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SOLVING A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS AND EXISTENCE OF ASYMPTOTICALLY STABLE SOLUTIONS

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Abstract. The paper is devoted to the study of a system of nonlinear integral equations. First, this system is reduced to a fixed point problem of a nonlinear integral operator and hence we can give suitable assumptions and using a fixed point theorem of Krasnosel'skii type in order to obtain the existence of solutions. Next, we prove the existence of asymptotically stable solutions for the above system. In order to illustrate the results, an example is also presented.

1. Introduction

In this paper, we consider the solvability and the existence of asymptotically stable solutions for the following system of nonlinear integral equations

$$\begin{cases} x_1(t) = p(t) + f(t,x_1(t),x_2(t)) + \int_0^t V(t,s,x_1(s),x_2(s))ds, \\ x_2(t) = q(t) + g(t,x_1(t),x_2(t)) + \int_0^\infty G(t,s,x_1(s),x_2(s))ds, \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}_+ = [0,\infty)$, $p,\ q : \mathbb{R}_+ \to E$; $f,\ g : \mathbb{R}_+ \times E^2 \to E$; $G : \mathbb{R}_+ \times \mathbb{R}_+ \times E^2 \to E$; $V : \Delta \times E^2 \to E$ are supposed to be continuous, $\Delta = \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : s \leqslant t\}$ and E is a Banach space.

Nonlinear functional integral equations with bounded intervals or unbounded intervals have been studied extensively by many authors using various methods and techniques. There are many important results about the existence, stability and other properties of solutions, for example, we refer to [1]-[9], [13]-[17] and the references given therein.

In the case $E = \mathbb{R}^d$, some types of (1.1) have been studied by C. Avramescu and C. Vladimirescu [2], [3]. The authors have proved the existence of asymptotically stable solutions to the following integral equations

$$x(t) = q(t) + f(t,x(t)) + \int_0^t V(t,s)x(s)ds + \int_0^t G(t,s,x(s))ds,\ t \in \mathbb{R}_+, \tag{1.2}$$

or

$$x(t) = q(t) + \int_0^t K(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds,\ t \in \mathbb{R}_+, \tag{1.3}$$

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under suitable hypotheses. In the proofs, a fixed point theorem of Krasnosel'skii type is used, (see [2], [3]).

C. Avramescu and C. Vladimirescu [5] also have proved the existence of solutions to the following integral equations system

$$\begin{cases} x(t) = \int_0^t K(t,s,y(\mu_1(s)))x(\nu_1(t))ds + \int_0^t F(t,s,x(\sigma_1(s)),y(\theta_1(s)))ds, \\ y(t) = \int_0^T G(t,s,y(\mu_2(s)))ds + \int_0^T H(t,s,x(\sigma_2(s)),y(\theta_2(s)))ds, 0 \leqslant t \leqslant T, \end{cases} \tag{1.4}$$

where $K : \Delta \times \mathbb{R}^N \to \mathfrak{M}_N(\mathbb{R})$ is a continuous and bounded quadratic matrix function, $F : \Delta \times \mathbb{R}^{2N} \to \mathbb{R}^N$, $G : [0,T]^2 \times \mathbb{R}^N \to \mathbb{R}^N$, $H : [0,T]^2 \times \mathbb{R}^{2N} \to \mathbb{R}^N$, μ_1, ν_1, σ_1, θ_1, μ_2, σ_2, $\theta_2 : [0,T] \to [0,T]$, are continuous and bounded functions. By using the fixed point theorem of Krasnosel'skii, the authors stated and proved an existence result of solutions for a system of type

$$\begin{cases} x = A_1(x,y) + B_1(x,y), \\ y = A_2(y) + B_2(x,y) \end{cases}$$

and (1.4) is the application of this result, (see [5]).

Also applying a fixed point theorem of Krasnosel'skii type and giving the suitable assumptions, Dhage and Ntouyas [7], Purnaras [16] obtained some results on the existence of solutions to the following nonlinear functional integral equation

$$x(t) = q(t) + \int_0^{\mu(t)} k(t,s)f(s,x(\theta(s)))ds + \int_0^{\sigma(t)} v(t,s)g(s,x(\eta(s)))ds, \ t \in [0,1], \tag{1.5}$$

where $E = \mathbb{R}$, $0 \leqslant \mu(t) \leqslant t$; $0 \leqslant \sigma(t) \leqslant t$; $0 \leqslant \theta(t) \leqslant t$; $0 \leqslant \eta(t) \leqslant t$, for all $t \in [0,1]$. Purnaras also shows that the technique used in [16] can be applied to yield existence results for the following equation

$$\begin{aligned} x(t) = q(t) &+ \int_{\alpha(t)}^{\mu(t)} k(t,s)f(s,x(\theta(s)))ds \\ &+ \int_{\beta(t)}^{\lambda(t)} \widehat{k}(t,s)F\left(s,x(\nu(s)), \int_0^{\sigma(s)} k_0 s,v,x(\eta(v)))dv\right)ds, \ t \in [0,1]. \end{aligned} \tag{1.6}$$

In case the Banach space E is arbitrary, recently in [13], [15], we also have proved the existence of asymptotically stable solutions to the following integral equations

$$x(t) = q(t) + f(t,x(t)) + \int_0^t V(t,s,x(s))ds + \int_0^t G(t,s,x(s))ds, t \in \mathbb{R}_+, \tag{1.7}$$

or

$$x(t) = q(t) + f(t,x(t)) + \int_0^t V(t,s,x(s))ds + \int_0^\infty G(t,s,x(s))ds, t \in \mathbb{R}_+, \tag{1.8}$$

by using the fixed point theorem of Krasnosel'skii type as follows.

THEOREM A. [13, L. T. P. Ngoc and N. T. Long]
Let $(X, |\cdot|_n)$ *be a Fréchet space and let* U, $C : X \to X$ *be two operators. Assume that*

(i) U *is a* k_n-*contraction operator,* $k_n \in [0,1)$ *(depending on* n*), with respect to a family of seminorms* $\|\cdot\|_n$ *equivalent with the family* $|\cdot|_n$*;*

(ii) C *is completely continuous;*

(iii) $\lim\limits_{|x|_n \to \infty} \frac{|Cx|_n}{|x|_n} = 0, \ \forall n \in \mathbb{N}.$

Then $U + C$ *has a fixed point.*

By choosing suitable spaces and establishing corresponding operators in the Fréchet space, also applying Theorem A, we improve the existence results of (1.7) and (1.8) for (1.1), in which the given functions satisfy conditions specified later. The results are obtained by combination of the arguments in [13], some techniques in [2], [3] with appropriate modifications and the arguments of density as in [15].

It is known that the space of continuous functions on a noncompact interval cannot be organized always as a Banach space, but it can be organized as a Fréchet space if we use suitable seminorms and define the corresponding metric [see [1] - [6], ([18], p.32, p.52)].

The paper consists of four sections. First, in section 2, the system (1.1) is reduced to a fixed point problem of a nonlinear integral operator and then we prove the existence of solutions. Next, in section 3, we prove the existence of asymptotically stable solutions. Remark that in order to obtain this result, here we need not the condition $V(t,s,0,0) = 0$, for all $(t,s) \in \Delta$ as in [13], [15]. Finally, section 4 presents an illustrated example.

2. Existence of solutions

Let $(E, |\cdot|_E)$ be a Banach space. Then $E^2 = E \times E$ is also a Banach space with the norm $|\cdot|$ defined as follows

$$|u| = |u_1|_E + |u_2|_E, \ u = (u_1, u_2) \in E^2.$$

Let $X = C(\mathbb{R}_+; E^2)$ be the space of all continuous functions on $\mathbb{R}_+$ to E^2 equipped with the numerable family of seminorms

$$|x|_n = \sup_{t \in [0,n]} |x(t)|, \ n \geqslant 1; \ |x(t)| = |x_1(t)|_E + |x_2(t)|_E; \ x = (x_1, x_2) \in X.$$

Then $(X, \ |\cdot|_n)$ is complete in the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x-y|_n}{1 + |x-y|_n}$$

and X is the Fréchet space (it will be proved in the appendix).

Consider in X the other family of seminorms $\|\cdot\|_n$ is defined as follows

$$\|x\|_n = |x|_{\gamma_n} + |x|_{h_n}, \; n \in \mathbb{N},$$

where

$$|x|_{\gamma_n} = \sup_{t\in[0,\gamma_n]} |x(t)|, \quad |x|_{h_n} = \sup_{t\in[\gamma_n,n]} e^{-h_n(t-\gamma_n)} |x(t)|,$$

$\gamma_n \in (0,n)$ and $h_n > 0$ are arbitrary numbers, which is equivalent to $|x|_n$, since

$$e^{-h_n(n-\gamma_n)} |x|_n \leqslant \|x\|_n \leqslant 2|x|_n, \forall x \in X, \forall n \in \mathbb{N}.$$

We make the following assumptions.

