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SOME BASIC RANDOM FIXED POINT THEOREMS WITH PPF DEPENDENCE AND FUNCTIONAL RANDOM DIFFERENTIAL EQUATIONS

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Abstract. In this paper two basic random fixed point theorems with PPF dependence are proved for random operators in separable Banach spaces with different domain and range spaces. The obtained abstract results are applied to certain nonlinear functional random differential equations for proving the existence results for random solutions with PPF dependence.

1. Introduction

The study of the random fixed point theorems in abstract spaces is initiated by Spacek [11] and Hans [7] and are the stochastic generalizations of the classical fixed point theorems in separable Banach spaces. The research along this line gained momentum after the publication of the paper by Bharucha-Reid [2] and since then several random fixed point theorems have been proved in the literature. It is worthwhile to mention that these randoms fixed point theorems are useful for proving the existence results for random solutions of nonlinear random equations in separable Banach spaces. The details of this aspect along with some nice applications to random differential equations appear in an interesting paper of Itoh [9]. A common assumption among all these random fixed point theorems is that the operators in question map an abstract space into itself, i.e. the domain and the range of the operators are same. To the best of our knowledge, there is no discussion so far concerning the random fixed point theorems for the operators with different domains and the range spaces. The classical or deterministic fixed point theorems for the operators with respect to domain and range spaces are not same studied in Bernfield *et al.* [1], [3] and Drici *et al.* [5, 6] and Dhage [3] are called fixed point theorems with PPF dependence, because they are useful for proving the existence of solutions for certain functional differential equations which may depend upon the past, present and future consideration.

In this paper we blend the above two approaches and prove two basic random fixed point theorems for the operators in separable Banach spaces with PPF dependence and apply them to some nonlinear random differential equations for proving the existence

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and uniqueness of PPF dependent random solutions. The rest of the paper is organized as follows: In the following section we present some of the basic terminologies that will be used in the subsequent development of this paper. In Section 3 we prove the basic random fixed point theorems with PPF dependence and in Section 4 we apply them to initial value problems of random functional differential equations.

2. Preliminaries

Let $(\Omega, \mathscr{A})$ be a measurable space and let E be a separable Banach space with norm $\|\cdot\|_E$. We equip the Banach space E with a σ-algebra β_E of Borel subsets of E so that (E, β_E) becomes a measurable space. A mapping $x : \Omega \to E$ is called measurable if

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathscr{A} \tag{2.1}$$

for all Borel sets $B \in \beta_E$.

Given two Banach spaces E_1 and E_2, a mapping $Q : \Omega \times E_1 \to E_2$ is called a random operator if $Q(\omega, x)$ is measurable in ω for all $x \in E_1$. We also denote a random operator Q on E_1 by $Q(\omega)x = Q(\omega, x)$. A random operator $Q(\omega)$ is called continuous on E if $Q(\omega, x)$ is continuous in x for each $\omega \in \Omega$. Similarly, Q is called compact on $\Omega \times E_1$ if $Q(\Omega \times E_1)$ is a relatively compact subset of E_2. Finally, $Q(\omega)$ is called compact on E_1 if $Q(\omega, E_1)$ is a relatively compact subset of E_2 for each $\omega \in \Omega$.

Given a closed and bounded interval $I = [a, b]$ in $\mathbb{R}$, the set of real numbers, for some $a, b \in \mathbb{R}$, $a < b$, let $E_0 = C(I, E)$ denote the Banach space of continuous E-valued functions defined on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|x\|_{E_0} = \sup_{t \in I} \|x(t)\|_E. \tag{2.2}$$

For a fixed $c \in I$, a *Razumkhin* or *Minimal* class of functions in E_0 is defined as

$$\mathscr{R}_c = \big\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\big\}. \tag{2.3}$$

The class $\mathscr{R}_c$ is algebraically closed with respect to difference if $\phi - \xi \in \mathscr{R}_c$ whenever $\phi, \xi \in \mathscr{R}_c$ Similarly, $\mathscr{R}_c$ is topologically closed if it is closed w.r.t. the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Let $Q : \Omega \times E_0 \to E$ be a random operator. A measurable function $\xi^* : \Omega \to E_0$ is called a PPF dependent random fixed point of the random operator $Q(\omega)$ if

$$Q(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$$

for some $c \in I$. Any mathematical statement that guarantees the existence of PPF dependent random fixed point of the random operator $Q(\omega)$ is a random fixed point theorem with PPF dependence or a PPF dependent random fixed point theorem.

The following theorem is often times used in the study of nonlinear discontinuous random differential equations. We also need this result in the subsequent part of this paper.

THEOREM 2.1. (Carathéodory) *Let* $Q:\Omega\times E_1\to E_2$ *be a mapping such that* $Q(\omega,x)$ *is measurable in* ω *for each* $x\in E$ *and* $Q(\omega,x)$ *is continuous in* x *for each* $\omega\in\Omega$. *Then the map* $(\omega,x)\mapsto Q(\omega,x)$ *is jointly measurable.*

In the following section we prove two basic random fixed point theorems with PPF dependence for random operators in a separable Banach space satisfying certain contraction and compactness type conditions.

3. PPF Dependent Random Fixed Point Theory

We need the following definitions in what follows.

DEFINITION 3.1. A random operator $Q:\Omega\times E_0\to E$ is called a random contraction if for each $\omega\in\Omega$,

$$\|Q(\omega,\xi)-Q(\omega,\eta)\|_E\leqslant\lambda(\omega)\,\|\xi-\eta\|_{E_0} \tag{3.1}$$

for all $\xi,\eta\in E_0$, where $\lambda:\Omega\to\mathbb{R}_+$ is a measurable function satisfying $0\leqslant\lambda(\omega)<1$ for all $\omega\in\Omega$.

