Differential
Equations
& Applications
Volume 4, Number 2 (2012), 197–219

EXISTENCE OF POSITIVE SOLUTIONS FOR SECOND ORDER MULTI–POINT BOUNDARY VALUE PROBLEMS

JAMES S. W. WONG

(*Communicated by S. K. Ntouyas*)

Abstract. We study the second order nonlinear differential equation:

$$(E) \quad y'' + h(t)f(y) = 0,\ 0 < t < 1,\ h(t) \in L^1(0,1),\ f(y) \in C(\mathbb{R}, \mathbb{R}_+)$$

subject to multi-point boundary conditions:

(i) $y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i),\ y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i)$,

(ii) $y'(0) = \sum_{i=1}^{m} \alpha_i y'(\xi_i),\ y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i)$,

(iii) $y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i),\ y'(1) = \sum_{i=1}^{m} \beta_i y'(\xi_i)$,

where $0 < \xi_1 < \cdots < \xi_m < 1, \alpha_i \geqslant 0, \beta_i \geqslant 0, i = 1, 2, \cdots, m$. We also assume that $h(t)$ is non-negative and can be singular at $t = 0$ or $t = 1$ or both and $\alpha_1 + \cdots + \alpha_m < 1$, $\beta_1 + \cdots + \beta_m < 1$. We prove existence theorems for positive solutions of (E) (i), (E) (ii) and (E) (iii) when the limit $f(y)/y$ as $y \to 0$ and $y \to \infty$ do not necessarily exist. Our results extend recent results of Zhang and Sun [18] for the boundary value problem (E)(i) when $\alpha_i \equiv 0$ for all $i = 1, \ldots, m$.

1. Introduction

We are interested in the existence of positive solutions to the second order differential equation

$$y'' + h(t)f(y) = 0, \qquad 0 < t < 1, \tag{1.1}$$

where $h(t) \in L^1(0,1)$ and $f(y) \in C\big(\mathbb{R}, \mathbb{R}_+\big)$ are non-negative functions subject to multi-point boundary condition of the following kind:

$$y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i) = \langle \alpha, y(\xi) \rangle; \ \ y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) = \langle \beta, y(\xi) \rangle, \tag{1.2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ and α_i, β_i are non-negative real numbers. Here α, β denote m-vectors $\alpha = (\alpha_1, \cdots, \alpha_m), \beta = (\beta_1, \cdots, \beta_m)$ and $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \cdots + \alpha_m\beta_m$ denotes the usual scalar product between 2 m-vectors. Likewise, $y(\xi)$ is a m-vector given by $(y(\xi_1), \cdots, y(\xi_m))$ where the components $y(\xi_i)$ is the value of $y(t)$ at $t = \xi_i$.

Mathematics subject classification (2010): 34B10, 34B15.
Keywords and phrases: boundary value problems, second order, multi-point boundary conditions.

Paper DEA-04-12

In addition to the m-point boundary condition (1.2), we also consider following 2 types of multi-point boundary conditions:

$$y'(0) = \sum_{i=1}^{m} \alpha_i y'(\xi_i) = \langle \alpha, y'(\xi) \rangle; \; y(1) = \sum_{i=1}^{m} \beta_i y(\xi_i) = \langle \beta, y(\xi) \rangle, \tag{1.3}$$

$$y(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i) = \langle \alpha, y(\xi) \rangle; \; y'(1) = \sum_{i=1}^{m} \beta_i y'(\xi_i) = \langle \beta, y'(\xi) \rangle. \tag{1.4}$$

We refer to (1.1), (1.2); (1.1), (1.3); (1.1), (1.4) as (BVP1), (BVP2), (BVP3) respectively. Furthermore, we denote

$$\overline{\alpha} = \sum_{i=1}^{m} \alpha_i = \langle \alpha, 1 \rangle \quad \text{and} \quad \overline{\beta} = \sum_{i=1}^{m} \beta_i = \langle \beta, 1 \rangle$$

for short and assume throughout this paper:

(H_0) $\alpha_i, \beta_i \geqslant 0$ for $i = 1, 2, \cdots, m$, $0 \leqslant \overline{\alpha} < 1$ and $0 \leqslant \overline{\beta} < 1$.

In an earlier paper with Kong [12], we present existence theorems for at least one positive solutions for (BVP_j), $j = 1, 2, 3$, in terms of the behavior of the nonlinear function $f(y)$ near $y = 0$ and $y = \infty$ when compared with the smallest eigenvalue of the associated linear problem, i.e. when $f(y) \equiv y$ in (1.1), subject to multi-point boundary conditions (1.2), (1.3) and (1.4). More specifically, we prove

THEOREM A. (Kong and Wong [12]) Assume either $f_\infty < \lambda_j < f_0$ or $f_0 < \lambda_j < f_\infty$, $j = 1, 2, 3$, where

$$f_0 = \lim_{y \to 0} f(y)/y, f_\infty = \lim_{y \to \infty} f(y)/y, \tag{1.5}$$

and λ_j is the smallest positive eigenvalue of

$$u'' + \lambda h(t) u = 0, \tag{1.6}$$

subject to boundary conditions $(\mathrm{BC})_j (j = 1, 2, 3)$ below:

$$\begin{aligned} &(\mathrm{BC})_1 \qquad u(0) = \langle \alpha, u(\xi) \rangle, \;\; u(1) = \langle \beta, u(\xi) \rangle; \\ &(\mathrm{BC})_2 \qquad u'(0) = \langle \alpha, u'(\xi) \rangle, \; u(1) = \langle \beta, u(\xi) \rangle; \\ &(\mathrm{BC})_3 \qquad u(0) = \langle \alpha, u(\xi) \rangle, \;\; u'(1) = \langle \beta, u'(\xi) \rangle. \end{aligned}$$

Then (BVP1), (BVP2), (BVP3) have at least one positive solution. □

Assumptions $f_\infty < \lambda_j < f_0$ and $f_0 < \lambda_j < f_\infty$ are referred to as sublinear and superlinear conditions motivated by the Emden-Fowler equation

$$y'' + h(t)|y|^{\gamma - 1} y = 0, \gamma > 0. \tag{1.7}$$

We refer the reader to [5], [6], [7], [10] on earlier results on the two-point boundary value problems related to (1.7), which are special cases of Theorem A.

We are interested in proving similar results for (BVPj), $j = 1,2,3$, when the limits f_0, f_∞ do not necessarily exist. Given any $h(t) \in L^1(0,1)$, which can be singular at $t = 0$, $t = 1$, or both, we provide upper and lower bounds on the nonlinear function $f(y)$ over certain specified finite intervals in $y \in \mathbb{R}_+$ so that boundary value problems (BVPj), $j = 1,2,3$, have at least one positive solution. A special case of such results for BVP1 is the following theorem:

THEOREM B. (Zhang and Sun [18]) Assume that (H_0) holds and that $\overline{\alpha} = 0$ in (1.2). Let $\tau \in \left(0, \frac{1}{2}\right)$ and r, R be positive real numbers, $0 < r < R$, satisfying one of the following conditions:

(a) $r \leqslant \tau^2(1-\tau)^2 R$ and

$$\begin{cases} f(u) \leqslant \sigma_1^{-1} h_0^{-1} r & \text{for } u \leqslant \tau^{-1}(1-\tau)^{-1} r, \\ f(u) \geqslant \sigma_2 h_\tau^{-1} R & \text{for } \tau(1-\tau)R \leqslant u \leqslant R; \text{ or} \end{cases}$$

(b) $R \geqslant \sigma_1 \sigma_2 h_0 h_\tau^{-1} r$ and

$$\begin{cases} f(u) \leqslant \sigma_1^{-1} h_0^{-1} R & \text{for } u \leqslant \tau^{-1}(1-\tau)^{-1} R, \\ f(u) \geqslant \sigma_2 h_\tau^{-1} r & \text{for } \tau(1-\tau) r \leqslant u \leqslant r, \end{cases}$$

where

$$\sigma_1 = \frac{1}{4} + \frac{\delta}{1-\delta}, \ \delta = \overline{\beta} < 1, \ \sigma_2 = \tau^{-2}(1-\tau)^{-1},$$
$$h_0 = \int_0^1 h(t)dt \ \text{ and } \ h_\tau = \int_\tau^{1-\tau} h(t)dt.$$

Then the boundary value problem (BVP1), i.e. $(1.1), (1.2)$ with $\alpha_i = 0$, $i = 1, \cdots, m$, has at least one positive solution. □

This paper is organized as follows. In section 2, we present abstract fixed point theorems due to Zhang and Sun [18] and Avery, Henderson, O'Regan [5]. Using these generalized Krasnoselskii's cone fixed point theorems, we prove generalizations of Theorem B for (BVPj), $j = 1,2,3$. In section 3, we introduce Hammerstein integral operators associated with (BVP1), (BVP2), (BVP3) whose fixed points correspond to positive solutions of these three boundary value problems. We establish upper and lower bounds for the kernels of these three Hammerstein integral operators. In section 4, we present extensions of Theorem B for (BVP1), followed by similar results for (BVP2), (BVP3) in section 5. In section 6, we discuss a right focal two-point problem studied in Avery, Henderson and O'Regan [2] [3] by different methods and show how our theorem related to (BVP3) can be applied to obtain improved results. Finally, in sections 7 and 8, we discuss examples of (BVP1), (BVP2), (BVP3).

2. Generalized Krasnoselskii Cone Fixed Point Theorems

In this section, we summarize the abstract results on generalized Krasnoselskii Cone Fixed Point Theorem proved by Zhang and Sun [18] and Avery, Henderson, O'Regan [4].

Let X be a real Banach space, $P \subseteq X$ is an ordered cone with $\theta \in P$ the zero element, $\Omega \subseteq X$ a bounded open subset, and $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator. A continuous functional $p : P \to [0,\infty)$ is called *convex* if

$$p\big(tx+(1-t)y\big) \leqslant tp(x)+(1-t)p(y),\ 0 \leqslant t \leqslant 1 \tag{2.1}$$

for all $x,y \in P$. Likewise a continuous functional $\varphi : P \to [0,\infty)$ is called *concave* if

$$\varphi\big(tx+(1-t)y\big) \geqslant t\varphi(x)+(1-t)\varphi(y),\ 0 \leqslant t \leqslant 1 \tag{2.2}$$

for all $x,y \in P$.