(A_1) $p, \; q \in C(\mathbb{R}_+;E)$;

(A_2) There exists a constant $L \in [0,1)$ such that $\forall x = (x_1,x_2), y = (y_1,y_2) \in E^2$, for all $t \in \mathbb{R}_+$,

$$|f(t,x_1,x_2) - f(t,y_1,y_2)|_E \leqslant \frac{L}{2}|x-y|, |g(t,x_1,x_2) - g(t,y_1,y_2)|_E \leqslant \frac{L}{2}|x-y|;$$

(A_3) There exists a continuous function $\omega_1 : \Delta \to \mathbb{R}_+$ such that for all $(t,s) \in \Delta$, for all $x = (x_1,x_2), y = (y_1,y_2) \in E^2$,

$$|V(t,s,x_1,x_2) - V(t,s,y_1,y_2)|_E \leqslant \omega_1(t,s)|x-y|;$$

(A_4) G is completely continuous such that for all bounded subsets $I_1, \; I_2$ of $\mathbb{R}_+$ and for any bounded subset J of E^2, for all $\varepsilon > 0$, there exists $\delta > 0$, such that $\forall t_1, \; t_2 \in I_1$,

$$|t_1 - t_2| < \delta \Longrightarrow |G(t_1,s,x_1,x_2) - G(t_2,s,x_1,x_2)|_E < \varepsilon, \forall s \in I_2, \; \forall x = (x_1,x_2) \in J;$$

(A_5) There exists a continuous function $\omega_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for each bounded subset I of $\mathbb{R}_+$,

$$\int_0^\infty \sup_{t\in I} \omega_2(t,s)ds < \infty,$$

and

$$|G(t,s,x_1,x_2)|_E \leqslant \omega_2(t,s), \; \forall (t,s) \in I \times \mathbb{R}_+, \; \forall x = (x_1,x_2) \in E^2.$$

THEOREM 1. *Let* (A_1)-(A_5) *hold. Then the system* (1.1) *has a solution on* $\mathbb{R}_+$.

Proof. The proof consists of four steps.

Step 1. In X, we consider the system

$$\begin{cases} x_1(t) = p(t) + f(t,x_1(t),x_2(t)) + \int_0^t V(t,s,x_1(s),x_2(s))ds, \\ x_2(t) = q(t) + g(t,x_1(t),x_2(t)), \; t \in \mathbb{R}_+. \end{cases} \tag{2.1}$$

We have the following lemma.

LEMMA 1. *Let* (A_1)-(A_3) *holds. Then the system* (2.1) *has a unique solution* $\xi=(\xi_1,\xi_2),\ \xi\in C(\mathbb{R}_+,E^2)$.

Proof. We rewrite (2.1) as follows

$$x(t)=\Phi x(t),\ t\in\mathbb{R}_+,$$

where

$$\begin{aligned}
&\Phi x(t)=(\Phi_1x(t),\Phi_2x(t)),\ x=(x_1,x_2)\in X=C(\mathbb{R}_+;E^2),\\
&\Phi_1x(t)=p(t)+f(t,x_1(t),x_2(t))+\int_0^t V(t,s,x_1(s),x_2(s))ds,\\
&\Phi_2x(t)=q(t)+g(t,x_1(t),x_2(t)),\ t\in\mathbb{R}_+.
\end{aligned}$$

By hypothesis $(A_2),(A_3)$, for all $x=(x_1,x_2),y=(y_1,y_2)\in X$, we have

$$\begin{aligned}
\Phi_1x(t)-\Phi_1y(t)&=f(t,x_1(t),x_2(t))-f(t,y_1(t),y_2(t))\\
&\quad+\int_0^t\left[V(t,s,x_1(s),x_2(s))-V(t,s,y_1(s),y_2(s))\right]ds,\\
|\Phi_1x(t)-\Phi_1y(t)|_E&\leqslant|f(t,x_1(t),x_2(t))-f(t,y_1(t),y_2(t))|_E\\
&\quad+\int_0^t|V(t,s,x_1(s),x_2(s))-V(t,s,y_1(s),y_2(s))|_E\,ds\\
&\leqslant\frac{L}{2}|x(t)-y(t)|+\int_0^t\omega_1(t,s)\,|x(s)-y(s)|\,ds.
\end{aligned}$$

Let $n\in\mathbb{N}$ be fixed. For all $t\in[0,\gamma_n]$, with $\gamma_n\in(0,n)$ chosen later, we have

$$\begin{aligned}
|\Phi_1x(t)-\Phi_1y(t)|_E&\leqslant\frac{L}{2}|x(t)-y(t)|+\int_0^t\omega_1(t,s)\,|x(s)-y(s)|\,ds\\
&\leqslant\frac{L}{2}|x-y|_{\gamma_n}+\gamma_n\tilde{\omega}_{1n}|x-y|_{\gamma_n}=\left(\frac{L}{2}+\gamma_n\tilde{\omega}_{1n}\right)|x-y|_{\gamma_n},
\end{aligned}$$

where

$$\tilde{\omega}_{1n}=\sup\{\omega_1(t,s):(t,s)\in\Delta_n\},\ \Delta_n=\{(t,s):0\leqslant s\leqslant t,0\leqslant t\leqslant n\}.$$

On the other hand, we also have

$$\begin{aligned}
|\Phi_2x(t)-\Phi_2y(t)|_E&=|g(t,x_1(t),x_2(t))-g(t,y_1(t),y_2(t))|_E\\
&\leqslant\frac{L}{2}|x(t)-y(t)|\leqslant\frac{L}{2}|x-y|_{\gamma_n}
\end{aligned}$$

for all $t\in[0,\gamma_n]$. Hence

$$\begin{aligned}
|\Phi x(t)-\Phi y(t)|&=|\Phi_1x(t)-\Phi_1y(t)|_E+|\Phi_2x(t)-\Phi_2y(t)|_E\\
&\leqslant\left(\frac{L}{2}+\gamma_n\tilde{\omega}_{1n}\right)|x-y|_{\gamma_n}+\frac{L}{2}|x-y|_{\gamma_n}\\
&=(L+\gamma_n\tilde{\omega}_{1n})\,|x-y|_{\gamma_n},\ t\in[0,\gamma_n].
\end{aligned}$$

So

$$|\Phi x-\Phi y|_{\gamma_n} \leqslant (L+\gamma_n\tilde{\omega}_{1n})\,|x-y|_{\gamma_n}.$$

For all $t\in[\gamma_n,n]$, similarly, we also have

$$\begin{aligned}
|\Phi_1 x(t)-\Phi_1 y(t)|_E &\leqslant \frac{L}{2}\,|x(t)-y(t)|+\tilde{\omega}_{1n}\int_0^{\gamma_n}|x(s)-y(s)|\,ds\\
&\quad+\tilde{\omega}_{1n}\int_{\gamma_n}^{t}|x(s)-y(s)|\,ds\\
&\leqslant \frac{L}{2}\,|x(t)-y(t)|+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\tilde{\omega}_{1n}\int_{\gamma_n}^{t}|x(s)-y(s)|\,ds.
\end{aligned}$$

Hence

$$\begin{aligned}
|\Phi x(t)-\Phi y(t)| &= |\Phi_1 x(t)-\Phi_1 y(t)|_E+|\Phi_2 x(t)-\Phi_2 y(t)|_E\\
&\leqslant \frac{L}{2}\,|x(t)-y(t)|+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\tilde{\omega}_{1n}\int_{\gamma_n}^{t}|x(s)-y(s)|\,ds\\
&\quad+\frac{L}{2}\,|x(t)-y(t)|\\
&\leqslant L\,|x(t)-y(t)|+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\tilde{\omega}_{1n}\int_{\gamma_n}^{t}|x(s)-y(s)|\,ds.
\end{aligned}$$

By the inequality

$$0<e^{-h_n(t-\gamma_n)}\leqslant 1,\ \forall t\in[\gamma_n,n],$$

with $h_n>0$ is also chosen later, we get

$$\begin{aligned}
|\Phi x(t)-\Phi y(t)|\,e^{-h_n(t-\gamma_n)} &\leqslant L\,|x(t)-y(t)|\,e^{-h_n(t-\gamma_n)}+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}\\
&\quad+\tilde{\omega}_{1n}e^{-h_n(t-\gamma_n)}\int_{\gamma_n}^{t}|x(s)-y(s)|\,ds\\
&\leqslant L\,|x-y|_{h_n}+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}\\
&\quad+\tilde{\omega}_{1n}\int_{\gamma_n}^{t}|x(s)-y(s)|\,e^{-h_n(s-\gamma_n)}e^{h_n(s-t)}ds\\
&\leqslant L\,|x-y|_{h_n}+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\tilde{\omega}_{1n}\,|x-y|_{h_n}\int_{\gamma_n}^{t}e^{h_n(s-t)}ds\\
&= L\,|x-y|_{h_n}+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}\\
&\quad+\frac{\tilde{\omega}_{1n}}{h_n}\,|x-y|_{h_n}\left(1-e^{h_n(\gamma_n-t)}\right)\\
&\leqslant L\,|x-y|_{h_n}+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\frac{\tilde{\omega}_{1n}}{h_n}\,|x-y|_{h_n}\\
&= \gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\left(L+\frac{\tilde{\omega}_{1n}}{h_n}\right)|x-y|_{h_n}.
\end{aligned}$$

Hence

$$|\Phi x-\Phi y|_{h_n}\leqslant \gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\left(L+\frac{\tilde{\omega}_{1n}}{h_n}\right)|x-y|_{h_n}.$$

Consequently,

$$\begin{aligned}\|\Phi x-\Phi y\|_n &= |\Phi x-\Phi y|_{\gamma_n}+|\Phi x-\Phi y|_{h_n}\\ &\leqslant (L+\gamma_n\tilde{\omega}_{1n})\,|x-y|_{\gamma_n}+\gamma_n\tilde{\omega}_{1n}\,|x-y|_{\gamma_n}+\left(L+\frac{\tilde{\omega}_{1n}}{h_n}\right)|x-y|_{h_n}\\ &\leqslant (L+2\gamma_n\tilde{\omega}_{1n})\,|x-y|_{\gamma_n}+\left(L+\frac{\tilde{\omega}_{1n}}{h_n}\right)|x-y|_{h_n}\leqslant L_n\,\|x-y\|_n,\end{aligned}$$

where $L_n=\max\left\{L+2\gamma_n\tilde{\omega}_{1n},\ L+\frac{\tilde{\omega}_{1n}}{h_n}\right\}$.