DEFINITION 3.2. A random operator $Q:\Omega\times E_0\to E$ is called a strong random contraction if for a given $c\in I$ and for each $\omega\in\Omega$,

$$\|Q(\omega,\xi)-Q(\omega,\eta)\|_E\leqslant\lambda(\omega)\,\|\xi(c,\omega)-\eta(c,\omega)\|_E \tag{3.2}$$

for all $\xi,\eta\in E_0$, where $\lambda:\Omega\to\mathbb{R}_+$ is a measurable function satisfying $0\leqslant\lambda(\omega)<1$ for all $\omega\in\Omega$.

REMARK 3.1. Notice that every strong random contraction is random contraction, but the converse may not be true.

Our first random fixed point theorem with PPF dependence is the following result.

THEOREM 3.1. *Let* $(\Omega,\mathscr{A})$ *be a measurable space and let* E *be a separable Banach space. If the random operator* $Q:\Omega\times E_0\to E$ *is a random contraction, then the following statements hold in* E.

(a) *If* $\mathscr{R}_c$ *is algebraically closed with respect to difference, then for a given* $\xi_0\in E_0$ *and for a given* $c\in I$, *every sequence* $\{\xi_n(\omega)\}$ *of measurable functions satisfying*

$$Q(\omega,\xi_n(\omega))=\xi_{n+1}(c,\omega) \tag{3.3}$$

and

$$\|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{E_0}=\|\xi_n(c,\omega)-\xi_{n+1}(c,\omega)\|_E \tag{3.4}$$

converges to a PPF dependent random fixed point of the random operator $Q(\omega)$, *i.e. there is a measurable function* $\xi^*:\Omega\to E_0$ *such that for each* $\omega\in\Omega$,

$$Q(\omega,\xi^*(\omega))=Q(\omega)\xi^*(\omega)=\xi^*(c,\omega).$$

(b) *Given* $\xi_0, \eta_0 \in E_0$, *let* $\{\xi_n(\omega)\}$ *and* $\{\eta_n(\omega)\}$ *be the sequences of iterates of measurable functions corresponding to* ξ_0 *and* η_0 *constructed as in (a). Then,*

$$\|\xi_n(\omega)-\eta_n(\omega)\|_{E_0} \leqslant \frac{1}{1-\lambda(\omega)}\left[\|\xi_0-\xi_1(\omega)\|_{E_0}+\|\xi_0-\xi_1(\omega)\|_{E_0}\right]+\|\xi_0-\eta_0\|_{E_0}.$$

If, in particular, $\xi_0 = \eta_0$, *and* $\{\xi_n(\omega)\} \neq \{\eta_n(\omega)\}$, *then*

$$\|\xi_n(\omega)-\eta_n(\omega)\|_{E_0} \leqslant \frac{2}{1-\lambda(\omega)}\|\xi_0-\xi_1(\omega)\|_{E_0}.$$

(c) *Finally, if* $\mathscr{R}_c$ *is topologically closed, then for a given* $\xi_0 \in E_0$, *every sequence* $\{\xi_n(\omega)\}$ *of iterates of* $Q(\omega)$ *constructed as in (a), converges to a unique PPF dependent random fixed point* $\xi^*(\omega)$ *of* $Q(\omega)$, *i.e. there is a unique measurable function* $\xi^* : \Omega \to E_0$ *such that* $Q(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$ *for all* $\omega \in \Omega$.

Proof. Let $\xi_0 \in E_0$ be arbitrary. By hypothesis, $Q(\omega, \xi_0) \in E$. Suppose that $Q(\omega, \xi_0) = x_1(\omega)$, where the function $x_1 : \Omega \to E$ is measurable. Choose a measurable function $\xi_1 : \Omega \to E_0$ such that $x_1(\omega) = \xi_1(c, \omega)$ and that

$$\|\xi_1(c,\omega)-\xi_0(c)\|_E = \|\xi_1(\omega)-\xi_0\|_{E_0}.$$

Define a sequence $\{\xi_n(\omega)\}$ of measurable functions from Ω into E_0 inductively so that

$$Q(\omega, \xi_n(\omega)) = \xi_{n+1}(c,\omega)$$

and

$$\|\xi_{n+1}(c,\omega)-\xi_n(c,\omega)\|_E = \|\xi_{n+1}(\omega)-\xi_n(\omega)\|_{E_0}$$

for all $\omega \in \Omega$.

We claim that the sequence $\{\xi_n(\omega)\}$ of measurable functions is Cauchy in E_0. Since $Q(\omega)$ is a random contraction, we have for each $\omega \in \Omega$,

$$\begin{aligned}\|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{E_0} &= \|\xi_n(c,\omega)-\xi_{n+1}(c,\omega)\|_E \\ &= \|Q(\omega,\xi_{n-1}(\omega))-Q(\omega,\xi_n(\omega))\|_E \\ &\leqslant \lambda(\omega)\|\xi_{n-1}(c,\omega)-\xi_n(c,\omega)\|_E \\ &\leqslant \lambda(\omega)\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{E_0}\end{aligned}$$

for all $n = 1, 2, \ldots$. By induction,

$$\|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{E_0} \leqslant \lambda^n(\omega)\|\xi_0-\xi_1(\omega)\|_{E_0} \tag{3.5}$$

