARY VALUE PROBLEMS GOVERNED BY ELLIPTIC POSSIBLY DEGENERATE OPERATORS

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(*Communicated by J.-M. Rakotoson*)

Abstract. This paper concerns with boundary value problems as

$$\begin{cases} \mathscr{L}u + a_0 u = f & \text{in } \Omega, \\ < \nabla u, \vec{\gamma} > + c_0|u|^{\mathrm{m}-1}u = g & \text{on } \partial\Omega, \end{cases}$$

where $\mathscr{L}$ is an elliptic possibly degenerate second order operator, a_0, c_0 are positive function, $\vec{\gamma}$ is an *oblique exterior vector* and $\mathrm{m} \geqslant 1$. By means of some arguments close to the Dynamics Programming we prove that the viscosity solution admits a *representation formula* that can be considered as an extension of probabilistic Brosamler formula of linear Neumann boundary value problems governed by uniformly elliptic operators. Although other generalizations are possible, by simplicity we limit this contribution to the presence of nonlinear terms exclusively on the boundary of the domain. We emphasize that any uniforme ellipticity assumption is required in the paper.

1. Introduction

In 1976, G. A. Brosamler [6], investigating the asymptotic behavior of the sample paths of positive recurrent diffusions, employed the probabilistic potential theory establishing a close connection with certain boundary problems. More precisely, Brosamler proved that any classical solution of the Neumann problem

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ \langle \mathrm{D}u(x), \vec{n}(x) \rangle = g(x), & x \in \partial\Omega, \end{cases} \tag{1}$$

admits the probabilistic representation

$$u(x) = \lim_{t\to\infty} \mathbb{E} \int_0^t g\big(\mathscr{X}_s\big) d\mathrm{L}_s, \quad x \in \overline{\Omega}, \tag{2}$$

Mathematics subject classification (2010): 35D40, 35J65, 60J60.
Keywords and phrases: non linear Neumann boundary condition, viscosity solutions, probabilistic approach, reflection diffusion.
Partially supported by the projects MTM 2008-06208 of DGISGPI (Spain) and the Research Group MOMAT (Ref. 910480) from Banco Santander and UCM.

Paper DEA-04-10

where Ω is a bounded open set of $\mathbb{R}^N$ with $\partial\Omega \in \mathscr{C}^3$, g is a continuous function on $\partial\Omega$ satisfying the compatibility condition $\int_{\partial\Omega} g(x)d\sigma_x = 0$, $\{\mathscr{X}_t\}_{t\geqslant 0}$ is a Brownian motion with reflection at the boundary and $\{\mathrm{L}_t\}_{t\geqslant 0}$ is the boundary local time for $\{\mathscr{X}_t\}_{t\geqslant 0}$ (see bellow for details). Since then these representation is known as the Brosamler formula. It is a kind of stationary version of the Feynmann–Kac formula.

More recently, A. Benchérif Madani and È. Pardoux, [5], have extended the Brosamler formula for the Neumann boundary problem

$$\begin{cases} -\dfrac{1}{2}\mathbf{Tr}\left(\mathscr{A}(x), \mathrm{D}^2 u(x)\right) + \langle \mathbf{a}(x), \nabla u(x)\rangle = f(x), & x \in \Omega, \\ \dfrac{1}{2}\langle \mathscr{A}\left(\nabla u(x)\right), \vec{n}(x)\rangle = g(x), & x \in \partial\Omega. \end{cases} \tag{3}$$

So, if Ω is a bounded open set of $\mathbb{R}^N$ with $\partial\Omega \in \mathscr{C}^{2,\alpha}$, $0 < \alpha < 1$, $\mathscr{A} \in \mathscr{C}^{1,\alpha}(\overline{\Omega})$ is a matricial uniformly elliptic function, $\mathbf{a}$, $f \in \mathscr{C}^{0,\alpha}(\overline{\Omega})$ and $g \in \mathscr{C}(\partial\Omega)$ satisfying the compatibility (or centering) condition

$$\int_\Omega f(x)p(x)dx + \int_{\partial\Omega} g(x)p(x)d\sigma_x = 0,$$

where p is the solution of the adjoint elliptic problem of (3), it is proved in [5] that any classical solution of (3) can be probabilistically represented by the formula

$$u(x) = \lim_{t\to\infty}\left[\int_0^t \mathbb{E}f\left(\mathscr{X}_s\right)dt + \int_0^t \mathbb{E}g\left(\mathscr{X}_s\right)d\mathrm{L}_s\right], \quad x \in \overline{\Omega}. \tag{4}$$

Here $\{\mathscr{X}_t\}_{t\geqslant 0}$ is the diffusion process involved to the PDE of (3) with reflection at the boundary and $\{\mathrm{L}_t\}_{t\geqslant 0}$ is the boundary local time for $\{\mathscr{X}_t\}_{t\geqslant 0}$ (again see bellow for details).

Our main goal in the paper is to show that a Brosamler formula is also available whenever the uniform ellipticity is not required on boundary value problems with nonlinear Neumann condition. It is is strongly motivated by reasoning close to Stochastic Optimal Control theory (see [9],[14], [25]). In order to simplify, we present the ideas on a king of one-control case, but they can be extended to more general control problems. So, we consider a $\mathbb{W}^{3,\infty}$ open bounded set $\Omega \subset \mathbb{R}^N$, $\mathrm{N} \geqslant 1$, whose unit outward normal vector is $\vec{n}(x)$ at each $x \in \partial\Omega$. Then, the *outward oblique directions* $\vec{\gamma}(x)$ are continuous functions on $\partial\Omega$, given by the property

$$\langle \vec{n}(x), \vec{\gamma}(x)\rangle > 0, \quad |\vec{\gamma}(x)| = 1, \quad x \in \partial\Omega.$$

We also consider two functions

$$\mathbf{a} : \mathbb{R}^N \to \mathbb{R}^N \quad \text{and} \quad \sigma : \mathbb{R}^N \to \mathscr{M}(\mathrm{N}\times\mathrm{M};\mathbb{R}),$$

satisfying

$$\begin{cases} |h(x) - h(x')| \leqslant C|x - x'|, & x, x' \in \mathbb{R}^N, \\ |h(x)| \leqslant C, & x \in \mathbb{R}^N, \end{cases} \tag{5}$$

for some positive constants C, with

$$h = \mathbf{a},\ \sigma.$$

On the other hand, on a probability space $(\mathscr{O},\mathscr{F},\mathbb{P})$ we consider a M-dimensional Brownian Motion $\{\mathscr{B}_t\}_{t\geqslant 0}$ and the relative *filtration*, $\{\mathscr{F}_t\}_{t\geqslant 0}\subset\mathscr{F}$, which is involved to $\{\mathscr{B}_t\}_{t\geqslant 0}$. In this probabilistic framework, we construct the *Skorohod problem*

$$\begin{cases} \mathrm{d}\mathscr{X}_t^x = -\mathbf{a}\big(\mathscr{X}_t^x\big)dt + \sigma\big(\mathscr{X}_t^x\big)d\mathscr{B}_t - \vec{\gamma}(\mathscr{X}_s^x)d\mathrm{L}_s^x, \quad t>0, \\ \mathscr{X}_0^x = x\in\overline{\Omega}, \end{cases} \tag{6}$$

where the boundary local time for $\{\mathscr{X}_t^x\}_{t\geqslant 0}$ is given by

$$\mathrm{L}_t^x = \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^t 1\mathrm{I}_{\Omega_\delta}\big(\mathscr{X}_s^x\big)ds, \quad \Omega_\delta = \{y\in\Omega:\ \mathrm{dist}(y,\partial\Omega)<\delta\}. \tag{7}$$

We send [18] or [20] for some properties of this process $\{\mathrm{L}_t^x\}_{t\geqslant 0}$. In [16] one proves the existence and uniqueness of the solutions of (6). In a rough sense the solutions are *stochastic trajectories* reflecting in coming the boundary $\partial\Omega$ to inside Ω. Next, for every couple of bounded Lipschitz continuous function $f:\ \Omega\to\mathbb{R}$, $g:\ \partial\Omega\to\mathbb{R}$ we construct

$$\begin{aligned} u(x) = \mathbb{E}\Bigg[&\int_0^\infty f\big(\mathscr{X}_t^x\big)\exp\left(-\int_0^t a_0\big(\mathscr{X}_s^x\big)ds\right)dt \\ &+\int_0^\infty g\big(\mathscr{X}_t^x\big)\exp\left(-\int_0^t a_0\big(\mathscr{X}_s^x\big)ds - \int_0^t c_0\big(\mathscr{X}_s^x\big)d\mathrm{L}_s^x\right)d\mathrm{L}_t^x\Bigg],\ x\in\overline{\Omega}. \end{aligned} \tag{8}$$

We note that since reflections hold, the *first exit time* of the trajectory $\mathscr{X}_t^x$ from $\overline{\Omega}$ of $\mathbb{R}^{\mathrm{N}}$ verifies

$$\tau_x \doteq \inf\{t\geqslant 0:\mathscr{X}_t^x\notin\overline{\Omega}\} = \infty.$$

In Remarks 1 and 7 one explains the terms of (8) from the Stochastic Optimal Control theory point of view (see also [9],[14], [25]).