Choose γ_n and h_n such that

$$0<\gamma_n<\min\left\{\frac{1-L}{2\tilde{\omega}_{1n}},n\right\}\quad\text{and}\quad h_n>\frac{\tilde{\omega}_{1n}}{1-L},$$

then we have $L_n<1$, so Φ is an L_n-contraction operator on X with respect to the family of seminorms $\|\cdot\|_n$. Based on the Banach contraction principle in an arbitrary Fréchet space, [see ([1], p.8), ([4], p.475), ([5], p.185)], we have the following lemma and the proof will be presented in the appendix.

LEMMA 2. *Let $(X,|\cdot|_n)$ be a Fréchet space and let $\Phi: X\to X$ be an L_n-contraction on X with respect to a family of seminorms $\|\cdot\|_n$ equivalent with $|\cdot|_n$. Then Φ has a unique fixed point in X.*

Remark 1. The existence of a fixed point in Lemma 2 can be also obtained from Theorem A especially when $Cx=0$ for all $x\in X$ (C is null-operator).

Applying Lemma 2, there is only a function $\xi\in X$ such that

$$\xi(t)=\Phi\xi(t),\ t\in\mathbb{R}_+.$$

Hence, Lemma 1 follows. □

By the transformation $x_1=y_1+\xi_1,\ x_2=y_2+\xi_2,$ we can write the system (1.1) in the form

$$\begin{aligned}y_1(t)+\xi_1(t)&=p(t)+f(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t))\\&\quad+\int_0^t V(t,s,y_1(s)+\xi_1(s),y_2(s)+\xi_2(s))ds,\\ y_2(t)+\xi_2(t)&=q(t)+g(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t))\\&\quad+\int_0^\infty G(t,s,y_1(s)+\xi_1(s),y_2(s)+\xi_2(s))ds,\end{aligned}$$

or

$$y(t)=Uy(t)+Cy(t),\ t\in\mathbb{R}_+, \tag{2.2}$$

where

$$\begin{cases} y=(y_1,y_2),\ Uy(t)=(U_1y(t),U_2y(t)),\ Cy(t)=(0,C_2y(t)), \\ U_1y(t)=p(t)+f(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t))-\xi_1(t) \\ \qquad\qquad +\int_0^t V(t,s,y_1(s)+\xi_1(s),y_2(s)+\xi_2(s))ds, \\ U_2y(t)=q(t)+g(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t))-\xi_2(t), \\ C_2y(t))=\int_0^\infty G(t,s,y_1(s)+\xi_1(s),y_2(s)+\xi_2(s))ds, t\in\mathbb{R}_+. \end{cases} \tag{2.3}$$

Step 2. The operator U is a k_n-contraction, $k_n\in[0,1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$ in which $\gamma_n,\ h_n$ are chosen suitably, $\gamma_n=\widehat{\gamma}_n,\ h_n=\widehat{h}_n$ as below.

Indeed, fixed an arbitrary positive integer $n\in\mathbb{N}$.

For all $t\in[0,\widehat{\gamma}_n]$, with $\widehat{\gamma}_n\in(0,n)$ chosen later, we have

$$\begin{aligned} U_1y(t)-U_1\tilde{y}(t) &= f(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t))-f(t,\tilde{y}_1(t)+\xi_1(t),\tilde{y}_2(t)+\xi_2(t)) \\ &\quad+\int_0^t V(t,s,y_1(s)+\xi_1(s),y_2(s)+\xi_2(s))ds \\ &\quad-\int_0^t V(t,s,\tilde{y}_1(s)+\xi_1(s),\tilde{y}_2(s)+\xi_2(s))ds \\ &= f(t,y(t)+\xi(t))-f(t,\tilde{y}(t)+\xi(t)) \\ &\quad+\int_0^t [V(t,s,y(s)+\xi(s))ds-V(t,s,\tilde{y}(s)+\xi(s))]\,ds, \end{aligned}$$

so

$$\begin{aligned} |U_1y(t)-U_1\tilde{y}(t)|_E &\leqslant \frac{L}{2}|y(t)-\tilde{y}(t)|+\int_0^t \omega_1(t,s)\,|y(s)-\tilde{y}(s)|\,ds \\ &\leqslant \left(\frac{L}{2}+\tilde{\omega}_{1n}\widehat{\gamma}_n\right)|y-\tilde{y}|_{\widehat{\gamma}_n}. \end{aligned} \tag{2.4}$$

For short, we can write

$$\begin{aligned} f(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t)) &\equiv f(t,y(t)+\xi(t)), \\ g(t,y_1(t)+\xi_1(t),y_2(t)+\xi_2(t)) &\equiv g(t,y(t)+\xi(t)), \end{aligned}$$

and it is similar to the other functions. On the other hand, we also have

$$\begin{aligned} |U_2y(t)-U_2\tilde{y}(t)|_E &= |g(t,y(t)+\xi(t))-g(t,\tilde{y}(t)+\xi(t))|_E \\ &\leqslant \frac{L}{2}|y(t)-\tilde{y}(t)|\leqslant\frac{L}{2}|y-\tilde{y}|_{\widehat{\gamma}_n}, \end{aligned} \tag{2.5}$$

for all $t\in[0,\widehat{\gamma}_n]$. This implies that

$$\begin{aligned} |Uy(t)-U\tilde{y}(t)| &= |U_1y(t)-U_1\tilde{y}(t)|_E+|U_2y(t)-U_2\tilde{y}(t)|_E \\ &\leqslant \left(\frac{L}{2}+\tilde{\omega}_{1n}\widehat{\gamma}_n\right)|y-\tilde{y}|_{\widehat{\gamma}_n}+\frac{L}{2}|y-\tilde{y}|_{\widehat{\gamma}_n} \\ &= (L+\tilde{\omega}_{1n}\widehat{\gamma}_n)\,|y-\tilde{y}|_{\widehat{\gamma}_n}. \end{aligned}$$

Hence

$$|Uy - U\tilde{y}|_{\widehat{\gamma}_n} \leqslant (L + \tilde{\omega}_{1n}\widehat{\gamma}_n)\,|y-\tilde{y}|_{\widehat{\gamma}_n}. \tag{2.6}$$

For all $t \in [\widehat{\gamma}_n, n],$ similarly, we also have

$$\begin{aligned} |U_1 y(t) - U_1\tilde{y}(t)|_E &\leqslant \frac{L}{2}\,|y(t)-\tilde{y}(t)| + \tilde{\omega}_{1n}\int_0^{\widehat{\gamma}_n}|y(s)-\tilde{y}(s)|\,ds \\ &\quad + \tilde{\omega}_{1n}\int_{\widehat{\gamma}_n}^t |y(s)-\tilde{y}(s)|\,ds \\ &\leqslant \frac{L}{2}\,|y(t)-\tilde{y}(t)| + \widehat{\gamma}_n\tilde{\omega}_{1n}\,|y-\tilde{y}|_{\widehat{\gamma}_n} + \tilde{\omega}_{1n}\int_{\widehat{\gamma}_n}^t |y(s)-\tilde{y}(s)|\,ds. \end{aligned} \tag{2.7}$$