for all $n \in \mathbb{N}$. If $m > n$, then by triangle inequality,

$$\begin{aligned}\|\xi_n(\omega)-\xi_m(\omega)\|_{E_0} &\leqslant \|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{E_0}+\|\xi_{n+1}(\omega)-\xi_{n+2}(\omega)\|_{E_0} \\ &\quad + \cdots \|\xi_{m-1}(\omega)-\xi_m(\omega)\|_{E_0} \\ &\leqslant \left[\lambda^n(\omega)+\lambda^{n+1}(\omega)+\cdots+\lambda^{m-1}(\omega)\right]\|\xi_0-\xi_1(\omega)\|_{E_0}\end{aligned}$$

$$\leqslant \frac{\lambda^n(\omega)}{1-\lambda(\omega)}\|\xi_0 - \xi_1(\omega)\|_{E_0}$$
$$\to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $\lim_{m,n\to 0}\|\xi_n(\omega) - \xi_m(\omega)\|_{E_0} \to 0$. This shows that $\{\xi_n(\omega)\}$ is a Cauchy sequence of measurable functions on Ω into E_0. Since E_0 is complete and separable Banach space, there is a measurable function $\xi^* : \Omega \to E_0$ such that $\lim_{n\to\infty}\xi_n(\omega) = \xi^*(\omega)$ for all $\omega \in \Omega$. Now, from (2.2) it follows that $\{\xi_n(c,\omega)\}$ is Cauchy and hence convergent in view of completeness of E. From continuity of the random operator $Q(\omega)$ it follows that

$$\begin{aligned}
Q(\omega,\xi^*(\omega)) &= Q\big(\omega, \lim_{n\to\infty}\xi_n(\omega)\big) \\
&= \lim_{n\to\infty} Q(\omega,\xi_n(\omega)) \\
&= \lim_{n\to\infty} \xi_{n+1}(c,\omega) \\
&= \xi^*(c,\omega)
\end{aligned}$$

for all $\omega \in \Omega$. Hence ξ^* is a random fixed point with PPF dependence of the random operator $Q(\omega)$ on E_0.

(b) Now, if $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ are any two sequences of measurable functions as constructed in (a). Then for each $\omega \in \Omega$,

$$\begin{aligned}
\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} &\leqslant \|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{E_0} + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0} \\
&\quad + \|\eta_{n-1}(\omega) - \eta_n(\omega)\|_{E_0} \\
&\leqslant \lambda^{n-1}(\omega)\big[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}\big] \\
&\quad + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0}.
\end{aligned}$$

for each n, $n = 1, 2, \ldots$. Therefore, by induction,

$$\begin{aligned}
\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} &\leqslant \lambda^{n-1}(\omega)\big[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}\big] \qquad (3.6)\\
&\quad + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0} \\
&\leqslant \big[\lambda^{n-1}(\omega) + \lambda^{n-2}(\omega)\big]\,\big[\|\xi_0 - \xi_1(\omega)\|_{E_0} \\
&\qquad\qquad + \|\eta_0 - \eta_1(\omega)\|_{E_0}\big] \qquad (3.7)\\
&\quad + \|\xi_{n-1}(\omega) - \eta_{n-1}(\omega)\|_{E_0} + \|\xi_{n-2}(\omega) - \eta_{n-2}(\omega)\|_{E_0} \\
&\leqslant \big[\lambda^{n-1}(\omega) + \lambda^{n-1}(\omega) + \cdots + 1\big]\big[\|\xi_0 - \xi_1(\omega)\|_{E_0} \\
&\qquad\qquad + \|\eta_0 - \eta_1(\omega)\|_{E_0}\big] \\
&\quad + \|\xi_0 - \eta_0\|_{E_0} \\
&\leqslant \frac{1}{1-\lambda(\omega)}\big[\|\xi_0 - \xi_1(\omega)\|_{E_0} + \|\eta_0 - \eta_1(\omega)\|_{E_0}\big] \\
&\quad + \|\xi_0 - \eta_0\|_{E_0}. \qquad (3.8)
\end{aligned}$$

In particular if, $\xi_0 = \eta_0$, then $Q(\omega, \xi_0) = Q(\omega, \eta_0)$ which implies that $\xi_1(c,\omega) = \eta_1(c,\omega)$. Hence, from (3.6)

$$\|\xi_n(\omega) - \eta_n(\omega)\|_{E_0} \leqslant \frac{2}{1-\lambda(\omega)} \|\xi_0 - \xi_1(\omega)\|_{E_0}.$$

(c) By part (a) above, the sequence $\{\xi_n(\omega)\}$ of measurable functions as constructed in (a) converges to a random fixed point $\xi^*(\omega)$ with PPF dependence. As $\mathscr{R}_c$ is topologically closed, $\xi^*(\omega) \in \mathscr{R}_c$. Suppose that $\eta^*(\omega) \neq \xi^*(\omega)$, $\omega \in \Omega$, be another random fixed point of $Q(\omega)$ in $\mathscr{R}_c$ with PPF dependence. Then,

$$\begin{aligned}\|\xi^*(\omega) - \eta^*(\omega)\|_{E_0} &= \|\xi^*(c,\omega) - \eta^*(c,\omega)\|_E \\ &= \|Q(\omega, \xi^*(\omega)) - Q(\omega, \eta^*(\omega))\|_E \\ &\leqslant \lambda(\omega) \|\xi^*(\omega) - \eta^*(\omega)\|_{E_0}\end{aligned}$$

which is a contradiction since $0 \leqslant \lambda(\omega) < 1$ for all $\omega \in \Omega$. This completes the proof.