LEMMA 1. ([18; p.582, Theorem 2.3). Suppose that $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator and $p : P \to [0,\infty)$ is a non-negative convex functional with $p(\theta) = 0$ and $p(x) > 0$ for $x \neq \theta$. If $p(Ax) \leqslant p(x)$ and $Ax \neq x$ for all $x \in P \cap \partial\Omega$, then the fixed point index $i(A,P \cap \Omega,P) = 1$.

LEMMA 2. ([18; p.581, Theorem 2.2]). Suppose that $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator and $\varphi : P \to [0,\infty)$ is a non-negative uniformly continuous convex functional with $\varphi(\theta) = 0$, $\varphi(x) > 0$ for $x \neq \theta$, and $\inf\limits_{x \in P\cap\partial\Omega} \varphi(x) > 0$. If $\varphi(Ax) \geqslant \varphi(x)$ and $Ax \neq x$ for all $x \in P \cap \partial\Omega$, then the fixed point index $i(A,P \cap \Omega,P) = 0$.

LEMMA 3. ([4; p.4-5]). Suppose that $A : P \cap \overline{\Omega} \to P$ is a completely continuous operator and $\psi : P \to [0,\infty)$ is a non-negative concave functional satisfying (i) $\sup\limits_{x \in p} \psi(x) = \infty$, (ii) if $\psi(Ax) \geqslant \psi(x)$ for all $x \in P \cap \partial\Omega$, with $x \neq Ax$, then the fixed point index $i(A;P \cap \Omega,P) = 0$.

LEMMA 4. ([4; p.6 Theorem 3.5]). Let Ω_1,Ω_2 be two bounded open subsets in X such that $\theta \in \Omega_1, \overline{\Omega}_1 \subseteq \Omega_2$ and P is an ordered cone in X. Suppose that $A : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to P$ is a completely continuous operator satisfying $Ax \neq x$ for $x \in P \cap \partial\left(\overline{\Omega}_2 \backslash \Omega_1\right)$ and

(a) $i(A,P \cap \Omega_1,P) = 1, i(A,P \cap \Omega_2,P) = 0$; or

(b) $i(A,P \cap \Omega_2,P) = 1, i(A,P \cap \Omega_1,P) = 0$

then $i(A,P \cap (\Omega_2 \backslash \overline{\Omega}_1),P) = 1$.

Proof. Note that $Ax \neq x$ for $x \in P \cap \left[\overline{\Omega}_2 \backslash (\Omega_1 \cup (\Omega_2 \backslash \overline{\Omega}_1))\right]$ implies by the additivity of the fixed point index ([4; p.3 (G2)])

$$i(A,P \cap \Omega_2,P) = i(A,P \cap (\Omega_2 \backslash \overline{\Omega}_1),P) + i(A,P \cap \Omega_1,P).$$

Using assumptions (a) and (b) of this Lemma, we conclude $i(A,P \cap (\Omega_2 \backslash \overline{\Omega}_1),P),P) = \pm 1$, proving that the operator A has a fixed point in $P \cap (\Omega_2 \backslash \overline{\Omega}_1)$. □

We first state for completeness the original Krasnoselskii fixed point theorem:

THEOREM I (Krasnoselskii - Guo [9]) If either

(a) (Expansion)

$$\|Au\| \leqslant \|u\| \text{ for all } u \in P \cap \partial\Omega_1, \text{ and } \|Au\| \geqslant \|u\| \text{ for all } u \in P \cap \partial\Omega_2; \text{ or}$$

(b) (Compression)

$$\|Au\| \geqslant \|u\| \text{ for all } u \in P \cap \partial\Omega_1, \text{ and } \|Au\| \leqslant \|u\| \text{ for all } u \in P \cap \partial\Omega_2;$$

then A has a fixed point in $P \cap \overline{\Omega}_2 \backslash \Omega_1$. □

Authors Zhang and Sun [18] showed that the norm can be replaced by the weaker assumption of convex functional in the following generalized Krasnoselskii fixed point theorem:

THEOREM C (Zhang-Sun [18]) Let p_1, p_2 be two non-negative uniformly continuous convex functionals on P satisfying $p_j(\theta) = 0$ and $p_j(x) > 0$ if $x \neq \theta$, $j = 1,2$. If any one of the following conditions is satisfied: either

(a) (Expansion). $p_1(Ax) \leqslant p_1(x)$ for all $x \in P \cap \partial\Omega_1$ and inf $\{p_2(x) : x \in P \cap \partial\Omega_2\} > 0$, $p_2(Ax) \geqslant p_2(x)$ for all $x \in P \cap \partial\Omega_2$; or

(b) (Compression). inf$\{p_1(x) : x \in P \cap \partial\Omega_1\} > 0$, $p_1(Ax) \geqslant p_1(x)$ for $x \in P \cap \partial\Omega_1$ and $p_2(Ax) \leqslant p_2(x)$ for all $x \in P \cap \partial\Omega_2$;

then the operator A has a fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$. □

REMARK 2.1. The original result of Zhang-Sun [18] used only one functional, i.e. $p_1 \equiv p_2$ in Theorem C above. However, we can apply Lemma 1 and Lemma 2 to p_1 and p_2 separately and combine with Lemma 4 on additivity of fixed point indices to complete the proof of Theorem C. □

In [4; Theorem 3.5, p.6], Avery, Henderson and O'Regan showed that Zhang-Sun Fixed Point Theorem has the following variant in which one of the functional is convex and the other concave. The proof is again based upon Lemmas 1-4.

THEOREM D (Avery, Henderson, O'Regan [4]) Let P be a cone in a real Banach space X and Ω_1, Ω_2 be two bounded open sets in X such that $\theta \in \Omega_1, \overline{\Omega}_1 \subseteq \Omega_2$. Suppose that $A : P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to P$ is a completely continuous operator, $p : P \to [0, \infty)$ is a convex continuous functional, $p(\theta) = 0$ and $p(x) > 0$ for $x \neq \theta$ and φ is a concave continuous functional, $\varphi(\theta) = 0$, $\varphi(x) > 0$ for $x \neq \theta$. If either one of the two conditions:

(a) (Expansion) $p(Ax) \leqslant p(x)$ for all $x \in P \cap \partial\Omega_1$ and φ the concave functional satisfies $\varphi(Ax) \geqslant \varphi(x)$ for all $x \in P \cap \partial\Omega_2$ and sup $\{\varphi(x) : x \in P\} = \infty$; or

(b) (Compression) $\varphi(Ax) \geqslant \varphi(x)$ for all $x \in P \cap \partial\Omega_1$ and p the convex functional satisfies $p(Ax) \leqslant p(x)$ for all $x \in P \cap \partial\Omega_2$ and in addition inf $\{p(x) : x \in P \cap \partial\Omega_2\} > 0$.

Then A has at least one fixed point in $P \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$. □

REMARK 2.2. The norm $\| \, . \, \|$ of a Banach space is a convex but not a concave functional, so Avery, Henderson, O'Regan fixed point theorem is not a generalization of the original Krasnoselskii theorem. □

3. Hammerstein integral operators

Let $X = C[0,1]$ be the Banach space with sup norm. In [12], we show that solutions of (BVP1), (BVP2), (BVP3) can be represented as fixed points of the following Hammerstein integral operators on X:

$$A_j y(t) = \int_0^1 K_j(t,s)h(s)f\big(y(s)\big)ds, j = 1,2,3. \tag{3.1}$$

Here the kernels $K_j(t,s)$ incorporating multi-point boundary conditions (1.2), (1.3), (1.4) are given by:

$$\begin{aligned} K_1(t,s) = g_1(t,s) &+ \frac{t}{\triangle}\{(1-\overline{\alpha})\langle\beta, g_1(\xi,s)\rangle - (1-\overline{\beta})\langle\alpha, g_1(\xi,s)\rangle\} \\ &+ \frac{1}{\triangle}\{\big(1-\langle\beta,\xi\rangle\big)\langle\alpha, g_1(\xi,s)\rangle + \langle\alpha,\xi\rangle\langle\beta, g_1(\xi s)\rangle\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} K_2(t,s) = g_2(t,s) + \frac{1}{D}\Big\{&(1-\overline{\alpha})\langle\beta, g_2(\xi,s)\rangle \\ &+ \langle\alpha, \chi_{[0,\xi]}(s)\rangle\big[\big(1-\langle\beta,\xi\rangle\big) - (1-\overline{\beta})t\big]\Big\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} K_3(t,s) = g_3(t,s) + \frac{1}{D}\Big\{&(1-\overline{\beta})\langle\alpha, g_3(\xi,s)\rangle \\ &+ \langle\beta, \chi_{[\xi,1]}(s)\rangle\left[\langle\alpha,\xi\rangle + (1-\overline{\alpha})t\right]\Big\}, \end{aligned} \tag{3.4}$$

where $\triangle = (1-\overline{\alpha})\,(1-\langle\beta,\xi\rangle) + (1-\overline{\beta})\langle\alpha,\xi\rangle$, $D = (1-\overline{\alpha})(1-\overline{\beta})$ and

$$g_1(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{3.5}$$

$$g_2(t,s) = \begin{cases} 1-s, & 0 \leqslant t \leqslant s \leqslant 1, \\ 1-t, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \tag{3.6}$$

$$g_3(t,s) = \begin{cases} t, & 0 \leqslant t \leqslant s \leqslant 1, \\ s, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{3.7}$$

Also, $\chi_I(s)$ denotes the characteristic function of $I \subseteq [0,1]$, i.e. $\chi_I(s) = 1$ if $s \in I$ and $\chi_I(s) = 0$ if $s \notin I$. We also denote

$$\begin{aligned} &\chi_{[0,\xi]}(s) = \big(\chi_{[0,\xi_1]}(s), \cdots, \chi_{[0,\xi_m]}(s)\big), \\ &\chi_{[\xi,1]}(s) = \big(\chi_{[\xi_1,1]}(s), \cdots, \chi_{[\xi_m,1]}(s)\big), \\ &g_j(\xi,s) = (g_j(\xi_1,s), g_j(\xi_2,s), \cdots, g_j(\xi_m,s)), j = 1,2,3, \end{aligned}$$

for short.