On the other hand, we also have

$$|U_2 y(t) - U_2\tilde{y}(t)|_E \leqslant \frac{L}{2}\,|y(t)-\tilde{y}(t)|.$$

We deduce that

$$\begin{aligned} |Uy(t) - U\tilde{y}(t)| &= |U_1 y(t) - U_1\tilde{y}(t)|_E + |U_2 y(t) - U_2\tilde{y}(t)|_E \\ &\leqslant \frac{L}{2}\,|y(t)-\tilde{y}(t)| + \widehat{\gamma}_n\tilde{\omega}_{1n}\,|y-\tilde{y}|_{\widehat{\gamma}_n} \\ &\qquad + \tilde{\omega}_{1n}\int_{\widehat{\gamma}_n}^t |y(s)-\tilde{y}(s)|\,ds + \frac{L}{2}\,|y(t)-\tilde{y}(t)| \\ &\leqslant L\,|y(t)-\tilde{y}(t)| + \widehat{\gamma}_n\tilde{\omega}_{1n}\,|y-\tilde{y}|_{\widehat{\gamma}_n} \\ &\qquad + \tilde{\omega}_{1n}\int_{\widehat{\gamma}_n}^t |y(s)-\tilde{y}(s)|\,ds, \forall t \in [\widehat{\gamma}_n, n]. \end{aligned} \tag{2.8}$$

By the inequality

$$0 < e^{-h_n(t-\widehat{\gamma}_n)} \leqslant 1, \forall t \in [\widehat{\gamma}_n, n],$$

in which $\widehat{h}_n > 0$ is also chosen later, we get

$$\begin{aligned} &|Uy(t) - U\tilde{y}(t)|\,e^{-\widehat{h}_n(t-\widehat{\gamma}_n)} \\ &\quad\leqslant L\,|y(t)-\tilde{y}(t)|\,e^{-\widehat{h}_n(t-\widehat{\gamma}_n)} + \tilde{\omega}_{1n}\widehat{\gamma}_n\,|y-\tilde{y}|_{\widehat{\gamma}_n} \\ &\qquad\qquad + \tilde{\omega}_{1n}\int_{\widehat{\gamma}_n}^t |y(s)-\tilde{y}(s)|\,e^{-\widehat{h}_n(t-\widehat{\gamma}_n)}ds \\ &\quad\leqslant L\,|y-\tilde{y}|_{\widehat{h}_n} + \tilde{\omega}_{1n}\widehat{\gamma}_n\,|y-\tilde{y}|_{\widehat{\gamma}_n} + \tilde{\omega}_{1n}\int_{\widehat{\gamma}_n}^t |y(s)-\tilde{y}(s)|\,e^{-\widehat{h}_n(s-\widehat{\gamma}_n)}e^{\widehat{h}_n(s-t)}ds \\ &\quad\leqslant L\,|y-\tilde{y}|_{\widehat{h}_n} + \tilde{\omega}_{1n}\widehat{\gamma}_n\,|y-\tilde{y}|_{\widehat{\gamma}_n} + \tilde{\omega}_{1n}\,|y-\tilde{y}|_{\widehat{h}_n}\int_{\widehat{\gamma}_n}^t e^{\widehat{h}_n(s-t)}ds \\ &\quad= L\,|y-\tilde{y}|_{\widehat{h}_n} + \tilde{\omega}_{1n}\widehat{\gamma}_n\,|y-\tilde{y}|_{\widehat{\gamma}_n} + \tilde{\omega}_{1n}\,|y-\tilde{y}|_{\widehat{h}_n}\frac{1}{\widehat{h}_n}\left(1-e^{\widehat{h}_n(\widehat{\gamma}_n-t)}\right) \\ &\quad\leqslant \tilde{\omega}_{1n}\widehat{\gamma}_n\,|y-\tilde{y}|_{\widehat{\gamma}_n} + \left(L + \frac{\tilde{\omega}_{1n}}{\widehat{h}_n}\right)|y-\tilde{y}|_{\widehat{h}_n}. \end{aligned} \tag{2.9}$$

We get

$$|Uy-U\tilde{y}|_{\widehat{h}_n} \leqslant \tilde{\omega}_{1n}\widehat{\gamma}_n |y-\tilde{y}|_{\widehat{\gamma}_n} + \left(L+\frac{\tilde{\omega}_{1n}}{\widehat{h}_n}\right)|y-\tilde{y}|_{\widehat{h}_n}. \tag{2.10}$$

Combining (2.6), (2.10), we deduce that

$$\begin{aligned}
\|Uy-U\tilde{y}\|_n &= |Uy-U\tilde{y}|_{\gamma_n} + |Uy-U\tilde{y}|_{\widehat{h}_n} \\
&\leqslant (L+\tilde{\omega}_{1n}\widehat{\gamma}_n)|y-\tilde{y}|_{\widehat{\gamma}_n} + \tilde{\omega}_{1n}\widehat{\gamma}_n|y-\tilde{y}|_{\widehat{\gamma}_n} + \left(L+\frac{\tilde{\omega}_{1n}}{\widehat{h}_n}\right)|y-\tilde{y}|_{\widehat{h}_n} \\
&\leqslant (L+2\tilde{\omega}_{1n}\widehat{\gamma}_n)|y-\tilde{y}|_{\widehat{\gamma}_n} + \left(L+\frac{\tilde{\omega}_{1n}}{\widehat{h}_n}\right)|y-\tilde{y}|_{\widehat{h}_n} \leqslant \tilde{k}_n\|y-\tilde{y}\|_n,
\end{aligned} \tag{2.11}$$

where $\tilde{k}_n = \max\left\{L+2\tilde{\omega}_{1n}\widehat{\gamma}_n, L+\frac{\tilde{\omega}_{1n}}{\widehat{h}_n}\right\}$. Choose

$$0<\widehat{\gamma}_n<\min\left\{\frac{1-L}{2\tilde{\omega}_{1n}}, n\right\} \quad \text{and} \quad \widehat{h}_n > \frac{\tilde{\omega}_{1n}}{1-L}. \tag{2.12}$$

Then we have $\tilde{k}_n<1$, by (2.11), U is a $\tilde{k}_n-$contraction operator with respect to a family of new seminorms $\|\cdot\|_n$.

Step 3. We show that $C: X \to X$ is completely continuous. It is obviously $C: X \to X$ is completely continuous if and only if $C_2 : X \to X_1 = C(\mathbb{R}_+;E)$ is completely continuous. We first show that C_2 is continuous.

For any $y_0 \in X$, let $\{y_m\}$ be a sequence in X such that $\lim\limits_{m\to\infty} y_m = y_0$. We recall that

$$\lim_{m\to\infty} y_m = y_0 \quad \text{in } X \text{ if and only if } \lim_{m\to\infty} |y_m - y_0|_n = 0,$$

i.e.

$$\lim_{m\to\infty} \sup_{t\in[0,n]} |y_m(t)-y_0(t)| = 0, \forall n \in \mathbb{N}.$$

Let $n\in\mathbb{N}$ be fixed. For any given $\varepsilon>0$, because of

$$\int_0^\infty \sup_{t\in[0,n]} \omega_2(t,s)ds < \infty,$$

there exists $T_n \in \mathbb{N}$, such that

$$\int_{T_n}^\infty \omega_2(t,s)ds \leqslant \int_{T_n}^\infty \sup_{t\in[0,n]} \omega_2(t,s)ds < \frac{\varepsilon}{8}, \ \forall t\in[0,n]. \tag{2.13}$$

Put

$$K = \{((y_{1m}+\xi_1)(s),\ (y_{2m}+\xi_2)(s)) : s\in[0,T_n],\ m\in\mathbb{Z}_+\}.$$

For short, we can write

$$K = \{(y_m+\xi)(s) : s\in[0,T_n],\ m\in\mathbb{Z}_+\}.$$

We have K is compact in E^2, since

$$\sup_{s\in[0,\,T_n]} |(y_m+\xi)(s)-(y_0+\xi)(s)| = \sup_{s\in[0,\,T_n]} |y_m(s)-y_0(s)|$$
$$= |y_m-y_0|_{T_n} \to 0 \text{ as } m\to\infty.$$

In detail, let $\{(y_{m_j}+\xi)(s_j)\}_j$ be a sequence in K. We can assume that there exists a subsequence of $\{s_j\}_j$, denoted by $\{s_j\}_j$, such that

$$\lim_{j\to\infty} s_j = s_0 \quad\text{and}\quad \lim_{j\to\infty} y_{m_j}+\xi = y_0+\xi.$$

We have

$$\begin{aligned}\big|(y_{m_j}+\xi)(s_j)-(y_0+\xi)(s_0)\big| &\leqslant \big|(y_{m_j}+\xi)(s_j)-(y_0+\xi)(s_j)\big| \\ &\quad + \big|(y_0+\xi)(s_j)-(y_0+\xi)(s_0)\ \big| \\ &\leqslant \big|y_{m_j}-y_0\big|_{T_n} + \big|(y_0+\xi)(s_j)-(y_0+\xi)(s_0)\ \big|,\end{aligned}$$

which shows that

$$\lim_{j\to\infty} (y_{m_j}+\xi)(s_j) = (y_0+\xi)(s_0) \text{ in } E^2.$$

It yields K is compact in E^2.