Classical arguments (see [12] or [20]) lead to think u as being a solution of a partial differential equation and a kind of Neumann boundary condition

$$\mathrm{B}\big(x,u,\mathrm{D}u\big) = 0 \quad \text{on } \Omega, \tag{9}$$

involving $\vec{\gamma}$, c_0 and g. Indeed, it is proved that u verifies, in some sense to be precised, the equation

$$\mathscr{L}u + a_0u = f \quad \text{in } \Omega, \tag{10}$$

where

$$\mathscr{L}u \doteq -\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}\cdot\mathrm{D}^2u\right) + \langle\mathbf{a},\mathrm{D}u\rangle.$$

Here $\mathscr{A}\doteq\sigma\sigma^t$ and σ^t is the transpose matrix of σ.

Certainly, some assumptions on the data must be required in order to prove that u satisfies (10)–(9) in a classical sense. An almost “unavoidable” hypothesis for that goal is the *non-degeneracy* of the diffusion: $\sigma(\cdot)$ must be a $\mathrm{N}\times\mathrm{N}$ matrix with $\sigma(\cdot)\geqslant\theta\mathrm{I}_{\mathrm{N}}$, for some $\theta>0$ on $\overline{\Omega}$. Unfortunately, the condition does not hold in some important examples of the applications and consequently the regularity the function u can not be guaranteed. The viscosity solution notion is adequate in order to remove the *non-degeneracy* hypothesis. We send the monograph by M. G. Crandall, M. G, H. Ishii and P. L. Lions [7] to understand how semi-continuous functions can solve (10) in that framework. We note that linearity of the operator $\mathscr{L}$ can be lost for this kind of solutions. From now on, sometimes we will drop the term viscosity which is an artifact of the origin of this theory motivated by the consistency of the notion with the method of vanishing viscosity, mainly for first order equations. Therefore we refer to viscosity sub-, super- and solutions as sub-, super- and solutions, respectively.

The possible non regularity interferes strongly with the condition (9). Furthermore, one may construct examples in which u is continuous on $\overline{\Omega}$ but (9) does not hold (see the ideas of [4] or [13]). This is the reason for which (9) is generalized to a boundary condition where from the control point of view the possible behaviors of the dynamical system and the strategy of the controller must be considered. The *relaxed Neumann boundary conditions* are

$$\min\big\{\mathscr{L}u+a_0u-f,\mathrm{B}(x,u,\mathrm{D}u)\big\}\leqslant 0\quad\text{on }\partial\Omega \tag{11}$$

and

$$\max\big\{\mathscr{L}u+a_0u-f,\mathrm{B}(x,u,\mathrm{D}u)\big\}\geqslant 0\quad\text{on }\partial\Omega \tag{12}$$

in the viscosity sense. Conditions (11) and (12) arise when we pass to limit smooth solutions of the classical Neumann boundary value problem in the *vanishing viscosity method* (see [7] for an introduction of the so–called *half–relaxed limits method*). Roughly, these relaxed conditions mean that the PDE holds up to the boundary if (9) does not hold in the ordinary viscosity sense. In this note the boundary is

$$\mathrm{B}(x,r,p)=\langle p,\vec{\gamma}(x)\rangle+c_0(x)|r|^{\mathrm{m}-1}r-g(x),\ \ (x,r,p)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^{\mathrm{N}},\ (\mathrm{m}\geqslant 1), \tag{13}$$

that satisfies the *obliqueness*

$$\mathrm{B}(x,r,p+\eta\vec{n}(x))-\mathrm{B}(x,s,p)=\eta\langle\vec{n}(x),\vec{\gamma}(x)\rangle+c_0(x)\big(|r|^{\mathrm{m}-1}r-|s|^{\mathrm{m}-1}s\big), \tag{14}$$

very useful in uniqueness results, provided $\eta>0,\ c_0(x)\geqslant 0$. In fact, this assumption enables to prove that *relaxed Neumann boundary conditions* becomes *ordinary Neumann boundary conditions* (see the comments in Remark 6 below).

So that, our interest here is to prove that (8) provides the Brosamler formula to the *Neumann value boundary problem* corresponding to

$$\begin{cases}\mathscr{L}u+a_0u=f & \text{in }\Omega,\\ \langle\mathrm{D}u,\vec{\gamma}\rangle+c_0u=g & \text{on }\Omega.\end{cases}$$

We note that this boundary condition coincides with (13) for $\mathrm{m}=1$. In Section 3 we prove that (8) is, in fact, the unique continuous solution whenever

$$\text{the data }\sigma,\ \mathbf{a},\ a_0>0,\ c_0>0,\ f\text{ and }g\text{ satisfy (5) in their arguments,} \tag{15}$$

see Theorem 3 below. We emphasize that $\mathscr{A} \doteq \sigma\sigma^t$ is the simple structural assumption needed on the leading part of $\mathscr{L}$ (see Remark 3 below) and no uniformly elliptic assumption is required in the paper. The proof uses an approach by means of solutions in the whole space obtained in the Section 2. The uniqueness is based on the obliqueness (14) (see [1]). Certainly, complementary regularity for solutions are available under more strict conditions on the data. In particular, as it is well known, classical solutions are obtained if we assume that $\mathscr{L}$ is uniformly elliptic operator (see, for instance, [11, Theorem 6.31] or [17]).

In Section 4 one studies the implicit Brosamler formula

$$\begin{aligned} u(x) =&\mathbb{E}\Big[\int_0^\infty f\big(\mathscr{X}_t^x\big)\exp\Big(-\int_0^t a_0\big(\mathscr{X}_s^x\big)ds\Big)dt \\ &\qquad + \int_0^\infty g\big(\mathscr{X}_t^x\big)\exp\Big(-\int_0^t a_0\big(\mathscr{X}_s^x\big)ds \\ &\qquad\qquad - \int_0^t c_0\big(\mathscr{X}_s^x\big)\big|u\big(\mathscr{X}_s^x\big)\big|^{\mathrm{m}-1}d\mathrm{L}_s^x\Big)d\mathrm{L}_t^x\Big], \end{aligned} \tag{16}$$

for $x \in \overline{\Omega}$, of the nonlinear boundary problem

$$\begin{cases} \mathscr{L}u + a_0 u = f & \text{in } \Omega, \\ \langle \nabla u, \vec{\gamma} \rangle + c_0 |u|^{\mathrm{m}-1} u = g & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{m} > 1$. By means of a Fixed Point Theorem it is proved that (16) is a solution (see Theorem 4 below). In fact, since the boundary condition is governed by (13) for $\mathrm{m} > 1$, (16) is the unique continuous solution. Other extensions, including non linearities in the interior operator or on the boundary operator are available, but they are not considered in this note.

One final word, in order to simplify the exposition, we omit here the study of complementary regularity based on the PDE theory. For this topic we refer, for instance, [9], [11] or [25].