In [16], Webb and Lan gave a frame work for proving existence of fixed points of Hammerstein integral operators $A_j, j = 1,2,3$ as given by (3.1) in terms of the kernel functions $K_j(t,s)$, $j = 1,2,3$, namely

(C_1) $K_j(t,s) \geqslant 0$ for $0 \leqslant t \leqslant s \leqslant 1$ is a measurable function on $[0,1]\times[0,1]$ satisfying for each $\tau \in [0,1]$,

$$\lim_{t\to\tau}\left|K_j(t,s)-K_j(\tau,s)\right| = 0, \text{ a.e. in } s\in[0,1],$$

where a.e. means *almost everywhere*.

(C_2) There exist bounded measurable functions $\Phi_j(s), s\in[0,1], j=1,2,3,$ such that for all $t,s\in[0,1]$,

$$\Phi_j(t)\Phi_j(s) \leqslant g_j(t,s) \leqslant K_j(t,s) \leqslant \nu_j\Phi_j(s), \tag{3.8}$$

where:

$$(a)\begin{cases}\Phi_1(s)=g_1(s,s)=s(1-s),\ D=(1-\overline{\alpha})(1-\overline{\beta});\\ \nu_1=\frac{1}{\triangle}\left\{1-\langle\beta,\xi\rangle+\langle\alpha,\xi\rangle+\max\left(\overline{\beta}-\overline{\alpha},0\right)\right\},\end{cases} \tag{3.9}$$

$$(b)\ \Phi_2(s)=g_2(s,s)=(1-s);\ \nu_2=\{D(1-\xi_m)\}^{-1}, \tag{3.10}$$

$$(c)\ \Phi_3(s)=g_3(s,s)=s; \nu_3=\{D\xi_1\}^{-1}. \tag{3.11}$$

It is easy to see that (C_1) is satisfied by $K_j(t,s), j=1,2,3$ as given by (3.2), (3.3), (3.4). The inequalities (3.8)-(3.11) for $K_j(t,s), j=1,2,3$ in (C_2) are proved in [12]. It therefore follows that Hammerstein integral operators $A_j, j=1,2,3,$ defined by (3.1) are completely continuous.

In proving our main results in sections 4 and 5, we also require lower bounds of the kernels $K_j(t,s),\ j=1,2,3$, over the following specified subintervals:

$$I_1=[\tau,1-\tau], 0<\tau<\frac{1}{2}, I_2=[0,\tau], I_3=[\tau,1],\ 0<\tau<1. \tag{3.12}$$

For any given τ, the lower bounds of $g_j(t,s)$ given in terms of $c_j(\tau),\overline{c}_j(\tau)$ and $g_j(s,s)$ are:

$$\max_{t\in I_j} g_j(t,s) \geqslant c_j(\tau)g_j(s.s), s\in[0,1] \tag{3.13}$$

and

$$\min_{t\in I_j} g_j(t,s) \geqslant \overline{c}_j(\tau)g_j(s,s),\ s\in[0,1], \tag{3.14}$$

where $c_1(\tau)=1-\tau, 0<\tau<\frac{1}{2}; c_2(\tau)=c_3(\tau)=1, 0<\tau<1$; and $\overline{c}_1(\tau)=\tau, 0<\tau<\frac{1}{2}; \overline{c}_2(\tau)=1-\tau, 0<\tau<1$; $\overline{c}_3(\tau)=\tau, 0<\tau<1$.

4. Main results-(BVP1)

We begin with an existence theorem for (BVP1).

THEOREM 4.1. *Assume that* (H_0) *holds. Let* r,R,τ_1,τ_2 *be positive real numbers satisfying* $0<r<R, 0<\tau_1\leqslant\tau_2<\frac{1}{2}$. *Suppose that the nonlinear function* $f(y)$ *satisfies one of the following two set of conditions:*

(I) (Expansion) (i) $R \geqslant [\tau_1\tau_2(1-\tau_1)(1-\tau_2)]^{-1} r$,

(ii) $f(u) \leqslant M_1 r$ *if* $0 \leqslant u \leqslant \nu_1 \tau_1^{-1}(1-\tau_1)^{-1} r$, *and*

(iii) $f(u) \geqslant m_1(\tau_2)R$ *if* $\nu_1^{-1}\tau_2(1-\tau_2)R \leqslant u \leqslant R$; *or*

(II) (Compression) (i) $R \geqslant m_1(\tau_2)M_1^{-1} r$,

(ii) $f(u) \leqslant M_1 R$ *if* $0 \leqslant u \leqslant \nu_1\tau_1^{-1}(1-\tau_1)^{-1}R$ *and*

(iii) $f(u) \geqslant m_1(\tau_2)r$ *if* $\nu_1^{-1}\tau_2(1-\tau_2)r \leqslant u \leqslant r$,

where $M_1 = \nu_1^{-1}h_1^{-1}, h_1 = \int_0^1 s(1-s)h(s)ds$ *and*

$$m_1(\tau) = \left\{(1-\tau)\int_\tau^{1-\tau} s(1-s)h(s)ds\right\}^{-1}. \tag{4.1}$$

Then the multi-point boundary value problem (BVP1) has a positive solution.

Proof. Denote

$$P_1 = \{u \in C[0,1] :\ u \text{ concave and non-negative, } u(t) \geqslant \nu_1^{-1}t(1-t)\|u\|, \forall t \in [0,1]\}.$$

We claim $A_1 : P_1 \to P_1$. Let $y \in P_1$, by (3.1), (3.8) we note that for $t, t_0 \in [0,1]$,

$$\begin{aligned} A_1y(t) &= \int_0^1 K_1(t,s)h(s)f\big(y(s)\big)ds \\ &\geqslant t(1-t)\int_0^1 g_1(s,s)h(s)f\big(y(s)\big)ds \\ &\geqslant \nu_1^{-1}t(1-t)\int_0^1 K_1(t_0,s)h(s)f\big(y(s)\big)ds \\ &= \nu_1^{-1}t(1-t)A_1y(t_0), \end{aligned}$$

which implies $A_1y(t) \geqslant \nu_1^{-1}t(1-t)\|A_1y\|$, proving $A_1(P_1) \subseteq P_1$.

Define the convex continuous functional $p_1(x)$ on P_1 and an open subset Ω_r of $C[0,1]$ by

$$p_1(u) = \max_{\tau_1 \leqslant t \leqslant 1-\tau_1} u(t),\ \Omega_r = \{u : u \geqslant 0, p_1(u) < r\}.$$

We show $p_1(Ay) \leqslant p_1(y)$ for $y \in P_1 \cap \partial\Omega_r$ where $\partial\Omega_r = \{y : p_1(y) = r\}$. Note that for $y \in P_1 \cap \partial\Omega_r$ there exists $t_1 \in [\tau_1, 1-\tau_1]$ such that $y(t_1) = p_1(y) = r$. Note that $y \in P_1$ implies

$$y(s) \leqslant \|y\| \leqslant t_1^{-1}(1-t_1)^{-1}\nu_1\, y(t_1) \leqslant \tau_1^{-1}(1-\tau_1)^{-1}\nu_1 r \tag{4.2}$$

for $s \in [0,1]$. Using (4.2) and (I) (ii), we obtain from (3.1) and $M_1 = \nu_1^{-1}h_1^{-1}$,

$$p_1(Ay) \leqslant \nu_1 M_1 r\int_0^1 g_1(s,s)h(s)ds = r = p_1(y), \tag{4.3}$$

so $p_1(A_1y) \leqslant p_1(y)$ for $y \in P_1 \cap \partial\Omega_r$. Suppose that $A_1y \neq y$ for $y \in P_1 \cap \partial\Omega_r$ for otherwise A_1 has a non-zero fixed point which is a positive solution of (BVP1). Next we apply Lemma 1. Using (4.3) we deduce that the fixed index of A_1 is given by $i(A_1, P_1 \cap \Omega_r, P_1) = 1$.

Next define $p_2(y)$ on P_1 another convex continuous functional by

$$p_2(y) = \max_{\tau_2 \leqslant t \leqslant 1-\tau_2} y(t),$$

which is non-negative, $p_2(y) > 0$ if $y \neq \theta$. Let $y \in P_1 \cap \partial\Omega_R$, where $\Omega_R = \{y : p_2(y) < R\}$, then $\inf_{y \in P_1 \cap \partial\Omega_R} p_2(y) = R > 0$. Again suppose $A_1 y \neq y$ for all $y \in P_1 \cap \partial\Omega_R$.

We note for any $y \in P_1 \cap \partial\Omega_R$, there exists $t_2 \in [\tau_2, 1-\tau_2]$ so that

$$\begin{aligned} R = p_2(y) = \max_{\tau_2 \leqslant t \leqslant 1-\tau_2} y(t) = y(t_2) &\geqslant \nu_1^{-1} t_2(1-t_2)\|y\| \\ &\geqslant \nu_1^{-1}\tau_2(1-\tau_2)\|y\| \geqslant \nu_1^{-1}\tau_2(1-\tau_2)R. \end{aligned} \tag{4.4}$$

From (4.4) and (I) (iii), we have by (3.12), (3.13)

$$\begin{aligned} p_2(A_1 y) &\geqslant \int_{\tau_2}^{1-\tau_2} \max_{\tau_2 \leqslant t \leqslant 1-\tau_2} K_1(t,s)h(s)f\big(y(s)\big)ds \\ &\geqslant (1-\tau_2)\left(\int_{\tau_2}^{1-\tau_2} s(1-s)h(s)ds\right) m_1(\tau_2)R = p_2(y), \end{aligned} \tag{4.5}$$

using the definition of $m_1(\tau)$ given by (4.1). Now apply Lemma 2 to A_1 with p_2 as the convex functional p satisfying (4.5) we obtain the fixed point index $i(A_1, P_1 \cap \Omega_R, P_1) = 0$. Using Lemmas 1, 2 and 4 concerning the additivity property of fixed point index, we conclude $i(A_1, P_1 \cap (\Omega_R \backslash \overline{\Omega}_r), P_1) = 1$. This establishes the existence of a fixed point of A_1 in $P_1 \cap (\Omega_R \backslash \overline{\Omega}_r)$, which is a positive solution of (BVP1).

The proof of part (II) Compression is similar and we omit the details.

We now give a corollary to Theorem 4.1 for the special case of (BVP1) when $\overline{\alpha} = 0$ which improves upon Theorem B of Zhang-Sun cited in section 1.