REMARK 3.2. If the Razumikhin class $\mathscr{R}_c$ of functions in E_0 is not topologically closed, then the sequence $\{\xi_n(\omega)\}$ of measurable functions as constructed in hypothesis (a) of Theorem 3.1 may converge to a random fixed point with PPF dependence of the random operator $Q(\omega)$ outside the set $\mathscr{R}_c$ which may not be unique.

Next, we prove a hybrid fixed point theorem with PPF dependence for the random operators satisfying mixed Lipschitz and compactness conditions in separable Banach spaces.

THEOREM 3.2. *Let $(\Omega, \mathscr{A})$ be a measurable space and let E be a separable Banach space. Suppose that $A : \Omega \times E_0 \to E$ and $B : \Omega \times E \to E$ are two continuous random operators satisfying for each $\omega \in \Omega$,*

(a) *$A(\omega)$ is strong random contraction, and*

(b) *B is compact is compact on $\Omega \times E$.*

If $\mathscr{R}_c$ is a topologically and algebraically closed with respect to difference, then for a given $c \in I$, the random operator equation

$$A(\omega, \xi(\omega)) + B(\omega, \xi(\omega, c)) = \xi(\omega, c) \tag{3.9}$$

has a random solution with PPF dependence, i.e. for a given $c \in I$, there is a measurable function $\xi^* : \Omega \to E_0$ such that

$$A(\omega, \xi^*(\omega)) + B(\omega, \xi^*(c,\omega)) = \xi^*(c,\omega).$$

for all $\omega \in \Omega$.

Proof. Let $\eta : \Omega \to E$ be a fixed measurable function and define a mapping $T_{\eta(\omega)} : \Omega \times E_0 \to E$ by

$$T_{\eta(\omega)}(\omega, \xi(\omega)) = A(\omega, \xi(\omega)) + B(\omega, \xi(\omega, c)). \tag{3.10}$$

If $\xi_1, \xi_2 : \Omega \to E_0$ are measurable, then

$$\begin{aligned}\|T_{\eta(\omega)}(\omega, \xi_1(\omega)) - T_{\eta(\omega)}(\omega, \xi_2(\omega))\|_E &= \|A(\omega, \xi_1(\omega)) - A(\omega, \xi_2(\omega))\|_E \\ &\leqslant \lambda(\omega)\|\xi_1(c,\omega) - \xi_2(c,\omega)\|_{E_0} \\ &\leqslant \lambda(\omega)\|\xi_1(\omega) - \xi_2(\omega)\|_{E_0}\end{aligned}$$

for all $\omega \in \Omega$, where $\lambda : \Omega \to \mathbb{R}_+$ be a measurable function such that $0 \leqslant \lambda(\omega) < 1$. This shows that $Q(\omega)$ is a random contraction on E_0. Hence, by Theorem 3.1, $T_{\eta(\omega)}(\omega)$ has a unique PPF dependent random fixed point, that is, there is a unique measurable function $\xi^* : \Omega \to E_0$ such that

$$T_{\eta(\omega)}(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$$

for all $\omega \in \Omega$. As a result,

$$A(\omega, \xi^*(\omega)) + B(\omega, \eta(\omega)) = \xi^*(c, \omega). \tag{3.11}$$

Define a random mapping $Q : \Omega \times E \to E$ by

$$Q(\omega, \eta(\omega)) = \xi^*(c, \omega) \tag{3.12}$$

for all $\omega \in \Omega$. If $\eta_1, \eta_2 : \Omega \to E$ be two measurable functions, then there are measurable functions $\xi_1^*, \xi_2^* : \Omega \to E_0$ such that

$$Q(\omega, \eta_1(\omega)) = \xi_1^*(c, \omega) \quad \text{and} \quad Q(\omega, \eta_2(\omega)) = \xi_2^*(c, \omega)$$

for all $\omega \in \Omega$. Therefore, for each $\omega \in \Omega$,

$$\begin{aligned}&\|Q(\omega, \eta_1(\omega)) - Q(\omega, \eta_2(\omega))\|_E \\ &\qquad \leqslant \|A(\omega, \xi_1^*(\omega)) - A(\omega, \xi_2^*(\omega))\|_E \\ &\qquad\qquad + \|B(\omega, \eta_1(\omega)) - B(\omega, \eta_2(\omega))\|_E \\ &\qquad \leqslant \lambda(\omega)\|\xi_1^*(\omega) - \xi_1^*(\omega)\|_{E_0} + \|B(\omega, \eta_1(\omega)) - B(\omega, \eta_2(\omega))\|_E \\ &\qquad \leqslant \lambda(\omega)\|Q(\omega, \eta_1(\omega)) - Q(\omega, \eta_2(\omega))\|_E \\ &\qquad\qquad + \|B(\omega, \eta_1(\omega)) - B(\omega, \eta_2(\omega))\|_E.\end{aligned}$$

Hence,

$$\begin{aligned}&\|Q(\omega, \eta_1(\omega)) - Q(\omega, \eta_2(\omega))\|_E \\ &\qquad \leqslant \left[1 - \lambda(\omega)\right]^{-1}\|B(\omega, \eta_1(\omega)) - B(\omega, \eta_2(\omega))\|_E.\end{aligned} \tag{3.13}$$