2. The problem in the whole space

Given

$$\mathbf{a} : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}} \quad \text{and} \quad \sigma : \mathbb{R}^{\mathrm{N}} \to \mathscr{M}(\mathrm{N} \times \mathrm{M}; \mathbb{R}),$$

functions satisfying (5), we may consider the unique, in the probability sense, solution of the *stochastic differential equation*

$$\text{(SDE)} \qquad \begin{cases} \mathrm{d}\mathscr{X}_t^x = -\mathbf{a}\big(\mathscr{X}_t^x\big)\mathrm{d}t + \sigma\big(\mathscr{X}_t^x\big)\mathrm{d}\mathscr{B}_t, \ t > 0, \\ \mathscr{X}_0^x = x \in \mathbb{R}^{\mathrm{N}}, \end{cases}$$

(see, for instance, [18] for details). Given any couple of continuous functions

$$f,\ a_0 : \mathbb{R}^{\mathrm{N}} \to \mathbb{R},$$

one defines

$$u(x)=\mathbb{E}\left[\int_0^\infty f\left(\mathscr{X}_t\right)\exp\left(-\int_0^t a_0\left(\mathscr{X}_s^x\right)ds\right)dt\right],\quad x\in\mathbb{R}^{\mathrm{N}} \tag{17}$$

provided that f is bounded.

REMARK 1. In a framework close to Control Theory, the function $u(x)$ is the optimal value of an one-control problem where $f\left(\mathscr{X}_t^x\right)$ denotes the *payment per time unit* and

$$\exp\left(-\int_0^t a_0\left(\mathscr{X}_s^x\right)ds\right)$$

an *actualization term*. □

In order to characterize the function u we employ a classical argument

PROPOSITION 1. (Dynamics Programming Principle) *For every $t\geqslant 0$ and $x\in\mathbb{R}^{\mathrm{N}}$ one satisfies*

$$u(x)=\mathbb{E}\left[\int_0^t f\left(\mathscr{X}_s^x\right)\exp\left(-\int_0^s a_0\left(\mathscr{X}_r^x\right)dr\right)\mathrm{d}s\right.$$
$$\left.+u\left(\mathscr{X}_t^x\right)\exp\left(-\int_0^t a_0\left(\mathscr{X}_s^x\right)ds\right)\right]. \tag{18}$$

REMARK 2. Property 18 is the mathematical expression of the well-known *heuristic principle* introduced by R. Bellman. □

From assumptions we have

$$-\infty<u_*(x)\leqslant u(x)\leqslant u^*(x)<+\infty,\quad x\in\mathbb{R}^{\mathrm{N}},$$

where u_*, respectively u^*, is the lower semi-continuous, respectively upper semi-continuous, envelop of u. Next we argue as in [2] or [8]. For some fix and arbitrary point $x_0\in\mathbb{R}^{\mathrm{N}}$ we consider $\varphi\in\mathscr{C}^2(\mathbb{R}^{\mathrm{N}})$ such that

$$(u^*-\varphi)(x_0)\geqslant(u^*-\varphi)(x),\quad x\in\mathbb{R}^{\mathrm{N}}.$$

Replacing φ by

$$\widehat{\varphi}(x)=\varphi(x)+(u^*-\varphi)(x_0)+|x-x_0|^2$$

if necessary, we may assume

$$0=(u^*-\varphi)(x_0)>(u^*-\varphi)(x),\quad x\in\mathbb{R}^{\mathrm{N}}.$$

If $\{x_\varepsilon\}_\varepsilon\subset\mathbb{R}^{\mathrm{N}}$ satisfies

$$\{x_\varepsilon\}\to x_0\quad\text{as } \varepsilon\to 0,$$
$$u(x_\varepsilon)\to u^*(x_0)\quad\text{as } \varepsilon\to 0,$$

we construct the *auxiliar time*

$$t_\varepsilon^2 = |u(x_\varepsilon) - \varphi(x_\varepsilon)|,$$

for which

$$\varphi(x_\varepsilon) = u(x_\varepsilon) + o(t_\varepsilon).$$

So,

$$\varphi(x_\varepsilon) \leqslant \mathbb{E}\Big[\int_0^{t_\varepsilon} f\big(\mathscr{X}_s^{x_\varepsilon}\big)\exp\Big(-\int_0^s a_0\big(\mathscr{X}_r^x\big)dr\Big)ds + \varphi\big(\mathscr{X}_{t_\varepsilon}^{x_\varepsilon}\big)\exp\Big(-\int_0^{t_\varepsilon} a_0\big(\mathscr{X}_s^{x_\varepsilon}\big)ds\Big)\Big] + o(t_\varepsilon) \quad (19)$$

(see (18)). Regularity of φ enables us to apply Ito's Rule to

$$\Phi(t) = \varphi\big(\mathscr{X}_t^x\big)\exp\Big(-\int_0^t a_0\big(\mathscr{X}_s^x\big)ds\Big)$$

and to obtain

$$d\Phi(t) = \exp\Big(-\int_0^t a_0\big(\mathscr{X}_s^{x_\varepsilon}\big)ds\Big)\Big\{\big[-a_0\big(\mathscr{X}_t^{x_\varepsilon}\big)\varphi\big(\mathscr{X}_t^{x_\varepsilon}\big) - \mathscr{L}\varphi\big(\mathscr{X}_t^{x_\varepsilon}\big)\big]dt + \langle\sigma\big(\mathscr{X}_t^{x_\varepsilon}\big)\mathrm{d}\mathscr{B}_t, \nabla\varphi\big(\mathscr{X}_t^{x_\varepsilon}\big)\rangle\Big\}$$

for the operator

$$\mathscr{L}\varphi(y) = -\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}(y)\cdot \mathrm{D}^2\varphi(y)\right) + \langle \mathbf{a}(y), \mathrm{D}\varphi(y)\rangle,$$
$$\mathscr{A}_{ij}(y) = \sum_{l=1}^{\mathrm{N}} \sigma_{il}(y)\sigma_{lj}(y),\ 1 \leqslant i,j \leqslant \mathrm{N}.$$

Then we get to

$$\mathbb{E}\Big[\varphi\big(\mathscr{X}_{t_\varepsilon}^{x_\varepsilon}\big)\exp\Big(-\int_0^{t_\varepsilon} a_0\big(\mathscr{X}_s^{x_\varepsilon}\big)ds\Big)\Big] = \varphi(x_\varepsilon) + \mathbb{E}\Big[\int_0^{t_\varepsilon}\exp\Big(-\int_0^s a_0\big(\mathscr{X}_r^{x_\varepsilon}\big)dr\Big)\Big(-a_0\big(\mathscr{X}_s^{x_\varepsilon}\big)\varphi\big(\mathscr{X}_s^{x_\varepsilon}\big) - \mathscr{L}\varphi\big(\mathscr{X}_t^{x_\varepsilon}\big)\Big)dt\Big]$$

and from (19)

$$o(t_\varepsilon) \leqslant \big[f(x_\varepsilon) - a_0(x_\varepsilon)\varphi(x_\varepsilon) - \mathscr{L}\varphi(x_\varepsilon)\big]\mathbb{E}\left[\frac{1-\exp(-a_0(x_\varepsilon)t_\varepsilon)}{a_0(x_\varepsilon)}\right].$$

Letting $\varepsilon \to 0$ we conclude

$$\mathscr{L}\varphi(x_0) + a_0(x_0)u^*(x_0) \leqslant f(x_0).$$

By an analogous reasoning one proves the u is also a super-solution, therefore one concludes

THEOREM 1. *Under the assumption* (5)*, for* $\mathbf{a}$, σ *and* a_0 *the function* u*, given in* (17)*, is a solution of* $\mathscr{L}u + a_0 u = f \quad in\ \mathbb{R}^N$.