COROLLARY 4.1. *Assume that (H_0) holds. Let r, R, τ be positive real numbers satisfying $0 < r < R$, $0 < \tau < \frac{1}{2}$. Suppose that the nonlinear function $f(y)$ satisfies one of the following two assumptions:*

(I) (Expansion) *Let $R \geqslant \tau^{-2}(1-\tau)^{-2} r$,*

(i) *$f(u) \leqslant M_1 r$ if $0 \leqslant u \leqslant \tau^{-1}(1-\tau)^{-1} r$, and*

(ii) *$f(u) \geqslant m_1(\tau)R$ if $\tau(1-\tau)R \leqslant u \leqslant R$; or*

(II) (Compression) *Let $R \geqslant m_1(\tau)M_1^{-1} r$,*

(i) *$f(u) \leqslant M_1 R$ if $0 \leqslant u \leqslant \tau^{-1}(1-\tau)^{-1} R$, and*

(ii) *$f(u) \geqslant m_1(\tau) r$ if $\tau(1-\tau) r \leqslant u \leqslant r$,*

where $M_1, m_1(\tau)$ are defined as in Theorem 4.1. Then the (BVP1) with $\overline{\alpha} = 0$ has a positive solution.

Proof. When $\alpha_i = 0, i = 1, \cdots, m$, the kernel $K_1(t,s)$ satisfies for any $t_0 \in [0,1]$:

$$K_1(t,s) = g_1(t,s) + t\left(1 - \langle \beta, \xi \rangle\right)^{-1} \langle \beta, g_1(\xi, s) \rangle$$

$$\geqslant g_1(t_0,s)t(1-t)+t(1-t)t_0(1-\langle\beta,\xi\rangle)^{-1}\langle\beta,g_1(\xi,s)\rangle$$
$$\geqslant t(1-t)K_1(t_0,s) \quad \text{for all } t,s\in[0,1]. \tag{4.6}$$

Using (4.6), we can show $A_1 : P_0 \to P_0$ where $P_0 = \{u \in C[0,1] : u(t) \geqslant 0$ and concave, $u(t) \geqslant t(1-t)\|u\|\}$. The proof is similar to that given for Theorem 4.1 with P_0 replacing P_1 with $\nu_1 = 1$. We again omit the details.

REMARK 4.1. We show how Corollary 4.1 improves upon Theorem B. Part (a) of r Theorem B requires
(i) $f(u) \leqslant \sigma_1^{-1}h_0^{-1}r$ for $0 \leqslant u \leqslant \tau^{-1}(1-\tau)^{-1}r$, where $\sigma_1 = \frac{1}{4}+\frac{\overline{\beta}}{1-\overline{\beta}}$, $h_0 = \int_0^1 h(s)ds$,
(ii) $f(u) \geqslant \sigma_2 h_\tau^{-1}R$ for $\tau(1-\tau)R \leqslant u \leqslant R$.
Part (I) of Corollary 4.1 requires $f(u) \leqslant M_1 r$ for $0 \leqslant u \leqslant \tau^{-1}(1-\tau)^{-1}r$, where $M_1 = \nu_1^{-1}h_1^{-1}, \nu_1 = 1+\frac{\overline{\beta}}{1-\langle\beta,\xi\rangle}$. Note that $h_1 \leqslant \frac{1}{4}h_0$, so

$$\nu_1 = 1+\frac{\overline{\beta}}{1-\langle\beta,\xi\rangle} < 1+\frac{4\overline{\beta}}{1-\overline{\beta}} = 4\left(\frac{1}{4}+\frac{\overline{\beta}}{1-\overline{\beta}}\right) = 4\sigma_1$$

hence $\sigma_2^{-1}h_0^{-1} < 4\nu_1^{-1}h_0^{-1} \leqslant \nu_1^{-1}h_1^{-1} = M_1$. Also, the lower bound required by Corollary 4.1 is $f(y) \geqslant m_1(\tau)R$. We observe

$$m_1(\tau) = \left((1-\tau)\int_\tau^{1-\tau} s(1-s)h(s)ds\right)^{-1}$$
$$\leqslant \tau^{-1}(1-\tau)^{-2}\left(\int_\tau^{1-\tau} h(s)ds\right)^{-1} < \sigma_2 h_\tau^{-1}.$$

As to part (b) of Theorem B, we only need to show that $R \geqslant \sigma_1\sigma_2 h_0 h_\tau^{-1} r$ implies $R \geqslant m_1(\tau)M_1^{-1}r$ which is clear from (4.1) and $M_1 = \nu_1^{-1}h_1^{-1}$. This show that Corollary 4.1 implies Theorem B.

Similar to the proof of Theorem 4.1 based upon the fixed point result in Theorem C, we now use Theorem D, the fixed point theorem of Avery, Henderson and O'Regan [4], to prove the following result:

THEOREM 4.2. *Assume that (H_0) holds. Let r, R, τ_1, τ_2 be positive real numbers satisfying $0 < r < R, 0 < \tau_1 \leqslant \tau_2 < \frac{1}{2}$. Suppose that the nonlinear function $f(y)$ satisfies one of the following two sets of conditions:*
(I) (Expansion) (i) $R \geqslant \tau_1^{-1}\tau_2^{-1}(1-\tau_1)^{-1}(1-\tau_2)^{-1}r$,
 (ii) $f(u) \leqslant M_1 r$ *if* $0 \leqslant u \leqslant \nu_1\tau_1^{-1}(1-\tau_1)^{-1}r$, *and*
 (iii) $f(u) \geqslant \widehat{m}_1(\tau_2)R$ *if* $\nu_1^{-1}\tau_2(1-\tau_2)R \leqslant u \leqslant R$; *or*
(II) (Compression) (i) $R \geqslant \widehat{m}_1(\tau_2)M_1^{-1}r$,
 (i) $f(u) \leqslant M_1 R$ *if* $0 \leqslant u \leqslant \nu_1\tau_1^{-1}(1-\tau_1)^{-1}R$, *and*
 (iii) $f(u) \geqslant \widehat{m}_1(\tau_2)r$ *if* $r \leqslant u \leqslant \nu_1\tau_2^{-1}(1-\tau_2)^{-1}r$,

where $M_1 = \nu_1^{-1}h_1^{-1}$, h_1 *is given in Theorem 4.1, and*

$$\widehat{m}_1(\tau) = \left\{\tau\int_{\tau}^{1-\tau} s(1-s)h(s)ds\right\}^{-1}. \tag{4.7}$$

Then the boundary value problem (1.1), (1.2) has a positive solution.

Proof. Let $P_1 = \{u \in C[0,1] : u(t) \geqslant 0, \text{concave}, u(t) \geqslant \nu_1^{-1}t(1-t)\|u\|, \forall t \in [0,1]\}$ as defined in Theorem 4.1. We have proved $A_1(P_1) \subseteq P_1$ in Theorem 4.1. Introduce continuous functionals p, φ on $C[0,1]$ as follows: for $y \in [0,1]$,

$$p(y) = \max_{\tau_1 \leqslant t \leqslant 1-\tau_1} y(t); \varphi(y) = \min_{\tau_2 \leqslant t \leqslant 1-\tau_2} y(t).$$

Note that p and φ are non-negative continuous functionals on P_1, p is convex and φ is concave.

We now prove part (II) Compression and omit the proof of part (I) Expansion which is similar to that of Theorem 4.1. Define open subsets Ω_R, U_r of $C[0,1]$ as follows:

$$\Omega_R = \{y \in C[0,1] : p(y) < R\}, U_r = \{y \in C[0,1] : \varphi(y) < r\}.$$

Let $y \in P_1 \cap \partial\Omega_R$ so there exists $t_3 \in [\tau_1, 1-\tau_1]$ such that $p(y) = y(t_3) = r$. For $y \in P_1 \cap \partial\Omega_R$, we have

$$y(s) \leqslant \|y\| \leqslant t_3^{-1}(1-t_3)^{-1}\nu_1 y(t_3) \leqslant \tau_1^{-1}(1-\tau_1)^{-1}\nu_1 r. \tag{4.8}$$

Using (4.3) and assumption (III) (ii) in (3.1), we note

$$p(A_1 y) \leqslant \nu_1 M_1 R \int_0^1 g_1(s,s)h(s)ds = R = p(y) \tag{4.9}$$

since $M_1 = \nu_1^{-1}h_1^{-1}$. Suppose that $A_1 y \neq y$ for $y \in P_1 \cap \partial\Omega_R$, for otherwise A_1 has a fixed point in $\partial\Omega_R$ which by (3.1) is a positive solution of (BVP1). Now apply (4.9) using Lemma 1, we obtain $i(A_1, P_1 \cap \partial\Omega_R, P_1) = 1$.

Note that φ satisfies $\sup\{\varphi(y) : y \in P_1\} = \infty$. Let $y \in P_1 \cap \partial U_r$, then there exists $t_4 \in [\tau_2, 1-\tau_2]$ such that $y(t_4) = \varphi(y) = r$. For $s \in [\tau_2, 1-\tau_2]$, we have

$$r \leqslant y(s) \leqslant \|y\| \leqslant \nu_1 t_4^{-1}(1-t_4)^{-1} y(t_4) \leqslant \nu\tau_2^{-1}(1-\tau_2)^{-1} r. \tag{4.10}$$

Using (4.10) in assumption (II) (iii) and (3.1), we have

$$\begin{aligned}\varphi(A_1 y) &\geqslant \min_{\tau_2 \leqslant t \leqslant 1-\tau_2} \int_{\tau_2}^{1-\tau_2} K_1(t,s)h(s)f\big(y(s)\big)ds \\ &\geqslant \int_{\tau_2}^{1-\tau_2} \widehat{m}_1(\tau_2) r s(1-s)h(s)ds = r = \varphi(y), \end{aligned} \tag{4.11}$$

from the definition of $\widehat{m}_1(\tau_2)$ in (4.7). Suppose further $y \neq A_1 y$, otherwise the fixed point of A_1 with $\varphi(y) = r$ gives a positive solution of (BVP1) by (3.1). We now use (4.11) in Lemma 3 and obtain $i(A, P_1 \cap \partial U_r, P_1) = 0$. Combining (4.9) and (4.11) and using Lemmas 1,3 and 4, we conclude $i(A, P_1 \cap (\Omega_R \backslash \overline{U}_r), P_1) = 1$, proving that A_1 has a fixed point in $\Omega_R \backslash \overline{U}_r$ in P_1. This provides a positive solution to (BVP1) and completes the proof.