Since B is compact on $\Omega \times E$, for each $\omega \in \Omega$, the sequence $\{B(\omega, \eta_n(\omega))\}$ has a convergent subsequence for any sequence $\{\eta_n(\omega)\}$ in E. Without loss of generality, we call the subsequence to be same sequence. As a result, $\{B(\omega, \eta_n(\omega))\}$ is Cauchy sequence. Hence from (3.13) it follows that $\{Q(\omega, \eta_n(\omega))\}$ is also a Cauchy sequence in E. Since E is complete, $\{Q(\omega, \eta_n(\omega))\}$ and consequently, every subsequence of it is convergent. Therefore, it follows that $Q:\Omega\times E\to E$ is a compact and continuous random operator. Set $S=\overline{\text{conv}}(Q(\Omega\times E))$. Now we apply random version of Schauder fixed point principle (see Himmelberg [8] and Itoh [9]) to the operator $Q:\Omega\times S\to S$ to yield that $Q(\omega)$ has a random fixed point, that is, there is a measurable mapping $\xi^*:\Omega\to E_0$ such that

$$Q(\omega,\xi^*(c,\omega))=\xi^*(c,\omega)$$

or, equivalently,

$$A(\omega,\xi^*(\omega))+B(\omega,\xi^*(c,\omega))=\xi^*(c,\omega).$$

This completes the proof.

4. Functional Random Differential Equations

In this section, we apply the abstract results of the previous section to initial value problems (IVP) of the functional random differential equations for proving the existence of PPF dependent random solutions under some Lipschitz and compactness type conditions.

Given the closed and bounded intervals $I_0=[-r,0]$ and $I=[0,T]$ in $\mathbb{R}$, the set of real numbers, for some real numbers $r>0$, $T>0$, let $\mathscr{C}$ denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathscr{C}$ with the supremum norm $\|\cdot\|_{\mathscr{C}}$ defined by

$$\|\xi\|_{\mathscr{C}}=\sup_{\theta\in I_0}|\xi(\theta)|. \tag{4.1}$$

It is clear that $\mathscr{C}$ is a Banach space with this norm called the history space of the problem under consideration.

For each $t\in I=[0,T]$, define a function $t\to x_t\in\mathscr{C}$ by

$$x_t(\theta)=x(t+\theta),\ \theta\in I_0, \tag{4.2}$$

where the argument θ represents the delay in the argument of solutions.

Now we are equipped with the necessary details to study the nonlinear problems of functional random differential equations for existence and uniqueness results.

4.1. IVP of functional random differential equations

Let $(\Omega,\mathscr{A})$ be a measurable space. By a mapping $x:\Omega\to C(J,\mathbb{R})$ we denote a function $x(t,\omega)$ which is continuous in the variable t for each $\omega\in\Omega$. In this case, we also write $x(t,\omega)=x(\omega)(t)$.

Given the measurable functions $\phi:\Omega\to\mathscr{C}$ and $x:\Omega\to C(I,\mathbb{R})$, consider an initial value problem of functional random differential equations of delay type (in short FRDE),

$$\left.\begin{aligned} x'(t,\omega)&=f(t,x_t(\omega),\omega)\\ x_0(\omega)&=\phi(\omega)\end{aligned}\right\} \tag{4.3}$$

for all $t \in I$ and $\omega \in \Omega$, where $f : I \times \mathscr{C} \times \Omega \to \mathbb{R}$.

By a random solution x of the FRDE (4.3) we mean a measurable function $x : \Omega \to C(J,\mathbb{R})$ that satisfies the equations in (4.3) on J, where $C(J,\mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

The functional random differential equation (4.3) is not new to the theory of nonlinear functional random differential equations and the existence and uniqueness theorems for FRDE (4.3) are obtained by using the random version of classical fixed point theorems of Schauder and Banach respectively. However, the novelty of the present paper lies in the nice applicability of our Theorem 2.1 for proving the existence of random solutions with PPF dependence for the FRDE (4.3) defined on J.

We consider the following hypotheses in what follows.

(H_1) The function $\omega \mapsto f(t,x,\omega)$ is measurable for each $t \in I$ and $x \in \mathscr{C}$ and the function $(t,x) \mapsto f(t,x,\omega)$ is jointly continuous for each $\omega \in \Omega$.

(H_2) There exists a real number $M_f > 0$ such that for each $\omega \in \Omega$,

$$|f(t,x,\omega)| \leqslant M_f$$

for all $t \in I$ and $x \in \mathscr{C}$.

(H_3) There exists real number $L > 0$ such that for each $\omega \in \Omega$,

$$|f(t,x,\omega) - f(t,y,\omega)| \leqslant L\|x-y\|_{\mathscr{C}}$$

for all $t \in I$ and $x, y \in \mathscr{C}$.

THEOREM 4.1. *Assume that the hypotheses (H_1) through (H_3) hold. Furthermore, if* $LT < 1$*, then the FRDE* (4.3) *has a unique PPF dependent random solution defined on* J.

Proof. Set $E = C(J,\mathbb{R})$. Then E is a Banach space with respect to the usual supremum norm $\|\cdot\|_E$ defined by

$$\|x\|_E = \sup_{t\in J} |x(t)|. \tag{4.4}$$

Clearly, E is a separable Banach space. Given a function $x \in C(J,\mathbb{R})$, define a mapping $\hat{x} : I \to \mathscr{C}$ by $\hat{x}(t) = x_t \in \mathscr{C}$ so that $\hat{x}(t)(0) = x_t(0) = x(t)$, $t \in I$ and $\hat{x}(0) = x_0$.

Define a set $\widehat{E}$ of functions by

$$\widehat{E} = \big\{\hat{x} = (x_t)_{t\in I} : x_t \in \mathscr{C}, x \in C(I,\mathbb{R}) \text{ and } x_0 = \phi\big\}. \tag{4.5}$$

Define a norm $\|\hat{x}\|_{\widehat{E}}$ in $\widehat{E}$ by

$$\|\hat{x}\|_{\widehat{E}} = \sup_{t\in I} \|x_t\|_{\mathscr{C}}. \tag{4.6}$$

Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = \mathscr{C}$. Next we show that $\widehat{E}$ is a Banach space. Consider a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{E}$. For simplicity of notations, we denote $\hat{x}_n(t) = x_t^n$. Then, $\{(x_t^n)_{t\in I}\}$

is a Cauchy sequence in $\mathscr{C}$ for each $t \in I$. This further implies that $\{x_t^m(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\{x_t^m(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is complete. Moreover, $\widehat{E}$ is a separable Banach space.