REMARK 3. Since

$$\sigma_{ik}\sigma_{jk}\xi_i\xi_j \geqslant -\frac{\sigma_{ik}^2\xi_i^2 + \sigma_{jk}^2\xi_j^2}{2} \quad \Rightarrow \quad \sum_{i,j=1;\ i<j}^{N} \sigma_{ik}\sigma_{jk}\xi_i\xi_j \geqslant -\sum_{i=1}^{N} \sigma_{ik}^2\xi_i^2,$$

the covariance matrix, $\mathscr{A}_{ij} = \sum_{k=1}^{M} \sigma_{ik}\sigma_{jk}$, is *elliptic possibly degenerate*, *i.e.*

$$\langle \mathscr{A}\xi, \xi \rangle \geqslant 0, \quad \xi \in \mathbb{R}^N.$$

Moreover, from $\sigma \not\equiv 0$ it verifies

$$\mathbf{Tr}\,\mathscr{A} > 0,$$

we say then that $\mathscr{A}(\cdot)$ is *elliptic quasi non–degenerate*. On the other hand, given $\mathscr{A}$ the property of to find some $\sigma : \mathbb{R}^N \to \mathscr{M}(N \times M; \mathbb{R})$ Lipschitz continuous functions solving

$$\mathscr{A} = \sigma\sigma^t$$

fails, in general, for any Lipschitz continuous positive semi–definite symmetric matricial function $\mathscr{A}$. However, we may consider the technicality: $\mathscr{A} \in W^{2,\infty}$ *implies that* $\sqrt{\mathscr{A}}$ *is uniformly Lipschitz continuous* (see [10] for some results of the factorization of non–negative definite matrices). □

Complementary regularity on the solution u whenever $\mathscr{L}$ is uniformly elliptic can be obtained (see [15]), but, as it was pointed out in Introduction, by simplicity we omit here the study of regularity based on the PDE theory. However, we provide two illustrative results derived directly from the construction of (17). First, we use the notation u_f in studying the dependence on the data of the function u, given in (17). So, we have

PROPOSITION 2. *Assumed the condition*

$$a_0(x) \geqslant \lambda > 0, \quad x \in \mathbb{R}^N, \tag{20}$$

one has

$$u_f(x) \leqslant \frac{\|f\|_\infty}{\lambda}, \quad x \in \mathbb{R}^N.$$

Proof. By definition one has

$$u_f(x) \leqslant \|f\|_\infty \int_0^\infty \exp(-\lambda t)\,dt \leqslant \frac{\|f\|_\infty}{\lambda}, \quad x \in \mathbb{R}^N. \square$$

Therefore, it follows

$$u_f(x) - u_{\widehat{f}}(x) \leqslant \|f-\widehat{f}\|_\infty \int_0^\infty \exp(-\lambda t)dt \leqslant \frac{\|f-\widehat{f}\|_\infty}{\lambda},$$

whence

$$\|u_f - u_{\widehat{f}}\|_\infty \leqslant \frac{\|f-\widehat{f}\|_\infty}{\lambda}, \quad x \in \mathbb{R}^{\mathrm{N}} \tag{21}$$

holds. The second one is

THEOREM 2. *Assume*

$$|\mathrm{D}\mathbf{a}(x) - \mathrm{D}\mathbf{a}(\widehat{x})| + |\mathrm{D}\sigma(x) - \mathrm{D}\sigma(\widehat{x})| \leqslant C|x-\widehat{x}|, \quad x,\ \widehat{x} \in \mathbb{R}^{\mathrm{N}}, \tag{22}$$

for some positive constant C. If f is semi-concave (*respectively semi-convex*) *the function u, given in* (17)*, is also semi-concave* (*respectively semi-convex*).

Proof. We recall that a function $\psi:\ \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ is semi-concave if $x \mapsto \psi(x) - K|x|^2$ is concave for some positive constant K. Consequently, by straightforward computations ψ is semi-concave if and only if

$$\mu\psi(x) + (1-\mu)\psi(y) - \psi(\mu x + (1-\mu)y) \leqslant K\mu(1-\mu)|x-y|^2, \quad x,y \in \mathbb{R}^{\mathrm{N}},\ 0<\mu<1.$$

On the other hand, $\psi:\ \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ is semi-convex if $-\psi$ is semi-concave. We only study the semi-concave case. In order to simplify we will assume $a_0(\cdot) \equiv \lambda$. Let $x_1, x_2 \in \mathbb{R}^{\mathrm{N}}$ and $0<\mu<1$ and we denote $x_\mu = \mu x_2 + (1-\mu)x_1$. Then

$$\begin{aligned}\mu u(x_2) + (1-\mu)u(x_1) - u(x_\mu) = \mathbb{E}\Big[\int_0^\infty &\mu f\big(\mathscr{X}_t^{x_2}\big)\exp(-\lambda t)dt \\ &+ \int_0^\infty (1-\mu) f\big(\mathscr{X}_t^{x_1}\big)\exp(-\lambda t)dt \\ &- \int_0^\infty f\big(\mathscr{X}_t^{x_\mu}\big)\exp(-\lambda t)dt\Big].\end{aligned}$$

Denoting

$$\mathscr{X}_t^{\mu} \doteq \mu\mathscr{X}_t^{x_2} + (1-\mu)\mathscr{X}_t^{x_1}$$

one has from assumptions

$$\begin{aligned}\mu u(x_2) + (1-\mu)u(x_1) - u(x_\mu) &\leqslant K\mu(1-\mu)\mathbb{E}\Big[\int_0^\infty \big|\mathscr{X}_t^{x_1} - \mathscr{X}_t^{x_2}\big|^2 \exp(-\lambda t)dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^\infty f\big(\mathscr{X}_t^{\mu}\big) - f\big(\mathscr{X}_t^{x_\mu}\big)\exp(-\lambda t)dt\Big] \\ &\leqslant \frac{K}{\lambda}\mu(1-\mu)\big|x_1 - x_2\big|^2\end{aligned}$$

$$+K\mathbb{E}\Big[\int_0^\infty \big|\mathscr{X}_t^{\mu}-\mathscr{X}_t^{x_\mu}\big|\exp(-\lambda t)dt\Big] \tag{23}$$

(here K is a generic positive constant). On the other hand, it follows

$$\begin{aligned}\big|\mu\mathbf{a}(x_2)+(1-\mu)\mathbf{a}(x_1)-\mathbf{a}(x_\mu)\big| &= \big|\mu\big(\mathbf{a}(x_2)-\mathbf{a}(x_\mu)\big)+(1-\mu)\big(\mathbf{a}(x_1)-\mathbf{a}(x_\mu)\big)\big|\\ &\leqslant \mu(1-\mu)\big|\langle\int_0^1 \big(\mathrm{D}\mathbf{a}\big(x_\mu+\theta(1-\mu)(x_2-x_1)\big)\\ &\qquad -\mathrm{D}\mathbf{a}\big(x_\mu+\theta\mu(x_1-x_2)\big)\big)d\theta, x_2-x_1\rangle\big|\\ &\leqslant K\mu(1-\mu)|x_1-x_2|^2.\end{aligned}$$

Analogously, one proves

$$\big|\mu\sigma(x_2)+(1-\mu)\sigma(x_1)-\sigma(x_\mu)\big|\leqslant K\mu(1-\mu)|x_1-x_2|^2.$$

By using on the *stochastic differential equation* (SDE) the Burkholder-Davis-Gundy inequality (see [18, Theorem 3.28]) we obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{[s,t]}\big|\mathscr{X}_t^{\mu}-\mathscr{X}_t^{x_\mu}\big|\Big]\\ &\leqslant K\mathbb{E}\Bigg[\sup_{r\in[s,t]}\Big|\int_s^r \big(\mu\mathbf{a}(\mathscr{X}_\tau^{x_2})+(1-\mu)\mathbf{a}\big(\mathscr{X}^{x_1}\big)-\mathbf{a}\big(\mathscr{X}^{\mu}\big)d\tau\Big|^2\Bigg]\\ &\qquad +K\mathbb{E}\Bigg[\sup_{r\in[s,t]}\Big|\int_s^r \big(\mathbf{a}\big(\mathscr{X}^{\mu}\big)-\mathbf{a}\big(\mathscr{X}_\tau^{x_\mu}\big)\big)d\tau\Big|^2\Bigg]\\ &\leqslant K\mathbb{E}\Bigg[\sup_{r\in[s,t]}\Big|\int_s^r \big(\mu\sigma(\mathscr{X}_\tau^{x_2})+(1-\mu)\sigma\big(\mathscr{X}_\tau^{x_1}\big)-\sigma\big(\mathscr{X}_\tau^{\mu}\big)\big)d\mathscr{B}_\tau\Big|^2\Bigg]\\ &\qquad +K\mathbb{E}\Bigg[\sup_{r\in[s,t]}\Big|\int_s^r \big(\sigma\big(\mathscr{X}_\tau^{\mu}\big)-\sigma\big(\mathscr{X}_\tau^{x_\mu}\big)\big)d\mathscr{B}_\tau\Big|^2\Bigg]\\ &\leqslant K\mu^2(1-\mu)^2\mathbb{E}\Big[\int_s^r \big|\mathscr{X}_\tau^{x_2}-\mathscr{X}_\tau^{x_1}\big|^4 d\tau\Big]+K\mathbb{E}\Big[\int_s^r \big|\mathscr{X}_\tau^{\mu}-\mathscr{X}_\tau^{x_\mu}\big|^2 d\tau\Big]\\ &\leqslant K\mu^2(1-\mu)^2\big|x_1-x_2\big|^4+2K\int_s^r \mathbb{E}\Big[\big|\mathscr{X}_\tau^{\mu}-\mathscr{X}_\tau^{x_\mu}\big|^2\Big]d\tau,\end{aligned}$$

therefore by Gronwall inequality one has

$$\mathbb{E}\big[\sup_{[s,t]}\big|\mathscr{X}_t^{\mu}-\mathscr{X}_t^{x_\mu}\big|\big]\leqslant K\mu(1-\mu)\big|x_1-x_2\big|^2. \tag{24}$$