5. Main results-(BVP2), (BVP3)

In this section, we present existence theorems for (BVP2), (BVP3) following the same approach as Theorems 4.1 and 4.2 for (BVP1) in the previous section.

THEOREM 5.1. *Let r, R, τ_1, τ_2 be positive real numbers, $0 < r < R, 0 < \tau_1 \leqslant \tau_2 < 1$. Assume that (H_0) holds. If the nonlinear function $f(y)$ satisfies one of the following two sets of assumptions $(j = 2,3)$:*

(I) (Expansion) (i) $R \geqslant \delta_j^{-1}\gamma_j^{-1} r$,

(i) $f(u) \leqslant M_j r$ *for* $0 \leqslant u \leqslant \gamma_j^{-1} r$, *and*

(ii) $f(u) \geqslant m_j(\tau_2) R$ *for* $\delta_j R \leqslant u \leqslant R$; *or*

(II) (Compression) (i) $R \geqslant m_j(\tau_2) M_j^{-1} r$,

(ii) $f(u) \leqslant M_j R$ *for* $0 \leqslant u \leqslant \gamma_j^{-1} R$, *and*

(iii) $f(u) \geqslant m_j(\tau_2) r$ *for* $\delta_j r \leqslant u \leqslant r$;

where $\gamma_2 = 1-\tau_1, \gamma_3 = \tau_1; \delta_2 = 1-\tau_2, \delta_3 = \tau_2, M_j = \nu_j^{-1} h_j^{-1}$ *(ν_j given by (3.10), (3.11) for $j = 1,2$) and* $h_2 = \int_0^1 (1-s)h(s)ds, h_3 = \int_0^1 \mathrm{s}h(s)ds$. *Moreover,*

$$m_2(\tau) = \left\{ \int_0^\tau (1-s)h(s)ds \right\}^{-1}, \; m_3(\tau) = \left\{ \int_\tau^1 \mathrm{s}h(s)ds \right\}^{-1}.$$

Then the boundary value problem (BVPj), $j = 2,3$, has a positive solution.

Proof. We define

$$P_2 = \{u \in C[0,1], u(t) \text{ non-negative and concave, } u(t) \geqslant (1-t)\|u\| \text{ for } 0 \leqslant t \leqslant 1\}.$$

We show $A_2 : P_2 \to P_2$, where the operator A_2 is defined by (3.1). Note from (3.3) that

$$K_2(t,s)g_2(t,s) = C_2(s)(1-t) + D_2(s),$$

where

$$C_2(s) = (1-\overline{\alpha})\langle \beta, g_2(\xi,s)\rangle / D, \quad D = (1-\overline{\alpha})(1-\overline{\beta}),$$
$$D_2(s) = \left[(1-\overline{\beta})(1-t) + \overline{\beta} - \langle \beta, \xi \rangle\right] \langle \alpha, \chi_{[0,\xi]}(s)\rangle / D,$$

with $C_2(s), D_2(s) \geqslant 0$ for all $s \in [0,1]$ by hypothesis (H_0). For any $t_0 \in [0,1]$, we then have

$$K_2(t,s) \geqslant C_2(s)(1-t_0)(1-t) + D_2(s)(1-t) = (1-t)K_2(t_0,s),$$

from which it follows $A_2 y(t) \geqslant (1-t)A_2 y(t_0)$, for all $t_0 \in [0,1]$. Hence $A_2 y(t) \geqslant (1-t)\|A_2 y\|$, proving $A_2(P_2) \subseteq P_2$.

Likewise, let $P_3 = \{u \in C[0,1], u(t)$ non-negative and concave, $u(t) \geqslant t\|u\|$ for $0 \leqslant t \leqslant 1.\}$

We first prove $A_3 : P_3 \to P_3$. Note from (3.4) that

$$K_3(t,s) = g_3(t,s) + C_3(s)t + D_3(s),$$

where

$$C_3(s) = (1-\overline{\alpha})\langle \beta, \chi_{[\xi,1]}(s)\rangle / D, \quad D = (1-\overline{\alpha})(1-\overline{\beta}),$$
$$D_3(s) = (1-\overline{\beta})\langle \alpha, g_3(\xi,s)\rangle + \langle \alpha, \xi\rangle\langle \beta, \chi_{[\xi,1]}(s)\rangle / D,$$

with $C_3(s), D_3(s) \geqslant 0$ for $s \in [0,1]$ by hypothesis (H_0). For any $t_0 \in [0,1]$, we have by $g_3(t,s) \geqslant t g_3(s,s) = ts$

$$K_3(t,s) \geqslant t\{g_3(t_0,s) + C_3(s)t_0 + D_3(s)\} = tK_3(t_0,s)$$

from which it follows $A_3y(t) \geqslant tA_3y(t_0)$ for $t_0 \in [0,1]$. This proves $A_3(P_3) \subseteq P_3$.

We only prove Part (I) Expansion for $j = 2$ and also Part (II) Compression for $j = 3$. The other two parts are similar to that of the proof of Theorem 4.1 and we omit the details.

PROOF OF THEOREM 5.1. We introduce non-negative continuous functionals $p_i : C[0,1] \to \mathbb{R}_+$ defined by

$$p_i(y) = \max_{0 \leqslant t \leqslant \tau_i} y(t), i = 1,2.$$

It is easy to see that both p_1 and p_2 are convex functionals on P_2. Denote by

$$\Omega_r^{(i)} = \{u \in C[0,1] : p_i(u) < r\},\ i = 1,2,$$
$$\Omega_R^{(i)} = \{u \in C[0,1] : p_i(u) < R\},\ i = 1,2,$$

open subsets in $C[0,1]$.

Let $y \in P_2 \cap \partial\Omega_r^{(1)}$ so $p_1(y) = r$. Then there exists $t_5 \in [0,\tau_1]$ such that $y(t_5) = r$. Note that for $s \in [0,1]$,

$$y(s)_{\leqslant} \|y\| \leqslant (1-t_5)^{-1} y(t_5) \leqslant (1-\tau_1)^{-1} r,$$

hence from (I) (ii) $f\big(y(s)\big) \leqslant M_2 r$. Using this in (3.1), we obtain

$$p_1(A_2y) = \max_{0 \leqslant t \leqslant \tau_1} \int_0^1 K_2(t,s)h(s)f\big(y(s)\big)ds$$
$$\leqslant M_2 r \nu_2 \int_0^1 g_2(s,s)h(s)ds = r = p_1(y) \tag{5.1}$$

since $M_2 = \nu_2^{-1} h_2^{-1}$. Next let $y \in P_2 \cap \partial\Omega_R^{(2)}$ so $p_2(y) = R$ Note that for $s \in [0,\tau_2]$, we have $R \geqslant y(s) \geqslant (1-s)\|y\| \geqslant (1-\tau_2)R$. From (3.8), (3.10), (3.13), we obtain

$$p_2(A_2y) = \max_{0 \leqslant t \leqslant \tau_2} \int_0^1 K_2(t,s)h(s)f\big(y(s)\big)ds$$

$$\geqslant \int_0^1 g_2(s,s)h(s)f\big(y(s)\big)ds \geqslant \int_0^{\tau_2}(1-s)h(s)f\big(y(s)\big)ds$$
$$\geqslant m_2(\tau_2)R\int_0^{\tau_2}(1-s)h(s)ds = R = p_2(y), \tag{5.2}$$

by definition of $m_2(\tau_2)$. Using (5.1), (5.2) in Theorem C, the fixed point theorem of Zhang and Sun [18], we conclude that (BVP2) has a positive solution.

To prove the Compression part (II) of Theorem 5.1 for $j=3$, we require $R \geqslant m_3(\tau_2)M_3^{-1}r$, and introduce non-negative continuous functionals $\varphi_i : C[0,1] \to \mathbb{R}_+$ defined by $\varphi_i(y) = \max_{\tau_i \leqslant t \leqslant 1} y(t), i = 1,2$, which are both convex functionals on P_3. Denote $\Lambda_r^{(i)} = \{u \in C[0,1] : \varphi_i(u) < r\}$ and $\Lambda_R^{(i)} = \{u \in C[0,1] : \varphi_i(u) < R\}, i = 1,2$, open subsets of $C[0,1]$.

Let $y \in P_3 \cap \partial\Lambda_R^{(1)}$, so $\varphi_1(y) = R$, then there exists $t_6 \in [\tau_1, 1]$ such that $y(t_6) = R$. Now for $s \in [0,1]$, we have

$$y(s) \leqslant \|y\| \leqslant t_6^{-1}y(t_6) \leqslant \tau_1^{-1}R,$$

which together with (II)(ii) yield

$$\varphi_1(A_3y) = \max_{\tau_1 \leqslant t \leqslant 1}\int_0^1 K_3(t,s)h(s)f\big(y(s)\big)ds$$
$$\leqslant \nu_3 M_3 R\int_0^1 sh(s)ds = R = \varphi_1(y) \tag{5.3}$$

since $M_3 = \nu_3^{-1}h_3^{-1}$. Next let $y \in P_3 \cap \partial\Omega_r^{(2)}$ so $\varphi_2(y) = r$ and $r \geqslant y(s) \geqslant s\|y\| \geqslant \tau_1\varphi_2(y) = \tau_1 r$ for $s \in [\tau_2, 1]$. Using assumption (II) (iii), we note by (3.12),

$$\varphi_2(A_3y) \geqq \int_{\tau_3}^1 g_3(s,s)h(s)f\big(y(s)\big)ds$$
$$\geqslant m_3(\tau_2)r\int_{\tau_3}^1 sh(s)ds = r = \varphi_2(y). \tag{5.4}$$

We apply Lemmas 1, 2 to operator A_3 with (5.3), (5.4) and conclude by the additivity Lemma 4 that A_3 has a fixed point in $P_3 \cap \big(\Lambda_R \backslash \overline{\Lambda}_r\big)$ which gives a positive solution of (BVP3).

We now apply Theorem D, the Avery, Henderson O'Regan Fixed Point Theorem, as in the case of Theorem 4.2 for (BVP1), to prove the following result for (BVP2) and (BVP3).