Now the FRDE (4.3) is equivalent to the nonlinear functional random integral equation (in short FRIE)

$$x(t,\omega) = \begin{cases} \phi(0,\omega) + \int_0^t f(s, x_s(\omega), \omega)\, ds, & \text{if } t \in I, \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \tag{4.7}$$

Given a measurable function $\hat{x} : \Omega \to \widehat{E}$, consider the operator $Q : \Omega \times \widehat{E} \to \mathbb{R}$ defined by

$$\begin{aligned} Q(\omega, \hat{x}(\omega)) &= Q(\omega, x_t(\omega)) \\ &= \begin{cases} \phi(0,\omega) + \int_0^t f(s, x_s(\omega), \omega)\, ds, & \text{if } t \in I, \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \end{aligned} \tag{4.8}$$

Then the FRIE (4.5) is equivalent to the random operator equation

$$Q(\omega, \hat{x}(\omega)) = \hat{x}(0,\omega) = \hat{x}(\omega)(0). \tag{4.9}$$

Define a sequence $\{\hat{x}_n(\omega)\}$ of measurable functions by

$$\left.\begin{aligned} &(i)\ Q(\omega, \hat{x}_n(\omega)) = \hat{x}_{n+1}(\omega)(0), \\ &(ii)\ \|\hat{x}_n(\omega) - \hat{x}_{n+1}(\omega)\|_{E_0} = \|\hat{x}_n(\omega)(0) - \hat{x}_{n+1}(\omega)(0)\|_E \end{aligned}\right\} \tag{4.10}$$

for $n = 1, 2, \ldots$.

We shall show that the operator Q satisfies condition (a) of Theorem 3.1 on $\Omega \times \widehat{E}$. Firstly, we show that Q is a random operator on $\Omega \times \widehat{E}$. Since hypothesis ($H_1$) holds, by Carathéodory theorem, the function $\omega \to f(t, x, \omega)$ is measurable for all $t \in I$ and $x \in \mathscr{C}$. As integral is the limit of the finite sum of measurable functions, the map

$$\omega \mapsto \int_0^t f(s, x_s(\omega), \omega)\, ds$$

is measurable. Again, sum of two measurable functions is measurable, so the map

$$\omega \mapsto \phi(0,\omega) + \int_0^t f(s, x_s(\omega), \omega)\, ds$$

is measurable. Hence, the operator $Q(\omega, \hat{x})$ is measurable in ω for each $\hat{x} \in \widehat{E}$. As a result, $Q(\omega)$ is a random operator on $\widehat{E}$ into E.

Secondly, we show that random operator $Q(\omega)$ is continuous on $\widehat{E}$. Let $\omega \in \Omega$ be fixed. We show that the continuity of the random operator $Q(\omega)$ in the following two cases:

Case I: Let $t \in [0,T]$ and let $\{\hat{x}_n(\omega)\}$ be a sequence of points in $\widehat{E}$ such that $\hat{x}_n(\omega) \to \hat{x}(\omega)$ as $n \to \infty$. Then, by dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} Q(\omega,\hat{x}_n(t,\omega)) &= \lim_{n\to\infty}\left(\phi(0,\omega)+\int_0^t f(s,x_s^n(\omega),\omega)\,ds\right)\\ &= \phi(0,\omega)+\lim_{n\to\infty}\left(\int_0^t f(s,x_s^n(\omega),\omega)\,ds\right)\\ &= \phi(0,\omega)+\left(\int_0^t \lim_{n\to\infty} f(s,x_s^n(\omega),\omega)\,ds\right)\\ &= \phi(0,\omega)+\int_0^t f(s,x_s(\omega),\omega)\,ds\\ &= Q(\omega,\hat{x}(t,\omega))\end{aligned}$$

for all $t \in [0,T]$ and for each fixed $\omega \in \Omega$.

Case II: Suppose that $t \in [-r,0]$. Then we have:

$$|Q(\omega,\hat{x}_n(\omega)) - Q(\omega,\hat{x}(\omega))| = |\phi(t,\omega)-\phi(t,\omega)| = 0$$

for each fixed $\omega \in \Omega$. Hence,

$$\lim_{n\to\infty} Q(\omega)\hat{x}_n(t,\omega) = Q(\omega)\hat{x}(t,\omega)$$

for all $t \in [-r,0]$ and $\omega \in \Omega$. Now combining the Case I with Case II, we conclude that $Q(\omega)$ is a pointwise continuous random operator on $\widehat{E}$ into itself.