Finally, (23) and (24) conclude the result. □

REMARK 4. It is clear that the semi-concavity (respectively semi-convexity) implies

$$\frac{\partial^2 u}{\partial\chi^2}\geqslant \text{(respectively }\leqslant\text{) } K \qquad \text{in } \mathscr{D}'(\mathbb{R}^{\mathrm{N}}) \qquad \text{for all } \chi,\ |\chi|^2=1.$$

The above proof follows argument of [25, Proposition 4.4.5]. □

3. The linear problem

There are many way to study the behavior of the reflection near the boundary (see, for instance, [12], [18], [19], [21] or [23]). Here we will employ the domain penalty method (see [3] or [16] for sharp details). So, assuming (15), for every $\delta > 0$ to be sending 0, we consider the equation in the whole space

$$\begin{cases} d\mathscr{X}_t^{\delta,x} = -\mathbf{a}(\mathscr{X}_t^{\delta,x})dt + \sigma(\mathscr{X}_t^{\delta,x})d\mathscr{B}_t - \dfrac{1}{\delta}\varphi(\mathscr{X}_t^{\delta,x})\vec{\gamma}(\mathscr{X}_t^{\delta,x})dt, \quad t > 0, \\ \mathscr{X}_0^{\delta,x} = x \in \mathbb{R}^{\mathrm{N}}, \end{cases}$$

where φ is a $\mathbb{W}^3$ extension of the function $\mathrm{dist}^4(\cdot,\overline{\Omega})$ to $\mathbb{R}^{\mathrm{N}}$, verifying

$$\langle \nabla\varphi(x), \vec{\gamma}(x)\rangle > 0, \quad x \in \mathbb{R}^{\mathrm{N}},$$

for Lipschitz extensions of the data $a_0, f,\ c_0,\ g$ y $\vec{\gamma}$ to the whole space $\mathbb{R}^{\mathrm{N}}$ satisfying (5). In [3, Section 3] one proves the convergence in law of $\{\mathscr{X}_t^{\delta,x}\}_{t\geqslant 0}$ to the unique solution $\{\mathscr{X}_t^{x}\}_{t\geqslant 0}$ of

$$\begin{cases} d\mathscr{X}_t^{x} = -\mathbf{a}(\mathscr{X}_t^{x})dt + \sigma(\mathscr{X}_t^{x})d\mathbb{B}_t - \vec{\gamma}(\mathscr{X}_t^{x})d\mathrm{L}_t^x, \quad t > 0, \\ \mathscr{X}_0^{x} = x \in \overline{\Omega}. \end{cases}$$

The proof uses the property

$$\limsup_{(y,y')\to(x,x');\delta\to 0} \mathbb{E}_{y,y'}\left[\left|\mathscr{X}_t^{\delta,y} - \mathscr{X}_t^{y'}\right|\right] = |x - x'|,\ x,x' \in \Omega, \tag{25}$$

therefore a better convergence holds.

REMARK 5. Really, in [3] the reflection term of the stochastic differential equation is denoted by other way. □

Next, we consider an extension $\psi \in \mathscr{C}^1(\mathbb{R}^{\mathrm{N}})$ of the function $\mathrm{dist}(z,\overline{\Omega})$ for which

$$\left\{ \begin{aligned} \frac{\varphi(z)}{\delta} &= \frac{\mathrm{dist}^4(z,\overline{\Omega})}{\delta}, \\ \frac{\psi(z)}{\delta} &= \frac{\mathrm{dist}(z,\overline{\Omega})}{\delta}, \end{aligned} \right. \quad \text{for } \mathrm{dist}(z,\overline{\Omega}) \leqslant \delta.$$

Then we construct the functions

$$\begin{cases} f_\delta(z) = f(z) + \dfrac{1}{\delta}\psi(z)g(z), \\ a_\delta(z) = a_0(z) + \dfrac{1}{\delta}\psi(z)c_0(z), \quad z \in \mathbb{R}^{\mathrm{N}}, \\ \mathbf{a}_\delta(z) = \mathbf{a}(z) + \dfrac{1}{\delta}\varphi(z)\vec{\gamma}(z), \end{cases}$$

that verify

$$\begin{cases} f_\delta(z) = f(z) \\ a_\delta(z) = a_0(z) \qquad z \in \overline{\Omega}, \\ \mathbf{a}_\delta(z) = \mathbf{a}(z), \end{cases}$$

independent on δ. Finally, we introduce

$$u_\delta(x) = \mathbb{E}\left[\int_0^\infty f_\delta\left(\mathscr{X}_t^{\delta,x}\right)\exp\left(-\int_0^t a_\delta\left(\mathscr{X}_s^{\delta,x}\right)ds\right)\mathrm{d}t\right], \quad x \in \mathbb{R}^{\mathrm{N}}, \tag{26}$$

for which

$$u_\delta(x) = \mathbb{E}\Bigg[\int_0^t f_\delta\left(\mathscr{X}_s^{\delta,x}\right)\exp\left(-\int_0^s a_\delta(\mathscr{X}_r^{\delta,x})\mathrm{d}r\right)\mathrm{d}s + u_\delta(\mathscr{X}_t^{\delta,x})\exp\left(-\int_0^t a_\delta(\mathscr{X}_s^{\delta,x})\mathrm{d}t\right)\Bigg], \tag{27}$$

holds for $t > 0$ and $x \in \mathbb{R}^{\mathrm{N}}$, as in Proposition 1 below. So that, the interior reasoning of Theorem 1 leads to

PROPOSITION 3. *Under assumptions on the data, for every* $\delta > 0$*, the function* u_δ*, given by* (26)*, is a solution of*

$$-\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}\cdot \mathrm{D}^2 u_\delta\right) + \langle \mathbf{a}_\delta, \nabla u_\delta\rangle + a_\delta u_\delta = f_\delta \quad \mathit{in}\ \mathbb{R}^{\mathrm{N}}.$$

Certainly, u_δ is bounded on $\overline{\Omega}$, uniformly in δ. Therefore we may construct the functions

$$\begin{aligned} \underline{u}(x) &= \liminf_{y_\delta \to x,\ y_\delta \in \mathbb{R}^{\mathrm{N}};\ \delta \to 0} u_\delta(y_\delta), \\ \overline{u}(x) &= \limsup_{y_\delta \to x,\ y_\delta \in \mathbb{R}^{\mathrm{N}};\ \delta \to 0} u_\delta(y_\delta), \end{aligned} \qquad x \in \overline{\Omega}, \tag{28}$$

moreover $\underline{u}$ is lower semi–continuous and $\overline{u}$ is upper semi–continuous and obvious inequality

$$\underline{u}(x) \leqslant \overline{u}(x), \quad x \in \overline{\Omega},$$

holds. Next, we prove that, in fact, they coincide in a continuous weak solution providing the Brosamler formula

THEOREM 3. *Under assumption* (15)*, the function* u *given in* (8) *is the unique continuous solution of the problem*

$$\begin{cases} \mathscr{L}u + a_0 u = f & \mathit{in}\ \Omega, \\ \langle \nabla u, \vec{\gamma}\rangle + c_0 u = g & \mathit{on}\ \partial\Omega. \end{cases} \tag{29}$$