THEOREM 5.2. *Let r, R, τ_1, τ_2 be positive real numbers, $0 < r < R, 0 < \tau_1 \leqslant \tau_2 < 1$. Assume that (H_0) holds. If the nonlinear function $f(y)$ satisfies one of the following two sets of assumptions (where $j = 2,3$):*

(I) (Expansion) (i) *$R \geqslant \gamma_j^{-1}r$,*

(ii) *$f(u) \leqslant M_j r$ for $0 \leqslant u \leqslant \gamma_j^{-1}r$, and*

(iii) $f(u) \geqslant \widehat{m}_j(\tau_2)R$ *for* $R \leqslant u \leqslant \delta_j^{-1}R$; *or*

(II) (Compression) (i) $R \geqslant \widehat{m}_j(\tau_2)M_j^{-1}r$,

(ii) $f(u) \leqslant M_jR$ *for* $0 \leqslant u \leqslant \gamma_j^{-1}R$, *and*

(iii) $f(u) \geqslant \widehat{m}_j(\tau_2)r$ *for* $r \leqslant u \leqslant \delta_j^{-1}r$;

where $\gamma_j, \delta_j, j = 2,3$, *are same as given in Theorem 5.1. Then the boundary value problem* $(BVP_j), j = 2,3$, *has a positive solution.*

Proof. We prove the Expansion Part (I) for $j = 3$ and Compression Part (II) for $j = 2$. From the proof of Theorem 5.1, we know $A_j(P_j) \subseteq P_j, j = 2,3$.

For (BVP3), introduce non-negative continuous functionals $p, \psi : C[0,1] \to \mathbb{R}_+$ defined by

$$p(y) = \max_{\tau_1 \leqslant t \leqslant 1} y(t), \psi(y) = \min_{\tau_2 \leqslant t \leqslant 1} y(t), 0 < \tau_1, \tau_2 < 1,$$

which are convex and concave functionals respectively on P_3. Denote open subsets in $C[0,1]$ by

$$\Omega_r = \{u : p(u) < r\} \quad \text{and} \quad U_R = \{u : \psi(u) < R\}.$$

Let $y \in P_3 \cap \delta\Omega_r$ so $p(y) = r$ and let $t_7 \in [\tau_1, 1]$ such that $y(t_7) = r$. Note that for $s \in [0,1]$, we have

$$y(s) \leqslant \|y\| \leqslant t_7^{-1}y(t_7) \leqslant \tau_1^{-1}r,$$

hence from (I) (ii), $f\big(y(s)\big) \leqslant M_3r$. Similar to the proof of (5.1), we conclude $p(A_3y) \leqslant p(y)$.

Next let $y \in P_3 \cap \partial U_R$ so $\psi(y) = R$ and let $t_8 \in [\tau_2, 1]$ such that $y(t_8) = R$. Note that for $s \in [\tau_1, 1]$,

$$R = \psi(y) \leqslant y(s) \leqslant \|y\| \leqslant t_8^{-1}y(t_8) \leqslant \tau_2^{-1}R,$$

hence from (I) (iii) $f\big(y(s)\big) \geqslant m_3(\tau_2)R$. Again following the same proof of (5.2), we observe from (3.8), (3.14),

$$\begin{aligned} \psi(A_3y) &= \min_{\tau_2 \leqslant t \leqslant 1} \int_0^1 K_3(t,s)h(s)f\big(y(s)\big)ds \\ &\geqslant \tau_2 \left(\int_{\tau_1}^1 g_3(s,s)h(s)ds \right) \widehat{m}_3(\tau_2)R = R = \psi(y). \end{aligned} \tag{5.5}$$

Now use $p(A_3y) \leqslant p(y)$ and $\psi(A_3y) \geqslant \psi(y)$ in Lemmas 1, 3 and 4 together with Theorem D we conclude that A_3 has a fixed point in $P_3 \cap \big(U_R \backslash \overline{\Omega}_r\big)$ which is a positive solution of (BVP3), proving Expansion part (I) for (BVP3).

Turning to the Compression part (II) for (BVP2), we have $R \geqslant \widehat{m}_2(\tau_2)M_2^{-1}r$. Introduce the following non-negative continuous functionals $\widehat{p}, \widehat{\psi} : C[0,1] \to \mathbb{R}_+$ defined by

$$\widehat{p}(y) = \max_{0 \leqslant t \leqslant \tau_1} y(t), \widehat{\psi}(y) = \min_{0 \leqslant t \leqslant \tau_2} y(t), 0 < \tau_1, \tau_2 < 1,$$

where $\widehat{p}$ is convex and $\widehat{\psi}$ is concave on P_2. Denote that open subsets in $C[0,1]$ by

$$\widehat{\Omega}_R = \{u : \widehat{p}(u) < R\} \text{ and } \widehat{D}_r = \{u : \widehat{\psi}(u) < r\}.$$

Let $y \in P_2 \cap \partial\widehat{\Omega}_R$ so $\widehat{p}(y) = R$ and let $t_9 \in [0, \tau_1]$ such that $y(t_9) = R$. Note that for $s \in [0,1]$, we have

$$y(s) \leqslant \|y\|\,(1-t_9)^{-1}\,y(t_9) \leqslant (1-\tau_1)^{-1}R,$$

which implies by assumption (II) (ii) $f\big(y(s)\big) \leqslant M_2 R$. Following a similar argument leading to (5.1), we can conclude $\widehat{p}(A_2 y) \leqslant \widehat{p}(y)$.

Next let $y \in P_2 \cap \partial\widehat{D}_r$ so $\widehat{\psi}(y) = r$ and let $t_{10} \in [0, \tau_1]$ such that $y(t_{10}) = \widehat{\psi}(y) = r$. Note that for $s \in [0, \tau_1]$, we have

$$r = \widehat{\psi}(y) \leqslant y(s) \leqslant \|y\| \leqslant (1-t_{10})^{-1}\,y(t_{10}) \leqslant (1-\tau_1)^{-1} r,$$

which implies by (II) (iii), $f\big(y(s)\big) \geqslant \widehat{m}_2(\tau_2) r$. Again by a similar argument as that of proving (5.5), we conclude that $\widehat{\psi}(A_2 y) \geqslant \widehat{\psi}(y)$. Now by an application of Theorem D, the fixed point theorem by Avery, Henderson and O'Regan [4], we conclude that A_2 has a fixed in $P_2 \cap \big(\Omega_R \backslash \overline{D}_r\big)$ which gives a positive solution of (BVP2).

6. A right focal two-point BVP

We consider the following right focal two-point boundary value problem discussed in Avery, Henderson, O'Regan [2], [3], which is a special case of (BVP3), i.e. (1.1), (1.4) with $h(t) \equiv 1, \alpha_i = \beta_i = 0, i = 1, 2, \cdots, m$:

$$y''(t) + f\big(y(t)\big) = 0, \quad 0 < t < 1, \tag{6.1}$$

$$y(0) = y'(1) = 0. \tag{6.2}$$

In [2], the authors proved a dual fixed point theorem to Theorem C by Zhang and Sun [18] where the role of convex and concave functionals were exchanged and used it to prove the following result for (6.1), (6.2):

THEOREM E ([2; p.8, Theorem 6.1]) Suppose that there exist positive numbers r and $\overline{R}$ with $0 < \frac{103}{25} r < \overline{R}$ and $f(y)$ is an increasing, concave function such that

$$\frac{16}{3} r \leqslant f(u) \leqslant \frac{32}{15}\overline{R},\ u \in \left[r, \frac{4\overline{R}}{3}\right] \tag{6.3}$$

then the BVP (6.1), (6.2) has at least one positive solution. □

We apply Theorem 5.2 (II) Compression to (6.1), (6.2) using assumption (ii), (iii) which require:

(i) $R \geqslant \widehat{m}_3(\tau_2) M_3^{-1} r$, and

(ii) $f(u) \leqslant M_3 R, 0 \leqslant u \leqslant \tau_1^{-1} R$ where $M_3 = \nu_3^{-1} h_3^{-1}$,

(iii) $f(u) \geqslant \widehat{m}_3(\tau_2) r \quad$ for $\quad r \leqslant u \leqslant \tau_2^{-1} r$.

In the case of (6.1), (6.2), $v_3 = 1, h_3 = \frac{1}{2}$ so $M_3 = 2$. Let $\tau_2 = \frac{1}{2}$, so $\widehat{m}_3(\frac{1}{2}) = 16/3$. Next let $\tau_1 = 4/5$, then Theorem 5.1 requires

$$\begin{cases} (1) \qquad R \geqslant 16/3 \cdot \frac{1}{2} r = 8/3r, \\ (2) \qquad f(u) \leqslant 2R \quad \text{for} \quad 0 \leqslant u \leqslant \frac{5}{4}R, \\ (3) \qquad f(u) \geqslant 16/3r \quad \text{for} \quad r \leqslant u \leqslant 2r. \end{cases} \tag{6.4}$$

Let $\overline{R} = \frac{15}{16}R$, so (6.3) becomes

$$\frac{16}{3} r \leqslant f(u) \leqslant 2R, \; r \leqslant u \leqslant 5/4R. \tag{6.5}$$

Clearly, $15/16R \geqslant 103/25r$ implies $R \geqslant \frac{8}{3}r$. Since $f(u)$ is required to be increasing, so (6.5) implies (2), (3) of (6.4). This shows that Theorem E can be deduced from Theorem 5.2 (II) Compression.

In [3], the same authors also proved a *Four functionals fixed point theorem* and used it to prove an existence result for the BVP (6.1), (6.2), namely.

THEOREM F ([3; p.1087, Theorem 10]) Suppose that there exist positive numbers r, R, M, B such that

$$\frac{320B}{128 - 131M} \leqslant 3R, \; \frac{16r}{3} \leqslant Mr + B, \; r \leqslant \frac{3R}{128} \leqslant 4r \tag{6.6}$$

and a continuous non-negative function $f(y)$ satisfying

$$\begin{cases} (a) \qquad f(u) \leqslant Mu + B \quad 0 \leqslant u \leqslant 2R, \\ (b) \qquad f(u) \geqslant \frac{16}{3} r, \quad r \leqslant u \leqslant 4r, \end{cases} \tag{6.7}$$

then the BVP (6.1), (6.2) has a positive solution. □

Again we can apply Theorem 5.2 (II) to show that (6.7) (a) (b) imply (6.4) (1), (2), (3). Firstly, $R > 128/3r$ in (6.6) implies $R > 8/3r$ in (6.4) (1). Also, (6.7) (b) implies (6.4)(3) over a smaller interval $[r, 2r]$. Finally, from (6.7) (a) and $320B \leqslant 3R(128 - 131M)$ in (6.6), we have for $0 \leqslant u \leqslant 5/4R$,

$$\begin{aligned} f(u) &\leqslant \frac{5}{4}MR + \frac{3R}{320}(128 - 131M) \\ &= \frac{384}{320}R + \frac{9}{320}MR \leqslant \frac{393}{320}R < 2R \end{aligned}$$

since $128 - 131M > 0$ implies $M < 1$. Thus (6.7)(a) implies (6.4) (2). This proves that Theorem F also follows from Theorem 5.2 (II).