Now we show that the family of functions $\{Q(\omega,\hat{x}_n(\omega)\}$ is a uniformly continuous set in E for a fixed $\omega \in \Omega$. We consider the following three cases:

Case I: Let $\varepsilon > 0$ and let $t_1,t_2 \in [0,T]$ be arbitrary. Then, we have

$$\begin{aligned}|Q(\omega,x_{t_1}^n(\omega)) - Q(\omega,x_{t_2}^n(\omega))| &\leqslant \left|\int_0^{t_1} f(s,x_s^n(\omega),\omega)\,ds - \int_0^{t_2} f(s,x_s^n(\omega),\omega)\,ds\right|\\ &\leqslant \left|\int_{t_2}^{t_1} |f(s,x_s^n(\omega),\omega)|\,ds\right|\\ &\leqslant M_f\,|t_1-t_2|.\end{aligned}$$

Choose $\delta_1 = \dfrac{\varepsilon}{2(M_f+1)} > 0$. Then, if $|t_1-t_2| < \delta_1$ implies

$$|Q(\omega,x_{t_1}^n(\omega)) - Q(\omega,x_{t_2}^n(\omega))| < \frac{M_f\,\varepsilon}{2(M_f+1)}$$

uniformly for $x_t^n = \hat{x}_n \in E_0$.

Case II: Let $t_1,t_2 \in [-r,0]$ be arbitrary. Since $t \mapsto \phi(\omega,t)$, is continuous on a compact $[-r,0]$, it is uniformly continuous there. Hence, for above $\varepsilon > 0$ there exists a $\delta_2 > 0$ such that $|t_1-t_2| < \delta_1$ implies

$$|Q(\omega,x_{t_1}^n(\omega)) - Q(\omega,x_{t_2}^n(\omega))| = |\phi(t_1,\omega)-\phi(t_2,\omega)| \leqslant \frac{\varepsilon}{2(M_f+1)}$$

uniformly for $\hat{x}_n \in E_0$.

Case III: Let $t_1 \in [-r,0]$ and $t_2 \in [0,T]$ be arbitrary. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, $|t_1 - t_2| < \delta$ implies

$$\begin{aligned}|Q(\omega,x_{t_1}^n(\omega)) - Q(\omega,x_{t_2}^n(\omega))| &\leqslant |Q(\omega,x_{t_1}^n(\omega)) - Q(\omega,x_0^n(\omega))| \\ &\quad + |Q(\omega,x_0^n(\omega)) - Q(\omega,x_{t_2}^n(\omega))| \\ &< \frac{M_f\varepsilon}{2(M_f+1)} + \frac{\varepsilon}{2(M_f+1)} = \varepsilon\end{aligned}$$

uniformly for $\hat{x}_n \in E_0$.

Thus, in all three cases, $|t_1 - t_2| < \delta$ implies

$$|Q(\omega,x_{t_1}^n(\omega)) - Q(\omega,x_{t_2}^n(\omega))| < \varepsilon$$

uniformly for all $t_1,t_2 \in J$ and $\hat{x}_n \in E_0$. This shows that $\{Q(\omega,\hat{x}_n)\}$ is a sequence of uniformly continuous functions on J. Hence, it converges uniformly on J. Hence, $Q(\omega,\hat{x})$ is a continuous random operator on $\widehat{E}$ for a fixed $\omega \in \Omega$.

Finally, we show that Q is random contraction on $\widehat{E}$. Let $\omega \in \Omega$ be fixed. Then,

$$\begin{aligned}\|Q(\omega,\hat{x}(\omega)) - Q(\omega,\hat{y}(\omega))\|_E &= \|Q\big(\omega,x_t(\omega)\big) - Q\big(\omega,y_t(\omega)\big)\|_E \\ &= \sup_{t\in I}\left|\int_0^t f(s,x_s(\omega),\omega)\,ds - \int_0^t f(s,y_s(\omega),\omega)\,ds\right| \\ &\leqslant \int_0^T L\|x_s(\omega) - y_s(\omega)\|_{\mathscr{C}}\,ds \\ &\leqslant \int_0^T L\|\hat{x}(\omega) - \hat{y}(\omega)\|_{\widehat{E}}\,ds \\ &\leqslant LT\|\hat{x}(\omega) - \hat{y}(\omega)\|_{\widehat{E}} \qquad (4.11)\end{aligned}$$

for all $\hat{x}(\omega),\hat{y}(\omega) \in \widehat{E}$. Hence, Q is a random contraction on $\widehat{E}$ with contraction constant $\alpha = LT < 1$.

Thus, the condition (a) of Theorem 3.1 is satisfied. Hence, an application of Theorem 3.1(a) yields that the functional random integral equation (4.9) has a random solution with PPF dependence defined on J. This further implies that the FRDE (4.3) has a PPF dependent random solution ξ^* defined on J and the sequence $\{\xi_n(\omega)\}$ of measurable functions constructed as in (4.10) converges to ξ^*. Moreover, here the Razumikhin class $\mathscr{R}_0$, $0 \in [-r,T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference, then by Theorem 3.1(c), ξ^* is a unique random solution with PPF dependence for the the FRDE (4.3) defined on J. This completes the proof.

4.2. IVP of hybrid random differential equations

Given the functions $\phi : \Omega \to \mathscr{C}$ and $x : \Omega \to C(I,\mathbb{R})$, consider an initial value problem of functional random differential equations of delay type (in short FRDE),

$$\left.\begin{aligned}x'(t,\omega) &= f(t,x_t(\omega),\omega) + g(t,x(t,\omega),\omega) \\ x_0(\omega) &= \phi(\omega)\end{aligned}\right\} \qquad (4.12)$$

for all $t \in I$ and $\omega \in \Omega$, where $f : I \times \mathscr{C} \times \Omega \to \mathbb{R}$ and $g : I \times \mathbb{R} \times \Omega \to \mathbb{R}$.

By a random solution x of the FRDE (4.12) we mean a measurable function $x : \Omega \to C(J, \mathbb{R})$ that satisfies the equations in (4.12) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

The functional random differential equation (4.12) is not new to the theory of nonlinear functional differential equations and the details of the classifications of different types of nonlinear differential equations appear in Dhage [4]. The existence theorems for the FRDE (4.12) are generally proved by using the hybrid fixed point theorems of Krasnoselskii and Dhage type. See for example, Krasnoselskii [10], Dhage [4] and the references given therein. In the following we prove an existence of PPF dependent random solutions for the FRDE (4.12) defined on J.