Proof. First of all we note that from limit operations (see [7]) the functions $\underline{u}(x)$ and $\overline{u}(x)$ are sub- and super-solutions, respectively, of

$$\mathscr{L}u + a_0 0 = f \quad \text{in } \Omega.$$

On the other hand, given $(p,\mathscr{X}) \in \mathscr{J}^{2,+}_{\overline{\Omega}}\overline{u}(x_0)$, $x_0 \in \partial\Omega$, and $\delta > 0$ there exists $x_\delta \in \mathbb{R}^{\mathrm{N}}$ and

$$\left(p_\delta, \mathscr{X}_\delta\right) \in \mathscr{J}^{2,+}_{\overline{\Omega}} u_\delta(x_\delta),$$

such that

$$\left\{\left(x_\delta, u_\delta(x_\delta), p_\delta, \mathscr{X}_\delta\right)\right\}_\delta \to (x_0, \overline{u}(x_0), p, \mathscr{X}) \quad \text{as } \delta \to 0$$

(see [7, Lemma 6.1 and Proposition 4.3]). Moreover, with no loss of generality, we may assume

$$x_\delta = x_0 + \delta^{\frac{1}{4}}\vec{n}(x_0).$$

Then

$$\nabla \mathrm{dist}(x_0, \overline{\Omega}) = \vec{n}(x_0),$$

and

$$\mathrm{dist}(x, \overline{\Omega}) = \langle \vec{n}(x_0), x - x_0\rangle + o(|x - x_0|)$$

imply

$$\frac{\varphi(x)}{\delta} = \frac{\left(\langle \vec{n}(x_0), x - x_0\rangle\right)^4}{\delta} + \frac{o(|x-x_0|)}{\delta} \quad \text{for } \delta \text{ small enough.}$$

Hence

$$\lim_{\delta\to 0} \frac{\varphi(x_0 + \delta^{\frac{1}{4}}\vec{n}(x_0))}{\delta} = |\vec{n}(x_0)|^8 = 1$$

shows

$$\lim_{\delta\to 0} \mathbf{a}_\delta(x_\delta) = \mathbf{a}(x_0) + \vec{\gamma}(x_0).$$

Analogously, reasoning with

$$x_\delta = x_0 + \delta\vec{n}(x_0),$$

it follows, from

$$\frac{\psi(x)}{\delta} = \frac{\langle \vec{n}(x_0), x - x_0\rangle}{\delta} + \frac{o(|x-x_0|)}{\delta} \quad \text{for } \delta \text{ small enough,}$$

the properties

$$\begin{cases} \lim\limits_{\delta\to 0} \dfrac{\psi(x_0 + \delta\vec{n}(x_0))}{\delta} = |\vec{n}(x_0)|^2 = 1, \\ \lim\limits_{\delta\to 0} f_\delta(x_\delta) = f(x_0) + g(x_0), \\ \lim\limits_{\delta\to 0} a_\delta(x_\delta) = a_0(x_0) + c_0(x_0). \end{cases}$$

So that,

$$-\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}\cdot \mathrm{D}^2 u_\delta\right) + \langle \mathbf{a}_\delta, \nabla u_\delta\rangle + a_\delta u_\delta \leqslant f_\delta \quad \text{in } \mathbb{R}^{\mathrm{N}},$$

leads to

$$\Big[-\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}\cdot \mathrm{D}^2\overline{u}\right)+\langle \mathbf{a},\nabla\overline{u}\rangle+a_0\overline{u}-f\Big]+\Big[\langle\nabla\overline{u},\vec{\gamma}\rangle+c_0\overline{u}-g\Big]\leqslant 0 \quad \text{on } \overline{\Omega},$$

in the viscosity sense, because

$$\begin{aligned}\liminf_{\delta\to 0}\Big(-\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}(x_\delta)\cdot\mathscr{X}_\delta\right)+\langle \mathbf{a}_\delta(x_\delta),p_\delta\rangle+a_\delta(x_\delta)u_\delta(x_\delta)-f_\delta(x_\delta)\Big)\\ =\Big[-\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}(x_0)\cdot\mathscr{X}\right)+\langle \mathbf{a}(x_0),p\rangle+a_0(x_0)\overline{u}(x_0)-f(x_0)\Big]\\ +\Big[\langle p,\vec{\gamma}(x_0)\rangle+c_0(x)\overline{u}(x_0)-g(x_0)\Big].\end{aligned}$$

So, it proves that $\overline{u}$ is a relaxed sub-solution of (29). A similar reasoning enables us to obtain

$$\Big[-\frac{1}{2}\mathbf{Tr}\left(\mathscr{A}\cdot \mathrm{D}^2\underline{u}\right)+\langle \mathbf{a},\nabla\underline{u}\rangle+a_0\underline{u}-f\Big]+\Big[\langle\nabla\underline{u},\vec{\gamma},\rangle+c_0\underline{u}-g\Big]\geqslant 0 \quad \text{on } \overline{\Omega}$$

that, now, proves that $\underline{u}$ is a relaxed super-solution of (29). Moreover, since the boundary operator

$$\mathrm{B}(x,r,p)=\langle p,\vec{\gamma}(x)\rangle+c_0(x)r-g(x)$$

satisfies the obliqueness (14) for $\mathrm{m}=1$, one concludes

$$\overline{u}(x)\leqslant\underline{u}(x),\quad x\in\overline{\Omega},$$

(see [1, Theorem 2.2]), thus, $u(x)=\overline{u}(x)=\underline{u}(x)$, $x\in\overline{\Omega}$, is a continuous solution of (29). In fact, it is the unique solution of (29) (see [1, Theorem 2.2]). Finally, the convergence in law of $\{\mathscr{X}_t^{\delta,x}\}_{t\geqslant 0}$ to the unique solution $\{\mathscr{X}_t^{x}\}_{t\geqslant 0}$ and the regularity of the data implies that the function u is given by the formula (8). □

REMARK 6. In order to understand the relaxed Neumann boundary conditions (see (11) and (12)) a main question arises. How the equation holds on the boundary? Some authors have studied the question. See, for instance, [1] or [7]. Essentially, if $u-\varphi$ attains a local maximum at some $x_0\in\partial\Omega$, as we consider for the viscosity sub-solutions, the same holds for $u-\varphi-\psi\big(\mathrm{dist}(\cdot,\partial\Omega)\big)$ whenever ψ is a smooth function and $\psi(0)=0$. Then the regularity of the boundary $\partial\Omega$ and suitable obliqueness enable to construct a sharp test function for which

$$\min\{\mathscr{L}u+a_0u-f,\mathrm{B}(x,u,\mathrm{D}u)\}\leqslant 0 \quad \text{on } \partial\Omega$$

becomes

$$\mathrm{B}(x,u,\mathrm{D}u)\leqslant 0 \quad \text{on } \partial\Omega.$$

In an analogous way, for super-solutions one may construct a sharp test function for which

$$\max\{\mathscr{L}u+a_0u-f,\mathrm{B}(x,u,\mathrm{D}u)\}\geqslant 0 \quad \text{on } \partial\Omega$$

becomes

$$\mathrm{B}(x,u,\mathrm{D}u) \geqslant 0 \quad \text{on } \partial\Omega$$

i.e. the *relaxed Neumann boundary condition* becomes

$$\mathrm{B}(x,u,\mathrm{D}u) = 0 \quad \text{on } \partial\Omega,$$

in the ordinary viscosity sense. We send [1] for details. We also note that if the boundary operator governs Dirichlet boundary conditions

$$\mathrm{B}(x,r,p) = r - g(x) \quad \text{on } \partial\Omega,$$

the relative relaxed Dirichlet boundary condition becomes the ordinary Dirichlet boundary condition under a simple and well known assumption: *the boundary $\partial\Omega$ must consist of regular boundary points*, as it was proved, for instance, in [2], [8], [10] or [24]. □

REMARK 7. In the Brosamler formula (8) we can understand that the *boundary payment* becomes less active with time guided by the cumulative rate

$$\exp\left(-\int_0^t a_0\left(\mathscr{X}_s^x\right)ds - \int_0^t c_0\left(\mathscr{X}_s^x\right)d\mathrm{L}_s^x\right)$$

per unit time, according to there were reflections produced on the boundary until the process "died near to the boundary", after of a possibly infinite number of reflections. □

Here, we end with an application of Theorem 3 to be used in certain estimate of the next section.