For $\varepsilon>0$ be given as above, by G is continuous on the compact set $[0,n]\times[0,T_n]\times K$, there exists $\delta>0$ such that for every $u,\ v\in K,\ |u-v|<\delta$,

$$|G(t,s,u)-G(t,s,v)|_E < \frac{\varepsilon}{4T_n},\ \forall (t,s)\in[0,n]\times[0,T_n].$$

By

$$\sup_{s\in[0,\,T_n]} |(y_m+\xi)(s)-(y_0+\xi)(s)| = |y_m-y_0|_{T_n} \to 0 \text{ as } m\to\infty,$$

there exists m_0 such that for $m>m_0$,

$$|(y_m+\xi)(s)-(y_0+\xi)(s)| < \delta,\ \forall s\in[0,T_n].$$

This implies that for all $t\in[0,n]$, for all $m>m_0$,

$$\begin{aligned}|C_2y_m(t)-C_2y_0(t)|_E &\leqslant \int_0^{T_n} |G(t,s,(y_m+\xi)(s))-G(t,s,(y_0+\xi)(s))|_E\,ds \\ &\quad + 2\int_{T_n}^{\infty} \omega_2(t,s)ds < T_n\frac{\varepsilon}{4T_n}+2\frac{\varepsilon}{8}=\frac{\varepsilon}{2},\end{aligned}$$

so $\sup_{t\in[0,n]} |C_2y_m(t)-C_2y_0(t)|_E<\varepsilon$, for all $m>m_0$, and the continuity of C_2 is proved.

It remains to show that C_2 maps bounded sets into relatively compact sets. Let us recall the following condition for the relative compactness of a subset in X.

LEMMA 3. *Let $C(\mathbb{R}_+;E)$ be the Fréchet space defined as above and A be a subset of $C(\mathbb{R}_+;E)$. For each $n \in \mathbb{N}$, let $C([0,n];E)$ be the Banach space of all continuous functions $u:[0,n] \to E$ with the norm*

$$\sup_{t\in[0,n]} |u(t)|_E \quad \textit{and} \quad A_n = \{x|_{[0,n]} : x \in A\}.$$

The set A in $C(\mathbb{R}_+;E)$ is relatively compact if and only if for each $n \in \mathbb{N}$, A_n is equicontinuous in $C([0,n];E)$ and for every $s \in [0,n]$, the set $A_n(s) = \{x(s) : x \in A_n\}$ is relatively compact in E.

This condition was stated in [11] and was proved in detail in [13]. The proof follows from the Ascoli-Arzela's Theorem, (see [12], p.211). $\square$

Now we continue with the proof. Let Ω be a bounded subset of $X = C(\mathbb{R}_+;E^2)$. We have to prove that for $n \in \mathbb{N}$:

(a) the set $(C_2\Omega)_n$ is equicontinuous in $C([0,n];E)$,

(b) for every $t \in [0,n]$, the set $(C_2\Omega)_n(t) = \{C_2y|_{[0,n]}(t) : y \in \Omega\}$ is relatively compact in E.

Let $n \in \mathbb{N}$ be fixed. Consider any $\varepsilon > 0$ given. Then, there exists $T_n \in \mathbb{N}$ (T_n is big enough) such that (2.13) is valid.

Proof of (a). For any $y \in \Omega$, for all $t_1,\ t_2 \in [0,n]$, by (A_5),

$$\begin{aligned} |C_2y(t_1) - C_2y(t_2)|_E \leqslant \int_0^{T_n} |G(t_1,s,(y+\xi)(s)) - G(t_2,s,(y+\xi)(s))|_E\, ds \\ + \int_{T_n}^{\infty} (\omega_2(t_1,s) + \omega_2(t_2,s))\, ds. \end{aligned} \tag{2.14}$$

According to (2.13), (2.14) and (A_4), $(C_2\Omega)_n$ is equicontinuous on $C([0,n];E)$.

Proof of (b). Let $\{C_2y_k|_{[0,n]}(t)\}_k$, $y_k \in \Omega$, be a sequence in $(C_2\Omega)_n(t)$. We shall show that there exists a convergent subsequence of $\{C_2y_k|_{[0,n]}(t)\}_k$.

Put $S = \{(y+\xi)(s) : y \in \Omega,\ s \in [0,T_n]\}$. Then S is bounded in E^2. Since G is completely continuous, the set $G([0,n]\times[0,T_n]\times S)$ is relatively compact in E. Let $\widehat{Q} = \mathbb{Q} \cap [0,T_n]$ be the set of rational numbers in $[0,T_n]$. Then $\widehat{Q}$ is countable and has form $\widehat{Q} = \{s_m\}$.

For $m = 1$, the sequence $\{G(t,s_1,(y_k+\xi)(s_1))\}_k$ belongs to $G([0,n]\times[0,T_n]\times S)$, that is relatively compact in E, so there exists a subsequence of $\{y_k\}$, denoted by $\{y_k^{(1)}\}_k$, such that

$$\left\{G(t,s_1,(y_k^{(1)}+\xi)(s_1))\right\}_k \text{ converges in } E.$$

For $m = 2$, similarly, there exists a subsequence of $\{y_k^{(1)}\}_k$, denoted by $\{y_k^{(2)}\}_k$, such that

$$\left\{G(t,s_2,(y_k^{(2)}+\xi)(s_2))\right\}_k \text{ converges in } E.$$

Therefore, for all $m \in \mathbb{N}$, by induction, we can establish a subsequence $\{y_k^{(m+1)}\}_k$ of $\{y_k^{(m)}\}_k$, such that

$$\left\{G(t, s_{m+1}, (y_k^{(m+1)} + \xi)(s_{m+1}))\right\}_k \text{ converges in } E.$$

Put $z_k = y_k^{(k)}$. Then $\{z_k\}_k$ is a subsequence of $\{y_k\}_k$ and $\{G(t, s_m, (z_k + \xi)(s_m))\}_k$ converges in E, for all $s_m \in \widehat{Q}$. Then, there exists $k_0 \geqslant 1$ (only depening on ε) such that for all $k,\ l \geqslant k_0$,

$$|G(t, s_m, (z_k + \xi)(s_m)) - G(t, s_m, (z_l + \xi)(s_m))|_E < \frac{\varepsilon}{8T_n} \text{ for all } s_m \in \widehat{Q}. \tag{2.15}$$

For each $s \in [0, T_n]$, there exists the sequence $\{s_m\}$, $s_m \in \widehat{Q}$, $m = 1, 2, \ldots$, such that $\lim_{m\to\infty} s_m = s$. By the continuity of the functions G, ξ, z_k, z_l, passing to the limit in (2.15), we obtain that for all $k,\ l \geqslant k_0$,

$$|G(t, s, (z_k + \xi)(s)) - G(t, s, (z_l + \xi)(s))|_E \leqslant \tfrac{\varepsilon}{8T_n}, \text{ for all } s \in [0, T_n]. \tag{2.16}$$

It follows that for every $t \in [0, n]$, for all $k,\ l \geqslant k_0$, we have

$$\begin{aligned} |C_2 z_k(t) - C_2 z_l(t)|_E &\leqslant \int_0^{T_n} |G(t, s, (z_k + \xi)(s)) - G(t, s, (z_l + \xi)(s))|_E\, ds \\ &\quad + \int_{T_n}^{\infty} |G(t, s, (z_k + \xi)(s)) - G(t, s, (z_l + \xi)(s))|_E\, ds \\ &\leqslant \frac{3\varepsilon}{8} + \frac{2\varepsilon}{8} < \varepsilon. \end{aligned}$$

It implies that $\{C_2 z_k|_{[0,n]}(t)\}_k$ is the Cauchy sequence in the Banach E, the convegence of $\{C_2 z_k|_{[0,n]}(t)\}_k$ follows. Note that $\{C_2 z_k|_{[0,n]}(t)\}_k$ is a subsequence of $\{C_2 y_k|_{[0,n]}(t)\}_k$. Then, $(C_2\Omega)_n(t)$ is relatively compact in E.

In view of Lemma 3, $C_2(\Omega)$ is relatively compact in $C(\mathbb{R}_+; E)$. Therefore, C_2 is completely continuous. Step 3 is proved.

Step 4. Finally, we show that $\forall n \in \mathbb{N}$,

$$\lim_{|y|_n \to \infty} \tfrac{|Cy|_n}{|y|_n} = 0.$$

By the assumption (A_5), for all $t \in [0, n]$, we get

$$\begin{aligned} |Cy(t)| = |C_2 y(t)|_E &\leqslant \int_0^{\infty} |G(t, s, (y + \xi)(s))|_E\, ds \\ &\leqslant \int_0^{\infty} \omega_2(t, s) ds \leqslant \int_0^{\infty} \sup_{t\in[0,n]} \omega_2(t, s) ds < \infty. \end{aligned}$$

It follows that

$$|Cy|_n = \sup_{t\in[0,n]} |Cy(t)|_n \leqslant \int_0^{\infty} \sup_{t\in[0,n]} \omega_2(t, s) ds < \infty$$

and then

$$\lim_{|y|_n \to \infty} \frac{|Cy|_n}{|y|_n} = 0.$$

By applying Theorem A, the operator $U + C$ has a fixed point y in X. Then the system (1.1) has a solution $(x_1, x_2) = (y_1 + \xi_1, y_2 + \xi_2)$ on $\mathbb{R}_+$. Theorem 1 is proved. $\square$

3. The asymptotically stable solutions

We now consider the asymptotically stable solutions of (1.1) defined as follows.