We consider the following hypothesis in what follows.

(H_4) There exists real number $L > 0$ such that

$$|f(t,x,\omega) - f(t,y,\omega)| \leqslant L\|x(0) - y(0)\|_{\mathscr{C}}$$

for all $t \in I$ and $x, y \in \mathscr{C}$.

(H_5) The function $\omega \mapsto g(t,x,\omega)$ is measurable for each $t \in I$ and $x \in \mathbb{R}$ and the function $(t,x) \mapsto g(t,x,\omega)$ is jointly continuous for each $\omega \in \Omega$.

(H_6) There exists a real number $M_g > 0$ such that for each $\omega \in \Omega$,

$$|g(t,x,\omega)| \leqslant M_g$$

for all $t \in I$ and $x \in \mathbb{R}$.

THEOREM 4.2. *Assume that the hypotheses (H_1)-(H_2) and (H_4) through (H_6) hold. Furthermore, if $LT < 1$, then the FRDE* (4.12) *has a PPF dependent random solution defined on J.*

Proof. Now the FRDE (4.12) is equivalent to the nonlinear functional random integral equation (in short FRIE)

$$x(t,\omega) = \begin{cases} \phi(0,\omega) + \int_0^t f(s,x_s(\omega),\omega)\,ds + \int_0^t g(s,x(s,\omega),\omega)\,ds, & \text{if } t \in I, \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \tag{4.13}$$

Define two separable Banach spaces E and $E_0 = \widehat{E}$ as in the proof of Theorem 4.1. Given a measurable function $\hat{x} : \Omega \to \widehat{E}$, consider the operators $A : \Omega \times \widehat{E} \to \mathbb{R}$ and $B : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} A(\omega, \hat{x}(t,\omega)) &= A(\omega, x_t(\omega)) \\ &= \begin{cases} \phi(0,\omega) + \int_0^t f(s,x_s(\omega),\omega)\,ds, & \text{if } t \in I, \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \end{aligned} \tag{4.14}$$

and

$$B(\omega,x(t,\omega)) = \begin{cases} \int_0^t g(s,x(s,\omega),\omega)\,ds, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{4.15}$$

Then the FRIE (4.13) is equivalent to the operator equation

$$A(\omega,\hat{x}(\omega)) + B(\omega,\hat{x}(0,\omega)) = \hat{x}(0,\omega) = \hat{x}(\omega)(0). \tag{4.16}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 3.2. It can be shown on the similar lines as in the proof of Theorem 4.1 that $A(\omega)$ and $B(\omega)$ are continuous random operators on $\widehat{E}$ and E respectively. Next, we prove that $A(\omega)$ is a strong contraction random operator on E_0. Let $\omega \in \Omega$ be fixed. Then,

$$\begin{aligned} \|A(\omega,\hat{x}(\omega)) - A(\omega,\hat{y}(\omega))\|_E &= \|A\big(\omega,x_t(\omega)\big) - A\big(\omega,y_t(\omega)\big)\|_E \\ &\leqslant \sup_{t\in I} \left| \int_0^t f(s,x_s(\omega),\omega)\,ds - \int_0^t f(s,y_s(\omega),\omega)\,ds \right| \\ &\leqslant \int_0^T L\|x_s(\omega)(0) - y_s(\omega)(0)\|_{\mathscr{C}}\,ds \\ &\leqslant \int_0^T L\|\hat{x}(\omega)(0) - \hat{y}(\omega)(0)\|_E\,ds \\ &\leqslant LT\,\|\hat{x}(\omega)(0) - \hat{y}(\omega)(0)\|_E \end{aligned} \tag{4.17}$$

for all $\hat{x}(\omega),\hat{y}(\omega) \in \widehat{E}$. Hence, A is a strong random contraction on $\widehat{E}$ with contraction constant $\alpha = LT < 1$. Next, we show that $B(\omega)$ is a compact random operator on E. Let $\{x_n(\omega)\}$ be a sequence of measurable functions on Ω into E. To finish, it is enough to show that $\{B(\omega,x_n(\omega))\}$ has a convergent subsequence. Now, using the standard arguments, it is shown that $\{B(\omega,x_n(\omega))\}$ is a uniformly bounded and equicntinuous set in E. Therefore, we apply Arzelá-Ascoli theorem and conclude that B is a compact random operator on $\Omega \times E$ into E. Again, the Razumikhin class $\mathscr{R}_0$, $0 \in [-r,T]$ is $C([0,T],\mathbb{R})$ which is topologically and algebraically closed with respect to difference. Thus, $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Theorem 3.2. Hence, the FRIE (4.5) and consequently FRDE (4.3) has a random solution with PPF dependence defined on J. This completes the proof.

5. Conclusion

Finally, we conclude this paper with the remark that the random fixed point theorems with PPF dependence proved here are very fundamental in the random fixed point theory involving geometric hypothesis of distance between the images and objects in question. However, using the principle that has been formulated in Theorems 3.1 and 3.2 several random fixed point theorems with PPF dependence can be proved in a separable Banach space which would be useful to study different types of the functional random differential equations those mentioned in Dhage [4]. In a forthcoming paper, we plan to prove some PPF dependent random fixed point theorems for the random operators satisfying certain generalized contraction conditions in separable Banach algebras and apply them to some random differential equations different from those considered in this paper.

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