7. The case when $h(t) = t^{-1/2}(1-t)^{-1/2}$

Zhang and Sun [18; Example 3.1] considered a three point problem with the specific function $h(t) = t^{-1/2}(1-t)^{-1/2} \in L^1(0,1)$ which is singular at both $t = 0$ and $t = 1$, namely

$$y'' + t^{-1/2}(1-t)^{-1/2} f(y) = 0,\ y(0) = 0,\ y(1) = \frac{1}{2} y(\eta), \tag{7.1}$$

where $0 < \eta < 1$. As an illustration of Theorem B, they give the following examples of $f(y)$ [18; p.585]:

$$f_1(u) = \begin{cases} (3/20\pi)u, & u \leqslant 16/3, \\ \frac{4}{35\pi}(57576u - 307065), & u > 16/3, \end{cases} \tag{7.2}$$

for part (a) Expansion, and

$$f_2(u) = \begin{cases} (1024/3\pi)u, & u \leqslant 3/16, \\ (8/7677\pi)(16u + 61413), & u > 16/3, \end{cases} \tag{7.3}$$

for part (b) Compression with which (7.1) have positive solution. Note that BVP (7.1) is a special case of (BVP1) with $\alpha_i = 0,\ i = 1, \cdots, m$ and $\beta_1 = \frac{1}{2}, \beta_i = 0, i = 2, \cdots, m$ and $\xi_1 = \eta, 0 < \eta < 1, \xi_i = 0, i = 2, \cdots, m$. In applying Theorem $B(a)$ Expansion part, Zhang and Sun [18 ; p.584] required with $\tau = \frac{1}{4}$,

(i) $R \geqslant \tau^{-2}(1-\tau)^{-2} r = 256/9r$; and take $R = 30$ with $r = 1$;

(ii) $f(y) \leqslant 4/5\pi$ for $0 \leqslant u \leqslant 16/3$;

(iii) $f(u) \geqslant 1920/\pi$ for $90/16 \leqslant u \leqslant 30$.

It is easy to verify that $f_1(u)$ given by (7.2) satisfies Theorem B(a) (i), (ii), (iiii) stated above.

In the Compression part Theorem B (b), it was required with $\tau = \frac{1}{4}$ that

(i) $R \geqslant (5/4)(64/3\pi)(3/\pi) = 80$; so take $R = 90$ with $r = 1$;

(ii) $f(u) \geqslant 72/\pi$ for $0 \leqslant u \leqslant 480$;

(iii) $f(u) \geqslant 64/\pi$ for $3/16 \leqslant u \leqslant 1$.

Similarly, example $f_2(u)$ given by (7.3) satisfies conditions required by Theorem B(b).

We now attempt to give simpler examples for (7.1) using Corollary 4.1. Here we note:

$$M_1 = \nu^{-1} h_1^{-1} \geqslant \frac{1}{2} \left(\int_0^1 \sqrt{s(1-s)} ds \right)^{-1} = \frac{4}{\pi}, \tag{7.4}$$

$$m_1\left(\frac{1}{4}\right) = \left\{ \frac{3}{4} \int_{\frac{1}{4}}^{3/4} \sqrt{s(1-s)} ds \right\} = \frac{4}{3} \left(\frac{\pi}{24} + \frac{\sqrt{3}}{16} \right)^{-1} = 5.5745 \tag{7.5}$$

and $\widehat{m}_1\left(\frac{1}{4}\right) = 3m_1\left(\frac{1}{4}\right) = 16.7236$. For the Expansion part of Corollary 4.1, take $\tau = \frac{1}{4}$ we require

(i) $R \geqslant \tau^{-2}(1-\tau)^{-2}r$, so let $r=1$ and $R=30$;

(ii) $f(u) \leqslant 4/\pi$ for $0 \leqslant u \leqslant 16/3$, (from (7.4));

(iii) $f(u) \geqslant 540/\pi \geqslant (5.5745)30$ for $90/16 \leqslant u \leqslant 40$ (from (7.5)).

We give the following simpler example:

$$\widetilde{f}_1(u) = \begin{cases} \frac{3}{4\pi}u, & 0 \leqslant u \leqslant 16/3, \\ \frac{1}{7\pi}(12864u - 68580), & u > 16/3. \end{cases} \tag{7.6}$$

It is easy to verify that $\widetilde{f}_1(u)$ satisfies conditions required by Corollary 4.1. Note that $f_1(u)$ given by (7.2) also satisfies the conditions required by Corollary 4.1, but $\widetilde{f}_1(u)$ does not satisfy the conditions required by Theorem B (a) of Zhang and Sun [18].

Turning to the Compression part, Theorem B(b) of Zhang and Sun [18], conditions required are:

(i) $R \geqslant \sigma_1\sigma_2 h_0 h_{1/4}^{-1} r = \frac{5}{4}\left(\frac{64}{3}\right)\pi \cdot \frac{3}{\pi} r = 80r$, so let $R = 90$ with $r=1$;

(ii) $f(u) \leqslant 4/5\pi(90) = 72/\pi, \quad 0 \leqslant u \leqslant 480$;

(iii) $f(u) \geqslant 64/\pi$, for $3/16 \leqslant u \leqslant 1$.

It is clear that $f_2(u)$ given by (7.3) satisfies (i), (ii), (iii) above. On the other hand, Corollary 4.1 requires:

(i) $R \geqslant m_1(\frac{1}{4})M_1^{-1}r$ which is satisfied with $r=1$, and $R=4$,

(ii) $f(u) \geqslant \frac{16}{3\pi}R = \frac{64}{3\pi}, 0 \leqslant u \leqslant 4$,

(iii) $f(u) \geqslant \frac{18}{\pi}r \geqslant (5.5745), 3/16 \leqslant u \leqslant 1$.

Consider the following example:

$$\widetilde{f}_2(u) = \frac{1}{183\pi}(160u + 3264), \qquad u \geqslant 0, \tag{7.7}$$

which satisfies conditions (i), (ii), (iii) of Corollary 4.1 since $\widetilde{f}_2(u)$ is required to satisfy upper and lower bounds over much smaller intervals, it is not strictly comparable with $f_2(u)$ given by (7.3). However, if we use $R = 27/8$ and $r = 1$, we can choose $f(u) = M_1R = m_1(1/4)r = 18/\pi$ for all $u \geqslant 0$ which is an even simpler example than $\widetilde{f}_2(u)$ by using Corollary 4.1. In section 8, we shall show that $f(u) \equiv m > 0$ in fact leads to a unique positive solution to $(\mathrm{BVP}_j), j = 1, 2, 3$.

We now give an example of (BVP2) to illustrate Theorem 5.2 again with $h(t) = t^{-1/2}(1-t)^{-1/2}$. Consider the four point boundary value problem:

$$\begin{cases} y'' + [t(1-t)]^{-1/2} f(y) = 0, \quad 0 < t < 1 \\ y'(0) = \frac{1}{3}y'\left(\frac{1}{4}\right),\ y(1) = \frac{1}{2}y(\frac{2}{3}). \end{cases} \tag{7.8}$$

This is a special case of (BVP2), where

$$\xi_1 = \frac{1}{4}\,\xi_2 = 2/3,\ \alpha_1 = 1/3,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = \frac{1}{2}.$$

Here

$$D = (1-\overline{\alpha})(1-\overline{\beta}) = \frac{1}{3},$$

$$\nu_2 = D^{-1}(1-\xi_2)^{-1} = 9,\ h_2 = \int_0^1 s^{-1/2}\sqrt{1-s}ds = \frac{\pi}{2}, \quad \text{so } M_2 = \nu_2^{-1}h_2^{-1} = \frac{2}{9\pi}.$$

Conditions required for the nonlinear function $f(y)$ by the Expansion Part of Theorem 5.2(I) with $\tau_1 = \tau_2 = 1/2$, $r = 1$ are
(i) $R = 4$,
(ii) $f(u) \leqslant \frac{2}{9\pi}$ for $0 \leqslant u \leqslant 2$,
(iii) $f(u) \geqslant \widehat{m}_2(\frac{1}{2}) \cdot 4$ for $4 \leqslant u \leqslant 8$,
where

$$\widehat{m}_2\left(\frac{1}{2}\right) = \left\{\frac{1}{2}\int_0^{1/2} s^{-1/2}\sqrt{1-3}ds\right\}^{-1} = 2\left(\frac{\pi}{4}+\frac{1}{2}\right)^{-1} = 1.5559.$$

Consider the following example:

$$f_3(u) = \begin{cases} \frac{2}{9\pi}u, & 0 \leqslant u \leqslant 2, \\ \frac{1}{9\pi}(89u-176), & u > 2, \end{cases}$$

which satisfies conditions (I)(i), (ii), (iii) of Theorem 5.1 (I).

REMARK 7.1. We note that $\tau = \frac{1}{4}$ is not the best possible choice for the minimum value of $\widehat{m}_1(\tau)$. Indeed, the minimum of $\widehat{m}_1(\tau)$, $0 < \tau < \frac{1}{2}$ is attained with $\tau = 0.2611$ and

$$\widehat{m}_1(0.2611) = 16.6803 < \widehat{m}_1(\frac{1}{4}) = 16.7258$$

so $\tau = 1/4$ gives a good approximation.