DEFINITION 1. A function x is said to be an *asymptotically stable solution* of (1.1) if for any solution $\bar{x}$ of (1.1),

$$\lim_{t \to \infty} |x(t) - \bar{x}(t)| = 0.$$

In this section, we assume (A_1)-(A_5) hold.

By Theorem 1, the system (1.1) has a solution on $\mathbb{R}_+$. On the other hand, if x is a solution of (1.1) then, as step 1 of the proof of Theorem 1, $y = x - \xi$ satisfies (2.2). This implies that for all $t \in \mathbb{R}_+$,

$$|y(t)| \leqslant |Uy(t)| + |Cy(t)|, \tag{3.1}$$

where $Uy(t),\ Cy(t)$ as in (2.3). Using (A_1)-(A_4) and note that

$$\begin{aligned} \xi_1(t) &= p(t) + f(t, \xi_1(t), \xi_2(t)) + \int_0^t V(t, s, \xi_1(s), \xi_2(s)) ds, \\ \xi_2(t) &= q(t) + g(t, \xi_1(t), \xi_2(t)), \end{aligned}$$

we obtain for all $t \in \mathbb{R}_+$,

$$\begin{aligned} |U_1 y(t)|_E &\leqslant |f(t, y_1(t) + \xi_1(t), y_2(t) + \xi_2(t)) - f(t, \xi_1(t), \xi_2(t))|_E \\ &\quad + \int_0^t |V(t, s, y_1(s) + \xi_1(s), y_2(s) + \xi_2(s)) - V(t, s, \xi_1(s), \xi_2(s))|_E \, ds \\ &\leqslant \frac{L}{2} |y(t)| + \int_0^t \omega_1(t, s) |y(s)| \, ds, \end{aligned}$$

and

$$|U_2 y(t)|_E = |g(t, y_1(t) + \xi_1(t), y_2(t) + \xi_2(t)) - g(t, \xi_1(t), \xi_2(t))|_E \leqslant \frac{L}{2} |y(t)|.$$

It follows that

$$|Uy(t)| = |U_1 y(t)|_E + |U_2 y(t)|_E \leqslant L |y(t)| + \int_0^t \omega_1(t, s) |y(s)| \, ds, \tag{3.2}$$

Combining (3.1), (3.2) and (A_5), for all $t \in \mathbb{R}_+$,

$$|y(t)| \leqslant L|y(t)| + \int_0^t \omega_1(t,s)|y(s)|ds + \int_0^\infty \omega_2(t,s)ds.$$

It follows that

$$|y(t)| \leqslant \frac{1}{1-L}\int_0^t \omega_1(t,s)|y(s)|ds + a(t),$$

where

$$a(t) = \frac{1}{1-L}\int_0^\infty \omega_2(t,s)ds.$$

Using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, $\forall a,b \in \mathbb{R}$, we get

$$|y(t)|^2 \leqslant \frac{2}{(1-L)^2}\int_0^t \omega_1^2(t,s)ds \int_0^t |y(s)|^2 ds + 2a^2(t). \tag{3.3}$$

If we put

$$v(t) = |y(t)|^2 \quad \text{and} \quad b(t) = \frac{2}{(1-L)^2}\int_0^t \omega_1^2(t,s)ds,$$

then (3.3) is rewritten as follows

$$v(t) \leqslant b(t)\int_0^t v(s)ds + 2a^2(t). \tag{3.4}$$

By (3.4), based on classical estimates, we obtain

$$|y(t)|^2 = v(t) \leqslant 2a^2(t) + 2b(t)\int_0^t a^2(s)\exp\left(\int_s^t b(u)du\right)ds,\ \forall t \in \mathbb{R}_+. \tag{3.5}$$

Then we have the following theorem about the asymptotically stable solutions.

THEOREM 2. *Let (A_1)-(A_5) hold. If*

$$\lim_{t\to\infty}\left[a^2(t) + b(t)\int_0^t a^2(s)\exp\left(\int_s^t b(u)du\right)ds\right] = 0, \tag{3.6}$$

where

$$a(t) = \frac{1}{1-L}\int_0^\infty \omega_2(t,s)ds \quad \text{and} \quad b(t) = \frac{2}{(1-L)^2}\int_0^t \omega_1^2(t,s)ds, \tag{3.7}$$

then every solution $x = (x_1,\, x_2)$ of the system (1.1) is an asymptotically stable solution.

Furthermore,

$$\lim_{t\to\infty}|x(t) - \xi(t)| = 0,$$

where ξ is a unique solution of (2.1). □

Remark 2. Assume that there exist functions $\alpha,\ \beta_1,\ \beta_2 \in C(\mathbb{R}_+;\mathbb{R}_+),$ such that

$$\begin{cases} \omega_1(t,s) \leqslant C\alpha(t)\beta_1(s),\ \omega_2(t,s) \leqslant C\alpha(t)\beta_2(s),\ \lim\limits_{t\to\infty}\alpha(t) = 0, \\ \int_0^\infty \alpha^2(t)dt < \infty,\ \int_0^\infty \beta_1^2(s)ds < \infty, \end{cases} \tag{3.8}$$

with C always indicating a constant independent of $t,\ s.$ Then (3.6) holds. Indeed, by (3.7), (3.8), we obtain:

$$a(t) = \frac{1}{1-L}\int_0^\infty \omega_2(t,s)ds \leqslant \frac{1}{1-L}C\alpha(t)\int_0^\infty \beta_2(s)ds \leqslant C_2\alpha(t), \tag{3.9}$$

$$b(t) = \frac{2}{(1-L)^2}\int_0^t \omega_1^2(t,s)ds \leqslant \frac{2}{(1-L)^2}C\alpha^2(t)\int_0^\infty \beta_1^2(s)ds \leqslant C_1\alpha^2(t), \tag{3.10}$$

and

$$\begin{aligned} b(t)\int_0^t a^2(s)\exp\left(\int_s^t b(u)du\right)ds &\leqslant C_2^2\alpha^2(t)\int_0^t C_1\alpha^2(s)\exp\left(\int_s^t C_1\alpha^2(u)du\right)ds \\ &= C_2^2\alpha^2(t)\left[\exp\left(\int_0^t C_1\alpha^2(u)du\right) - 1\right] \\ &\leqslant C_2^2\alpha^2(t)\exp\left(\int_0^\infty C_1\alpha^2(u)du\right) \\ &\leqslant Const.\alpha^2(t). \end{aligned} \tag{3.11}$$

Hence

$$a^2(t) + b(t)\int_0^t a^2(s)\exp\left(\int_s^t b(u)du\right)ds \leqslant C\alpha^2(t) \to 0, \text{ as } t\to\infty. \tag{3.12}$$

4. An example

Let us give an illustrated example for the results obtained as above.

Let $E = C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $u : [0,1] \to \mathbb{R}$ with the norm

$$\|u\| = \sup_{0\leqslant\eta\leqslant 1} |u(\eta)|,\ u \in E.$$

Then, for all $x \in X = C(\mathbb{R}_+;E),$ for any $t \in \mathbb{R}_+,$ $x(t)$ is an element of E and we denote

$$x(t)(\eta) = x(t,\eta),\ 0 \leqslant \eta \leqslant 1.$$

Consider (1.1) in the following form:

$$\begin{cases} x_1(t) = p(t) + f(t,x_1(t),x_2(t)) + \int_0^t V(t,s,x_1(s),x_2(s))ds, \\ x_2(t) = q(t) + g(t,x_1(t),x_2(t)) + \int_0^\infty G(t,s,x_1(s),x_2(s))ds, \end{cases} \tag{4.1}$$

where $p,\ q,\ f,\ g,\ V,\ G,$ are the continuous functions given respectively as follows:

(i) function $p:\mathbb{R}_+\to E,$ defined by

$$p(t)(\eta)=p(t,\eta)=\left(\frac{-k_1-k_2+e^{-2t}}{e^t+\eta}\right)e^{-2t},\ 0\leqslant\eta\leqslant 1,\ t\geqslant 0,$$

with $k_1,\ k_2$ are given constants such that $|k_1|<1/2,\ |k_2|<1/\pi;$

(ii) function $q:\mathbb{R}_+\to E,$ defined by

$$q(t)(\eta)=q(t,\eta)=-\frac{\tilde{k}_1+\tilde{k}_2}{e^{2t}+\eta},\ 0\leqslant\eta\leqslant 1,\ t\geqslant 0,$$

with $\tilde{k}_1,\ \tilde{k}_2$ are given constant such that $|\tilde{k}_1|<1/(4\pi),\ |\tilde{k}_2|<1/2;$