PROPOSITION 4. *Let $\mathrm{H}^\lambda_{\partial\Omega}$ be the relaxed solution of the boundary value problem*

$$\begin{cases} \mathscr{L}\mathrm{H}^\lambda_{\partial\Omega} + \lambda \mathrm{H}^\lambda_{\partial\Omega} = 0 & \text{in } \Omega, \\ \langle \nabla \mathrm{H}^\lambda_{\partial\Omega}, \vec{\gamma} \rangle = 1 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Then, we have the representation

$$\mathrm{H}^\lambda_{\partial\Omega}(x) = \mathbb{E}\left[\int_0^\infty \exp(-\lambda t)d\mathrm{L}_t^x\right] > 0, \quad x \in \Omega. \tag{31}$$

REMARK 8. Whenever $\mathscr{L}$ is uniformly elliptic, existence, uniqueness and regularity of function $\mathrm{H}^\lambda_{\partial\Omega}$ also follows from [15, Theorem I.1] and the positivity can be obtained by using Hopf's Principle (see [11]). We note that in any case

$$\lambda \mapsto \mathrm{H}^\lambda_{\partial\Omega}(\cdot)$$

is a decreasing and convex map. We also note that the borderline case

$$\mathrm{H}^0_{\partial\Omega}(x) = \mathbb{E}\left[\lim_{t\to\infty} \mathrm{L}^x_t\right] > 0, \quad x \in \Omega,$$

is a solution of

$$\begin{cases} \mathscr{L}\mathrm{H}^\lambda_{\partial\Omega} = 0 & \text{in } \Omega, \\ \langle \nabla \mathrm{H}^\lambda_{\partial\Omega}, \vec{\gamma}\rangle = 1 & \text{on } \partial\Omega. \ \square \end{cases}$$

Complementary regularity derived directly from the Brosamler formula can be obtained as in Section 2.

4. The nonlinear boundary problem

In this section we will assume the condition (15) as well as $c_0(x) > 0$, for which we will use the notation

$$a_0(x) \geqslant \lambda > 0,\ x \in \overline{\Omega}, \quad \text{and} \quad c_0(x) \geqslant \rho > 0,\ x \in \partial\Omega. \tag{32}$$

In what follows we are going to study the dependence on *cumulative actualization*

$$\exp\left(-\int_0^t a_0\left(\mathscr{X}^x_s\right)ds - \int_0^t c_0\left(\mathscr{X}^x_s\right)\left|v\left(\mathscr{X}^x_s\right)\right|^{\mathrm{m}-1} d\mathrm{L}^x_s\right)$$

for $\mathrm{m} > 1$ and $v \in \mathscr{C}(\overline{\Omega})$. More precisely, let us consider the application

$$\begin{aligned} \mathscr{T}v(x) = \mathbb{E}\Bigg[\int_0^\infty f\left(\mathscr{X}^x_t\right) \exp\left(-\int_0^t a_0\left(\mathscr{X}^x_s\right)ds\right)dt \\ + \int_0^\infty g\left(\mathscr{X}^x_t\right) \exp\Bigg(-\int_0^t a_0\left(\mathscr{X}^x_s\right)ds \\ - \int_0^t c_0\left(\mathscr{X}^x_s\right)\left|v\left(\mathscr{X}^x_s\right)\right|^{\mathrm{m}-1} d\mathrm{L}^x_s\Bigg) d\mathrm{L}^x_t\Bigg], \end{aligned} \tag{33}$$

for $x \in \overline{\Omega}$. Again Theorem 3 implies that $\mathscr{T}v$ is the unique solution of the boundary value problem

$$\begin{cases} \mathscr{L}\mathscr{T}v + a_0\mathscr{T}v = f & \text{in } \Omega \\ \langle \nabla\mathscr{T}v, \vec{\gamma}\rangle + c_0|v|^{\mathrm{m}-1}\mathscr{T}v = g & \text{on } \partial\Omega. \end{cases}$$

Our aim is clear: *show the existence of a fixed point* u

$$\mathscr{T}u(x) = u(x), \quad x \in \overline{\Omega}.$$

Straightforward computations on the definition of (33) lead to

$$|\mathscr{T}v(x)| \leqslant \sup_{\overline{\Omega}}|f| \int_0^{+\infty} \exp(-\lambda t)dt + \sup_{\partial\Omega}|g| \mathbb{E}_x\left[\int_0^\infty \exp(-\lambda t) d\mathrm{L}^x_t\right], \quad x \in \overline{\Omega},$$

whence (31) derives the estimate

$$|\mathscr{T}v(x)| \leqslant \frac{1}{\lambda}\sup_{\overline{\Omega}}|f| + \sup_{\partial\Omega}|g|\mathrm{H}^{\lambda}_{\partial\Omega}(x), \quad x\in\overline{\Omega}, \tag{34}$$

where the function $\mathrm{H}^{\lambda}_{\partial\Omega}$ was introduced in Proposition 4. That estimate gives the inclusion

$$\mathscr{T}\left(\mathscr{C}(\overline{\Omega})\right) \subseteq \overline{\mathbf{B}}_{\mathscr{R}}(0) \subset \mathscr{C}(\overline{\Omega}),$$

for

$$\mathscr{R} \doteq \frac{1}{\lambda}\sup_{\overline{\Omega}}|f| + \sup_{\partial\Omega}|g|\sup_{\overline{\Omega}}|\mathrm{H}^{\lambda}_{\partial\Omega}|.$$

The existence of a point fixed is now obtained through continuity of application $\mathscr{T}$.

THEOREM 4. *Let us assume the conditions* (15) *and* (32). *Then, for* $\mathrm{m}>1$ *the mapping* (33) *is uniformly continuous. As consequence, there exists a function* $u\in\mathscr{C}(\overline{\Omega})$, *such that*

$$\begin{cases} \|u\|_{\mathscr{C}(\overline{\Omega})} \leqslant \dfrac{1}{\lambda}\sup_{\overline{\Omega}}|f| + \sup_{\partial\Omega}|g|\sup_{\overline{\Omega}}|\mathrm{H}^{\lambda}_{\partial\Omega}|, \\ \mathscr{T}u = u \quad \text{in } \mathscr{C}(\overline{\Omega}), \end{cases}$$

given by the implicit Brosamler formula (16). *Moreover,* u *is the unique solution of the boundary value problem*

$$\begin{cases} \mathscr{L}u + a_0 u = f & \text{in } \Omega, \\ \langle\nabla u,\vec{\gamma}\rangle + c_0|u|^{\mathrm{m}-1}u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. Let $v,\hat{v}\in\mathscr{C}(\overline{\Omega})$ be two arbitrary functions. Then, for each $x\in\overline{\Omega}$ the inequality

$$(\mathscr{T}v-\mathscr{T}\hat{v})(x) \leqslant \rho\sup_{\partial\Omega}|g|\mathbb{E}\left[\int_0^{\infty}\exp(-\lambda t)\left(\int_0^t\left|\left|v\left(\mathscr{X}^x_s\right)\right|^{\mathrm{m}-1} - \left|\hat{v}\left(\mathscr{X}^x_s\right)\right|^{\mathrm{m}-1}\right|d\mathrm{L}^x_s\right)d\mathrm{L}^x_t\right]$$

holds. Moreover, from definition

$$\mathrm{L}^x_t = \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^t 1\!\mathrm{I}_{\Omega_\delta}\left(\mathscr{X}^x_s\right)ds,$$

given $0<\varepsilon<1$ there exists δ_ε, small enough, such that

$$\int_0^t d\mathrm{L}^x_s = \mathrm{L}^x_t \leqslant \varepsilon + \frac{1}{2\delta_\varepsilon}\int_0^t 1\!\mathrm{I}_{\Omega_\varepsilon}\left(\mathscr{X}^x_s\right)ds \leqslant 1+\frac{t}{2\delta_\varepsilon},$$

whence

$$(\mathscr{T}v-\mathscr{T}\hat{v})(x)\leqslant\Big(\rho\sup_{\partial\Omega}|g|\mathbb{E}\Big[\int_0^\infty \exp(-\lambda t)\,d\mathrm{L}_t^x + \frac{1}{2\delta_\varepsilon}\int_0^\infty t\exp(-\lambda t)\,d\mathrm{L}_t^x\Big]\Big)\mathrm{H}_{\mathrm{m}}(v,\hat{v}), \quad (35)$$

where

$$\mathrm{H}_{\mathrm{m}}(v,\hat{v})\doteq\begin{cases}\mathrm{C}_{\mathrm{m}}\|v-\hat{v}\|^{\mathrm{m}-1}, & \text{if } 1<\mathrm{m}\leqslant 2,\\ \mathrm{C}_{\mathrm{m}}(\|v\|^{\mathrm{m}-1}+\|\hat{v}\|^{\mathrm{m}-1})^{\frac{\mathrm{m}-2}{\mathrm{m}-1}}\|v-\hat{v}\|, & \text{if } 2\leqslant \mathrm{m},\end{cases} \quad (36)$$

where δ_ε and C_{m} are two positive constants independent on v and $\hat{v}$ (see (38) in the Appendix below). From the convexity

$$\lambda\mapsto \mathrm{H}^{\lambda}_{\partial\Omega}(\cdot)$$

(see the Remark 8) we obtain the inequality

$$\mathrm{H}^0_{\partial\Omega}(x)-\mathrm{H}^{\lambda}_{\partial\Omega}(x)\geqslant -\lambda\frac{\partial}{\partial\lambda}\mathrm{H}^{\lambda}_{\partial\Omega}(x)=\lambda\mathbb{E}\left[\int_0^\infty t\exp(-\lambda t)\,d\mathrm{L}_t^x\right]>0, \quad x\in\overline{\Omega}.$$