REMARK 7.2. In the case of example (7.8) for (BVP2), $\tau = \frac{1}{2}$ is also not the best choice for $\widehat{m}_2(\tau)$ which attains its minimum at $\tau_0 = 0.305663$ with value

$$\widehat{m}_2(0.305663) = (0.7266)^{-1} = 1.37627 < \widehat{m}_2(\frac{1}{2}) = 1.5559.$$

REMARK 7.3. When $h(t) = t^{-1/2}(1-t)^{-1/2}$, the lower bounds for $f(y)$, $\widehat{m}_2(\tau)$ and $\widehat{m}_3(\tau)$ in the case of (BVP2) and (BVP3), are related by the equality $\widehat{m}_2(\tau) = \widehat{m}_3(1-\tau)$, $0 < \tau < 1$. Using this, one can easily construct examples for (BVP3) following the approach in connection with $f_3(u)$ given above for the case of (BVP2).

8. Further example when $f(y) \equiv m > 0$

In the previous section, we have not given any example of $f(y)$ for the Compression part of Theorems 5.1 and 5.2. Unlike the Expansion part of our theorems for which we give examples in the form of linear functions, we can use $f(y) \equiv M_j R = \widehat{m}_j(\tau) r = m > 0,\ j = 1,2,3,$ as examples for the Compression part with any $0 < r < R$ and $M_j, \widehat{m}_j(\tau)$ as given in Theorems 5.1, 5.2. Indeed, when $f(y) = m$ then $f_0 = \infty,\ f_\infty = 0$ as defined by (1.5) so $f_\infty < \lambda_j < f_0$ and $f(y)$ is in the sublinear case, the existence of at least one positive solution is guaranteed by Theorem A. Here λ_j is the smallest eigenvalue of (1.6) subject to boundary conditions $(\mathrm{BC})_j, j = 1,2,3,$ as described in section 1. In this section, we show that if $f(y) = m$ then (BVP1), (BVP2), (BVP3) has a unique positive solution which can be explicitly given in terms of $h(t)$ and parameters involved in the boundary conditions, i.e. $\xi_i, \alpha_i, \beta_i, i = 1,2,\cdots,m$.

THEOREM 8.1. *Let $h(t) \in L^1(0,1)$, non-negative and $f(y) \equiv m$ where m is a positive real number for all $y \in \mathbb{R}$, then for $j = 1,2,3,$ the (BVP j) has a unique positive solution $y_j(t)$ given by $y_j(t) = -mH(t) + c_j t + d_j$ where $H(t) = \int_0^t (t-s)h(s)ds$ and c_j, d_j are given in terms of $\alpha_i, \beta_i, H(\xi_i)$ and $H'(\xi_i), i = 1,\cdots,m$.*

Proof. We can express the solutions of $y'' + mh(t) = 0$ subject to boundary conditions (1.2), (1.3), (1.4) by

$$y_j(t) = -mH(t) + c_j t + d_j, j = 1,2,3, \tag{8.1}$$

where

$$c_1 = \frac{m}{\Lambda}\{(1-\overline{\alpha})(H(1) - \langle\beta, H(\xi)\rangle - (1-\overline{\beta})\langle\alpha, H(\xi)\rangle\}, \tag{8.2}$$

$$d_1 = \frac{m}{\Lambda}\{\langle\alpha,\xi\rangle(H(1) - \langle\beta, H(\xi)\rangle) + (1 - \langle\beta,\xi\rangle)\langle\alpha, H(\xi)\rangle\}, \tag{8.3}$$

with

$$\Lambda = (1-\overline{\alpha})(1 - \langle\beta,\xi\rangle) + (1-\overline{\beta})\langle\alpha,\xi\rangle;$$

and

$$c_2 = -\frac{m}{1-\overline{\alpha}}\langle\alpha, H'(\xi)\rangle, d_2 = \frac{1}{1-\overline{\beta}}\{mH(1) - m\langle\beta, H(\xi)\rangle c_2[\langle\beta,\xi\rangle - 1]\}, \tag{8.4}$$

$$c_3 = -\frac{m}{1-\overline{\beta}}\{H'(1) - \langle\beta, H'(\xi)\rangle\}, d_3 = \frac{1}{1-\overline{\alpha}}\{-m\langle\alpha, H(\xi)\rangle c_3\langle\alpha,\xi\rangle\}, \tag{8.5}$$

with $H'(t) = \int_0^t h(s)ds$. Note that $h(t) \geqslant 0$ implies $\left(t^{-1}H(t)\right)' \geqslant 0$ so $H(t) \leqslant tH(1)$ for $0 \leqslant t \leqslant 1$. Using $H(\xi_i) \leqslant \xi_i H(1)$ in (8.2), it is easy to see that $d_1 \geqslant 0$ from (7.3). Now use (8.1), (8.2), we find

$$y_1(t) \geqslant -mH(t) + c_1 t > (-mH(1) + c_1)t, \quad 0 < t < 1,$$

and

$$c_1 - mH(1) \geqslant \frac{m}{\Lambda}\left\{(1-\overline{\alpha})(1 - \langle\beta,\xi\rangle + (1-\overline{\beta})\langle\alpha,\xi\rangle - \Lambda\right\}H(1) = 0.$$

Turning to $y_2(t)$, we note from (7.4) that $c_2 < 0$, so

$$y_2(t) > [-mH(1) + c_2]t + d_2 \geqslant -mH(1) + c_2 + d_2, \tag{8.6}$$

and

$$\begin{aligned} c_2 + d_3 &\geqslant c_2 + \frac{1}{1-\overline{\beta}}\{mH(1)(1 - \langle\beta,\xi\rangle) + c_2[\langle\beta,\xi\rangle - 1]\} \\ &\geqslant c_2\left(1 + \frac{\langle\beta,\xi\rangle - 1}{1-\overline{\beta}}\right) + mH(1) \geqslant mH(1). \end{aligned} \tag{8.7}$$

Combining (8.6) and (8.7), we obtain $y_2(t) > 0$, $0 < t < 1$.

Finally, we note from (8.5), $c_3 \geqslant mH'(1)$ which implies

$$d_3 \geqslant \frac{m}{1-\overline{\alpha}}\left\{-H(1)\langle\alpha,\xi\rangle + H'(1)\langle\alpha,\xi\rangle\right\} > 0.$$

Thus $y_3(t) = -mH(t) + c_3 t + d_3 \geqslant -mH(t) + mH'(1)t > 0$. This completes the proof of the theorem.

REMARK 8.1. In the special case $h(t) = t^{-1/2}(1-t)^{-1/2}$ considered in section 7, $H(t)$ has the explicit formula

$$H(t) = 2t\sin^{-1}\sqrt{t} - \sin^{-1}\sqrt{t} + t\sqrt{1-t^2}. \tag{8.8}$$

Using (8.8) in (8.1) with $c_j, d_j, j = 1,2,3$, we obtain explicit solutions to (BVP j), $j = 1,2,3$ when $f(t) \equiv m > 0$ and $h(t) = t^{-1/2}(1-t)^{-1/2}$. □

We close our discussion with additional remarks on related topics:

a) For a discussion on topological nature of Krasnoselskii fixed point theorem, we refer the reader to [13], [14].

b) It will be of interest to give extensions of results in this paper to higher order equations see e.g. [8], [11], and that with non-homogeneous boundary conditions e.g. [17].

c) Our results do not provide multiple solutions as results using Leggett-Williams type fixed point theorem, see e.g. Anderson, Avery, Henderson[1].

Acknowledgements. The author is grateful to two anonymous referees for their careful readings of the manuscript and suggested meaningful changes.

REFERENCES

[1] D. R. ANDERSON, R.I. AVERY, J. HENDERSON, *Functional expansion-compression fixed point theorem of Leggett Williams type*, Electron. J. Differ. Equ., No. 63 (2010), 1–9.

[2] R. I. AVERY, J. HENDERSON, D. O'REGAN, *A dual of compression-expansion fixed point theorem*, Fixed Point Theory and Applications, 2007, Art. ID 90715, 11 pages.

[3] R. I. AVERY, J. HENDERSON, D. O'REGAN, *Four functionals fixed point theorem*, Math. Comp. Modelling, **48** (2008), 1081–1089.

[4] R. I. AVERY, J. HENDERSON AND D. O'REGAN, *Functional compression-expansion fixed point theorem*, Electronic J. Diff. Equ. 2008, No. 22, 1–12.

[5] L. H. ERBE, *Boundary value problems for ordinary differential equations*, Rocky Mountain Math. J., **1** (1970), 709–729.

[6] L. H. ERBE, *Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems*, Math. Comput. Model, **32** (2000), 529–539.

[7] L. H. ERBE AND H. WANG, *On the existence of positive solutions of boundary value problems*, Proc. Amer. Math. Soc., **120** (1994), 743–748.

[8] J. R. GRAEF, L. KONG, Q. KONG AND J. S. W. WONG, *Higher order multi-point boundary value problems with sign-changing nonlinearities and non-homogeneous boundary conditions*, Elec. J. Qual. Diff. Equ., 2010, No. 28, 1–40.

[9] D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[10] G. B. GUSTAFSON AND K. SCHMITT, *Nonzero solutions of boundary value problems for second order ordinary and delay Differential equations*, J. Differ. Equation, **12** (1972), 129–147.

[11] L. KONG AND J. S. W. WONG, *Positive solutions for higher order multi-point boundary value problems with non-homogeneous boundary conditions*, J. Math. Anal. Appl., **367** (2010), 588–611.

[12] L. KONG AND J. S. W. WONG, *Optimal Existence Theorems for positive solutions of second order multi-point boundary value problems*, Communications in Applied Analysis, **16** (2012), 73–86.

[13] M. A. KRASNOSELSKII AND P. P. ZABREIKO, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Berlin, 1984.

[14] M. K. KWONG, *On Krasnoselskii's cone fixed point theorem*, Fixed point Theorem and applications, 2008, Art. ID164537, 18 pages.

[15] J. R. L. WEBB AND G. INFANTE, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc., **74** (2006), 673–693.

[16] J. R. L. WEBB AND K. Q. LAN, *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type*, Topol. Methods Nonlinear Analysis, **27** (2006), 91–115.

[17] J. S. W. WONG, *Positive solutions of second order multi-point boundary value problems with non-homogeneous boundary conditions*, Differ. Equ. Appl., **2** (2010), 345–375.

[18] G. ZHANG AND J. SUN, *A generalization of the cone expansion and compression fixed point theorem and applications*, Nonlinear Analysis, **67** (2007), 579–586.

(Received November 15, 2011)

James S. W. Wong
Institute of Mathematical Research
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
e-mail: jsww@chinneyhonkwok.com