(iii) function $f:\mathbb{R}_+\times E^2\to E,$ defined by

$$f(t,u_1,u_2)(\eta)=k_1e^{-2t}\left|u_1(\eta)\right|+\frac{k_2}{e^t+\eta}e^{-2t}\sin\left(\frac{\pi}{2}e^{2t}\left\|u_2\right\|\right),$$

for $0\leqslant\eta\leqslant 1,\ (t,u_1,u_2)\in\mathbb{R}_+\times E^2.$

(iv) function $g:\mathbb{R}_+\times E^2\to E,$

$$g(t,u_1,u_2)(\eta)=\frac{\tilde{k}_1}{e^{2t}+\eta}\cos\left(2\pi(e^t+\eta)u_1(\eta)\ \right)+\tilde{k}_2\left|u_2(\eta)\right|,$$

for $0\leqslant\eta\leqslant 1,(t,u_1,u_2)\in\mathbb{R}_+\times E^2.$

(v) Function $V:\Delta\times E^2\to E,\ \Delta=\{(t,s):0\leqslant s\leqslant t,\ t\geqslant 0\},$

$$V(t,s,u_1,u_2)(\eta)=\frac{4e^{-4s}}{e^t+\eta}\left[\sin\left(\pi(e^s+\eta)u_1(\eta)\right)+\sin\left(\frac{\pi}{2}(e^{2s}+\eta)u_2(\eta)\right)\ \right]$$

for $0\leqslant\eta\leqslant 1,\ (t,s,u_1,u_2)\in\Delta\times E^2.$

(vi) Function $G:\mathbb{R}_+^2\times E^2\to E,$

$$\begin{aligned}G(t,s,u_1,u_2)(\eta)=\frac{e^{-2s}}{e^{2t}+\eta}\Bigg[&\sin\left(\frac{\pi}{2}\int_0^1(e^s+\zeta)u_1(\zeta)d\zeta\right)\\&+\sin\left(\frac{\pi}{2}\int_0^1(e^{2s}+\zeta)u_2(\zeta)d\zeta\right)\Bigg],\end{aligned}$$

for $0\leqslant\eta\leqslant 1,\ (t,s,u_1,u_2)\in\mathbb{R}_+^2\times E^2.$

We will prove that (A_1)-(A_5) hold. The proofs of (A_1) is easy and hence, we omit the details. The condition (A_2) holds, since for all $u=(u_1,u_2)\in E^2,\ \tilde{u}=(\tilde{u}_1,\tilde{u}_2)\in E^2$ and $t\geqslant 0,$

$$\begin{aligned}\|f(t,u_1,u_2)-f(t,\tilde{u}_1,\tilde{u}_2)\|&\leqslant|k_1|\,\|u_1-\tilde{u}_1\|+\frac{\pi}{2}\,|k_2|\,\|u_2-\tilde{u}_2\|\\&\leqslant\frac{L}{2}\left[\|u_1-\tilde{u}_1\|+\|u_2-\tilde{u}_2\|\right]\end{aligned}$$

and

$$\begin{aligned}\|g(t,u_1,u_2)-g(t,\tilde{u}_1,\tilde{u}_2)\| &\leqslant 2\pi\left|\tilde{k}_1\right|\|u_1-\tilde{u}_1\|+\left|\tilde{k}_2\right|\|u_2-\tilde{u}_2\|\\ &\leqslant \frac{L}{2}\left[\|u_1-\tilde{u}_1\|+\|u_2-\tilde{u}_2\|\right],\end{aligned}$$

in which

$$0\leqslant \tfrac{L}{2}=\max\{|k_1|,\, \tfrac{\pi}{2}|k_2|,\, 2\pi\left|\tilde{k}_1\right|,\, \left|\tilde{k}_2\right|\}<\tfrac{1}{2}.$$

Assumption (A_3) holds because for all $(u_1,u_2)\in E^2,\ (\tilde{u}_1,\tilde{u}_2)\in E^2$ and $(t,s)\in\Delta$, $\forall\eta\in[0,1]$, we have:

$$|V(t,s,u_1,u_2)(\eta)-V(t,s,\tilde{u}_1,\tilde{u}_2)(\eta)|\leqslant \omega_1(t,s)\left[\|u_1-\tilde{u}_1\|+\|u_2-\tilde{u}_2\|\ \right]$$

in which $\omega_1(t,s)=8\pi e^{-t-2s}$.

Assumption (A_4) is also fulfilled, the proof is as below. First, we show $G:\mathbb{R}_+^2\times E^2\to E$ is continuous. For all $(t,s,u_1,u_2),\ (\tilde{t},\tilde{s},\tilde{u}_1,\tilde{u}_2)\in\mathbb{R}_+^2\times E^2$, we have

$$\begin{aligned}\|G(t,s,u_1,u_2)-G(\tilde{t},\tilde{s},\tilde{u}_1,\tilde{u}_2)\| &\leqslant 4\left[|s-\tilde{s}|+|t-\tilde{t}|\right]\\ &\quad+\frac{\pi}{2}\left[(e^s+1)\|u_1-\tilde{u}_1\|+|e^s-e^{\tilde{s}}|\|\tilde{u}_1\|\right]\\ &\quad+\frac{\pi}{2}\left[(e^{2s}+1)\|u_2-\tilde{u}_2\|+|e^{2s}-e^{2\tilde{s}}|\|\tilde{u}_2\|\right].\end{aligned}$$

Hence the continuity of G is proved. Next, we show $G:\mathbb{R}_+^2\times E^2\to E$ is compact. Let B is bounded in $\mathbb{R}_+^2\times E^2$, we have

$$\|G(t,s,u_1,u_2)\|\leqslant \omega_2(t,s)=2e^{-2t-2s}\leqslant 2\equiv M,\, \forall(t,s,u_1,u_2)\in B,$$

it implies that $G(B)$ is uniformly bounded in E. For all $\eta_1,\ \eta_2\in[0,1]$, for all $(t,s,u_1,u_2)\in B$, we have

$$\begin{aligned}|G(t,s,u_1,u_2)(\eta_1)-G(t,s,u_1,u_2)(\eta_2)| &\leqslant 2e^{-2s}\frac{|\eta_2-\eta_1|}{(e^{2t}+\eta_1)\,(e^{2t}+\eta_2)}\\ &\leqslant 2\ |\eta_2-\eta_1|,\end{aligned}$$

consequently $G(B)$ is equicontinuous. Finally, for all bounded subsets $I_1,\ I_2$ of $\mathbb{R}_+$ and for any bounded subsets J of E^2, for all $\varepsilon>0$, there exists $\delta>0$, such that

$$\forall t_1,t_2\in I_1,\, |t_1-t_2|<\delta\Longrightarrow \|G(t_1,s,u_1,u_2)-G(t_2,s,u_1,u_2)\|<\varepsilon,$$

for all $(s,u_1,u_2)\in I_2\times J$. We get the above property since

$$|G(t_1,s,u_1,u_2)(\eta)-G(t_2,s,u_1,u_2)(\eta)|\leqslant 2\tfrac{\left|e^{2t_2}-e^{2t_1}\right|}{(e^{2t_1}+\eta)(e^{2t_2}+\eta)}e^{-2s}\leqslant 4\,|t_1-t_2|,$$

$\forall\eta\in[0,1],\ \forall t_1,\ t_2\in I_1,\ \forall(s,u_1,u_2)\in I_2\times J.$

Assumption (A_5) is also clear since $\forall\eta \in [0,1],\ \forall(t,s) \in I\times\mathbb{R}_+,\ \forall(u_1,u_2)\in E^2$, we have:

$$|G(t,s,u_1,u_2)(\eta)| \leqslant \frac{2e^{-2s}}{e^{2t}+\eta} \leqslant 2e^{-2t-2s} = \omega_2(t,s),\ \int_0^\infty \sup_{t\in I}\omega_2(t,s)ds \leqslant 1 < \infty.$$

On the other hand, the condition (3.8) is satisfied. Indeed,

$$\omega_1(t,s) = 8\pi e^{-t-2s} \leqslant C\alpha(t)\beta_1(s),\ \ \omega_2(t,s) = 2e^{-2t-2s} \leqslant C\alpha(t)\beta_2(s),$$

where $\alpha(t) = e^{-t}$ and $\beta_1(s) = \beta_2(s) = e^{-2s}$, in which

$$\lim_{t\to\infty}\alpha(t) = 0,\ \int_0^\infty \alpha^2(t)dt < \infty,\ \int_0^\infty \beta_1^2(s)ds < \infty.$$

We conclude Theorems 2 holds for (4.1). For more details, we can compute to assert that $x = (x_1,x_2) : \mathbb{R}_+ \to E^2$, with

$$x_i(t)(\eta) = x_i(t,\eta) = \tfrac{1}{e^{it}+\eta},\ \forall\eta \in [0,1],\ i = 1,\ 2, \tag{4.2}$$

is the solution of (4.1). Furthermore

$$\lim_{t\to\infty}\left(\|x_1(t)\| + \|x_2(t)\|\right) = \lim_{t\to\infty}\left(e^{-t} + e^{-2t}\right) = 0.$$

So, it is clear that $\bar{x}(t) \equiv 0$ and x as in (4.2) are asymptotically stable solutions of (4.1).