Finally, previous arguments lead to

$$(\mathscr{T}v-\mathscr{T}\hat{v})(x)\leqslant\rho\sup_{\partial\Omega}|g|\sup_{\overline{\Omega}}\left(\mathrm{H}^{\lambda}_{\partial\Omega}+\frac{1}{2\lambda\delta_\varepsilon}|\mathrm{H}^0_{\partial\Omega}-\mathrm{H}^{\lambda}_{\partial\Omega}|\right)\mathrm{H}_{\mathrm{m}}(v,\hat{v}), \quad x\in\overline{\Omega},$$

that proves the uniform continuity of the mapping

$$\mathscr{T}:\mathscr{C}(\overline{\Omega})\to\mathscr{C}(\overline{\Omega})$$

(see the definition of H_{m} in (36)). So that, from

$$\mathscr{T}\left(\overline{\mathbf{B}}_{\mathscr{R}}(0)\right)\subseteq\overline{\mathbf{B}}_{\mathscr{R}}(0), \quad \mathbf{B}_{\mathscr{R}}(0)\subset\mathscr{C}(\overline{\Omega}),$$

with

$$\mathscr{R}=\frac{1}{\lambda}\sup_{\overline{\Omega}}|f|+\sup_{\partial\Omega}|g|\sup_{\overline{\Omega}}|\mathrm{H}^{\lambda}_{\partial\Omega}|$$

we obtain, through an extension of the Brouwer Fixed Point Theorem (see [11, Theorem 11.1]), the existence of a *fixed point*

$$u\in\overline{\mathbf{B}}_{\mathscr{R}}(0),\ \mathscr{T}u=u.$$

Definition of mapping $\mathscr{T}$ enables us to obtain the representation formula (16) and to prove that u is a solution of the boundary value problem. On the other hand, since the boundary condition

$$\mathrm{B}(x,r,p)=\langle p,\vec{\gamma}(x)\rangle+c_0(x)|r|^{\mathrm{m}-1}r-g(x), \quad (x,r,p)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^{\mathrm{N}},$$

satisfies the obliqueness (14), the function u is the unique continuous solution. □

REMARK 9. Theorem 4 also holds for the problem

$$\begin{cases} \mathscr{L}u + a_0 u = f & \text{in } \Omega, \\ \langle \nabla u, \vec{\gamma} \rangle + c_0 \Psi(u) u = g & \text{on } \partial\Omega, \end{cases}$$

assumed that for each $\mathrm{M} > 0$

$$r \mapsto \Psi(r), \qquad |r| \leqslant \mathrm{M},$$

is a positive uniform continuous function. Here the relative Brosamler formula is given by

$$\begin{aligned} u(x) = \mathbb{E}\Big[\int_0^\infty f\big(\mathscr{X}_t^x\big) \exp\Big(- \int_0^t a_0\big(\mathscr{X}_s^x\big) ds \Big) dt \\ + \int_0^\infty g\big(\mathscr{X}_t^x\big) \exp\Big(- \int_0^t a_0\big(\mathscr{X}_s^x\big) ds \\ - \int_0^t c_0\big(\mathscr{X}_s^x\big) \Psi\big(u\big(\mathscr{X}_s^x\big)\big) d\mathrm{L}_s^x \Big) d\mathrm{L}_t^x \Big], \end{aligned}$$

for $x \in \overline{\Omega}$. Moreover, if $r\Psi(r)$ is nondecreasing the boundary operator satisfies the obliqueness

$$\mathrm{B}(x, r, p + \eta \vec{n}(x)) - \mathrm{B}(x, s, p) = \eta \langle \vec{n}(x), \vec{\gamma}(x) \rangle + c_0(x)\big(\Psi(r) r - \Psi(s) s\big).$$

Therefore the function u is, in fact, the unique solution of the boundary problem. □

5. Appendix. On a technical inequality

In [22] one proves the inequality

$$\big\langle |\theta|^{\mathrm{p}-2}\theta - |\widehat{\theta}|^{\mathrm{p}-2}\widehat{\theta}, \theta - \widehat{\theta} \big\rangle \geqslant \begin{cases} \mathrm{C_p} |\theta - \widehat{\theta}|^{\mathrm{p}}, & \text{if } 2 \leqslant \mathrm{p}, \\ \mathrm{C_p} \dfrac{|\theta - \widehat{\theta}|^2}{\big(|\theta| + |\widehat{\theta}|\big)^{2-\mathrm{p}}}, & \text{if } 1 \leqslant \mathrm{p} < 2,\ |\theta| + |\widehat{\theta}| \neq 0, \end{cases} \tag{37}$$

where $\mathrm{C_p}$ is positive constant depending on p. For the choice

$$\theta = |\xi|^{\mathrm{m}-1},\ \widehat{\theta} = |\widehat{\xi}|^{\mathrm{m}-1} \quad \text{and} \quad (\mathrm{p}-1)(\mathrm{m}-1) = 1,$$

inequality (37) becomes

$$\langle |\xi| - |\widehat{\xi}|, |\xi|^{\mathrm{m}-1} - |\widehat{\xi}|^{\mathrm{m}-1} \rangle \geqslant \begin{cases} \mathrm{C_m} \big| |\xi|^{\mathrm{m}-1} - |\widehat{\xi}|^{\mathrm{m}-1} \big|^{\frac{\mathrm{m}}{\mathrm{m}-1}}, & \text{if } \mathrm{m} \leqslant 2, \\ \mathrm{C_m} \dfrac{\big| |\xi|^{\mathrm{m}-1} - |\widehat{\xi}|^{\mathrm{m}-1} \big|^2}{\big(|\xi|^{\mathrm{m}-1} + |\widehat{\xi}|^{\mathrm{m}-1}\big)^{\frac{\mathrm{m}-2}{\mathrm{m}-1}}}, & \text{if } 2 < \mathrm{m},\ |\xi| + |\widehat{\xi}| \neq 0. \end{cases}$$

For the case $m \leqslant 2$ one has

$$\left||\xi|^{m-1} - |\widehat{\xi}|^{m-1}\right| \leqslant C_m^{-1}\left(\langle|\xi| - |\widehat{\xi}|, |\xi|^{m-1} - |\widehat{\xi}|^{m-1}\rangle\right)^{\frac{m-1}{m}}$$

$$\leqslant C_m^{-1}\left(\frac{(|\xi| - |\widehat{\xi}|)^{m-1}}{m\varepsilon^m} + \varepsilon^{\frac{m}{m-1}}\frac{m-1}{m}\left||\xi|^{m-1} - |\widehat{\xi}|^{m-1}\right|\right),$$

by using Cauchy inequality. Then for ε small enough we derives

$$\left||\xi|^{m-1} - |\widehat{\xi}|^{m-1}\right| \leqslant \begin{cases} C_m\left|\xi - \widehat{\xi}\right|^{m-1}, & \text{if } m \leqslant 2, \\ C_m\left(|\xi|^{m-1} + |\widehat{\xi}|^{m-1}\right)^{\frac{m-2}{m-1}}\left|\xi - \widehat{\xi}\right|, & \text{if } 2 < m, \end{cases} \tag{38}$$

(the case $2 < m$ follows by straightforward computations).

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(Received August 31, 2011)

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