5. Appendix.

1. Let $(E,\ |\cdot|)$ be a Banach space. Let $X = C(\mathbb{R}_+;E)$ be the space of all continuous functions on $\mathbb{R}_+$ to E equipped with the numerable family of seminorms

$$|x|_n = \sup_{t\in[0,n]}|x(t)|,\ n\in\mathbb{N},\ x\in X.$$

Then $(X,\ |\cdot|_n)$ is complete in the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}\frac{|x-y|_n}{1+\ |x-y|_n}$$

and X is the Fréchet space. Indeed, we need show that every Cauchy sequence $\{x_k\}$ of X converges to a point of X. We only prove that

$$\lim_{k,\ h\to\infty} d(x_k,x_h) = 0 \Longleftrightarrow \lim_{k,h\to\infty}|x_k - x_h|_n = 0, \forall n\in\mathbb{N}.$$

(i) Let $\lim\limits_{k,\ h\to\infty} d(x_k,x_h) = 0$. Fixed $n\in\mathbb{N}$, we have

$$2^{-n}\frac{|x_k-x_h|_n}{1+\ |x_k-x_h|_n} \leqslant d(x_k,x_h) \to 0.$$

So

$$2^{-n}\frac{|x_k - x_h|_n}{1+ \ |x_k - x_h|_n} \to 0.$$

Then, by the property of the function $f(x) = 2^{-n}\frac{x}{1+x}$ on $[0,\infty),$ it implies that

$$\lim_{k,\, h\to\infty} |x_k - x_h|_n = 0.$$

(ii) Let $\lim\limits_{k,\, h\to\infty} |x_k - x_h|_n = 0,\ \forall n \in \mathbb{N}.$ For all $\varepsilon > 0,$ choose $n_0 \in \mathbb{N},$ such that

$$\sum\nolimits_{n=n_0+1}^{\infty} 2^{-n} < \frac{\varepsilon}{2}.$$

With $n \in \{1,2,...,n_0\},$ since $\lim\limits_{k,\, h\to\infty} |x_k - x_h|_n = 0,$ there exists $k_n \in \mathbb{N}$ such that

$$\forall k,\, h \geqslant k_n \Longrightarrow |x_k - x_h|_n \leqslant \tfrac{\varepsilon}{2}.$$

Setting $K_\varepsilon = \max\limits_{1\leqslant n\leqslant n_0} k_n.$ Then for all $\forall k,\ h \geqslant K_\varepsilon,$

$$\begin{aligned} d(x_k,x_h) &= \sum_{n=1}^{n_0} 2^{-n}\frac{|x_k - x_h|_n}{1+ \ |x_k - x_h|_n} + \sum_{n=n_0+1}^{\infty} 2^{-n}\frac{|x_k - x_h|_n}{1+ \ |x_k - x_h|_n} \\ &\leqslant \sum_{n=1}^{n_0} 2^{-n}\,|x_k - x_h|_n + \sum_{n=n_0+1}^{\infty} 2^{-n} \\ &\leqslant \sum_{n=1}^{n_0} 2^{-n}\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = (1-2^{-n_0})\frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

So

$$\lim_{k,\, h\to\infty} d(x_k,x_h) = 0.$$

This equivalence leads to the existence of a point of X which is a limit of a Cauchy sequence $\{x_k\}_k$. Indeed, every Cauchy sequence $\{x_k\}_k$ of $X = C(\mathbb{R}_+;E),$ for all $n \in \mathbb{N},$ $\{x_k|_{[0,n]}\}$ is the Cauchy sequence in the Banach $C([0,n];E).$ So, there exists $x^n \in C([0,n];E)$ such that $\{x_k|_{[0,n]}\}_k$ converges to x^n in $C([0,n];E)$ as $k \to \infty.$ By the uniqueness of the limit, it is easy to see that

$$x^n|_{[0,m]} = x^m, \forall m = 1,2,...,\ n.$$

It follows that $\{x_k\}_k$ converges to x in X, where x is defined by

$$x(t) = x^n(t) \text{ if } t \in [0,n], \forall n \in \mathbb{N}.$$

Consequently, X is the Fréchet space.

2. Proof of Lemma 2. Since $\Phi : X \to X$ is an L_n-contraction on X with respect to a family of seminorms $\|\cdot\|_n,$ for every $n \in \mathbb{N},$ there exists $L_n \in [0,1)$ such that

$\|\Phi x - \Phi y\|_n \leqslant L_n \|x-y\|_n$. Let $x_0 \in X$ be arbitrary, we construct by recurrence the sequence $\{x_k\}$ as follows

$$x_{k+1} = \Phi x_k,\ k \in \mathbb{N}.$$

Then for every $n \in \mathbb{N}$,

$$\|x_{k+1} - x_k\|_n \leqslant L_n \|x_k - x_{k-1}\|_n, \forall k \in \mathbb{N},$$

hence we obtain

$$\begin{aligned}\left\|x_{k+p} - x_k\right\|_n &\leqslant \left\|x_{k+p} - x_{k+p-1}\right\|_n + \left\|x_{k+p-1} - x_{k+p-2}\right\|_n + \ldots + \|x_{k+1} - x_k\|_n \\ &\leqslant L_n^k \left(L_n^{p-1} + L_n^{p-2} + \ldots + 1\right) \|x_1 - x_0\|_n \\ &\leqslant \frac{L_n^k}{1-L_n} \|x_1 - x_0\|_n, \forall k, p \in \mathbb{N}.\end{aligned}$$

Consequently, for every $n \in \mathbb{N}$, for all $p \in \mathbb{N}$,

$$\lim_{k\to\infty} \left\|x_{k+p} - x_k\right\|_n = 0 \Longleftrightarrow \lim_{k\to\infty} \left|x_{k+p} - x_k\right|_n = 0,$$

because $\|\cdot\|_n$ is equivalent with $|\cdot|_n$, which means that $\{x_k\}$ is a Cauchy sequence. Since $(X,\ |\cdot|_n)$ is complete, $\{x_k\}$ converges to a point x of X. It is clearly that x is a unique fixed point of Φ. $\square$

Remark 3. In general for $(X, \|\cdot\|_n)$ and $\Phi : X \to X$, the main assumption of Lemma 2 ($\forall n \in \mathbb{N},\ \exists L_n \in [0,1)$ such that $\|\Phi x - \Phi y\|_n \leqslant L_n \|x-y\|_n,\ \forall x,\ y \in X$) does not imply the conclusion:

$$\exists L \in [0,1) \text{ such that } d(\Phi x, \Phi y) \leqslant L d(x,y),\ \forall x, y \in X, \tag{*}$$

where the metric $d(x,y)$ is generated by the family of seminorms $\|\cdot\|_n$ in the way:

$$d(x,y) = \textstyle\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

In fact, if we put $X = C(\mathbb{R}_+; E)$, $E = \mathbb{R}$, $\Phi x = \frac{1}{a}x$, for all $x \in E$, $a > 1$, since $\|\cdot\|_n$ are seminorms, we have

$$\|\Phi x - \Phi y\|_n = \tfrac{1}{a} \|x-y\|_n, \forall n \in \mathbb{N}, \forall x, y \in X.$$

Obviously, Φ defined as above satisfies the main assumption of Lemma 2 for $L_n = \frac{1}{a} \in [0,1)$. Next, let Φ satisfy (*) and let β be such that $L < \beta < 1$. Also, let $x_0,\ y_0 \in X$ be defined by

$$y_0(t) = 0 \text{ and } x_0(t) = \tfrac{a\beta}{1-\beta} \text{ for all } t \in \mathbb{R}_+.$$

It is clearly that $x_0,\ y_0 \in X$. Moreover, we claim that

$$d(\Phi x_0, \Phi y_0) > \beta d(x_0, y_0). \tag{**}$$

Indeed,

$$d(x_0,y_0) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x_0 - y_0\|_n}{1 + \ \|x_0 - y_0\|_n} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x_0\|_n}{1 + \ \|x_0\|_n}$$
$$= \frac{\frac{a\beta}{1-\beta}}{1 + \frac{a\beta}{1-\beta}} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a\beta}{a\beta - \beta + 1},$$

and

$$d(\Phi x_0,\Phi y_0) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\Phi x_0 - \Phi y_0\|_n}{1 + \|\Phi x_0 - \Phi y_0\|_n} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\left\|\frac{1}{a} x_0\right\|_n}{1 + \left\|\frac{1}{a} x_0\right\|_n}$$
$$= \frac{\frac{1}{a}\frac{a\beta}{1-\beta}}{1 + \frac{1}{a}\frac{a\beta}{1-\beta}} \sum_{n=1}^{\infty} \frac{1}{2^n} = \beta.$$

Since $\beta < 1$ we have $1 > \frac{a\beta}{a\beta - \beta + 1}$ and therefore, previous two equalities for $d(x_0,y_0)$ and $d(\Phi x_0,\Phi y_0)$ prove (**).

However, from (*) and (**) we obtain: $d(\Phi x_0,\Phi y_0) \leqslant Ld(x_0,y_0) < \frac{L}{\beta} d(\Phi x_0,\Phi y_0)$ which implies that $\beta < L$ which is not possible since β has been chosen to satisfy $L < \beta < 1$. Therefore, such Φ satisfies the main assumption of Lemma 2 but does not satisfy the conclusion (*). For more details, let see Gabor [10]. $\square$

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