Differential
Equations
& Applications
Volume 4, Number 2 (2012), 319–337

NON–SIMULTANEOUS BLOW–UP FOR A SEMILINEAR PARABOLIC SYSTEM WITH LOCALIZED REACTION TERMS

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(*Communicated by Philippe Souplet*)

Abstract. In this paper, we study positive blow-up solutions of the semilinear parabolic system with localized reactions $u_t = \Delta u + v^r + u^p(0,t)$, $v_t = \Delta v + u^s + v^q(0,t)$ in the ball $B = \{x \in \mathbb{R}^N : |x| < R\}$, under the homogeneous Dirichlet boundary condition. It is shown that non-simultaneous blow-up may occur according to the value of p, q, r, and s ($p,q,r,s > 1$). We also investigate blow-up rates of all total blow-up solutions when simultaneous blow-up occurs.

1. Introduction

In this paper, we are concerned with the initial-boundary value problem of the following semilinear parabolic system with both local and nonlocal (localized) reaction terms:

$$\text{(P)} \quad \begin{cases} u_t = \Delta u + v^r + u^p(0,t), & (x,t) \in B \times (0,T), \\ v_t = \Delta v + u^s + v^q(0,t), & (x,t) \in B \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x \in B, \end{cases}$$

where B is the ball $\{x \in \mathbb{R} : |x| < R\}$ with boundary ∂B and $p,q,r,s > 1$.

Throughout this paper, we always assume that

$$\text{(A1)} \begin{cases} u_0, v_0 \in C(\bar{B}),\ u_0(R) = v_0(R) = 0, \\ \text{for } r = |x|,\ u_0(x) = u_0(r) \geqslant 0,\ v_0(x) = v_0(r) \geqslant 0, \\ u_0(r) \text{ and } v_0(r) \text{ are nonincreasing in } r \in [0,R], \end{cases}$$

and

$$\text{(A2)} \quad \begin{cases} \Delta u_0 + (1-\varepsilon)v_0^r + (1-\varepsilon\varphi_0(x))u_0^p(0) \geqslant 0,\ x \in \bar{B}, \\ \Delta v_0 + (1-\varepsilon)u_0^s + (1-\varepsilon\varphi_0(x))v_0^q(0) \geqslant 0,\ x \in \bar{B}, \end{cases}$$

Mathematics subject classification (2010): 34K51, 35B44.
Keywords and phrases: blow-up, parabolic system, simultaneous blow-up, non-simultaneous blow-up, total blow-up.

with $\varepsilon \in (0,1)$, where $\varphi_0 \in C^2(B) \cap C(\bar{B})$ is a radially symmetric function which satisfies $\varphi_0(R) = 0$, $\varphi_0'(r) < 0$ $(r \in (0,R])$ and

$$\max_{x \in \bar{B}} |\varphi_0(x)| \leqslant 1. \tag{1.1}$$

Under assumption (A1), problem (P) has a unique pair of solutions (u,v) such that $u,v \in C^{2,1}(B \times (0,T)) \cap C(\bar{B} \times [0,T))$, where T denotes the maximal existence time of (u,v). It is well known that for large initial data (u_0,v_0), (u,v) may blow up in a finite time T, that is,

$$\lim_{t \nearrow T} (\|u(t)\|_\infty + \|v(t)\|_\infty) = \infty,$$

where $\|\cdot\|_\infty$ is the usual L^∞-norm.

We define

$$S = \{x \in \bar{B} : \text{there exists a sequence } (x_n,t_n) \in B \times (0,T) \text{ such that}$$
$$x_n \to x, t_n \nearrow T,\ u(x_n,t_n) \to \infty \ \text{ or } \ v(x_n,t_n) \to \infty \text{ as } n \to \infty\}.$$

The set S is called "the blow-up set" of (u,v) and each x of S is called a "blow-up point" of (u,v). When $S = \bar{B}$, we call this blow-up phenomena "total blow-up" and when S consists of a single point of B, we call this "single point blow-up".

We say that the blow-up is "simultaneous" if

$$\limsup_{t \nearrow T} \|u(t)\|_\infty = \limsup_{t \nearrow T} \|v(t)\|_\infty = \infty \tag{1.2}$$

and it is called "non-simultaneous" if (1.2) does not hold, that is, if one of the two component (u,v) remains bounded on $B \times [0,T)$.

System (P) can be regarded as a combination of the following two systems: $u_t = \Delta u + v^r$, $v_t = \Delta v + u^s$ and $u_t = \Delta u + u^p(0,t)$, $v_t = \Delta v + v^q(0,t)$. It is shown by Escobedo-Herrero [3] that the first system allows the simultaneous blow-up generically. Furthermore, Souplet [9] proved that the blow-up set S consists of the origin, that is, the single point blow-up occurs. The second system is completely uncoupled for which it has been proved that all blow-up solutions blow up on the whole domain, that is, the total blow-up occurs [2, 4, 8] and it is obvious that the blow-up times for u and v differ in general. Hence one may easily expect that for our system, both simultaneous and non-simultaneous blow-up may occur and moreover for the simultaneous blow-up case, the single point blow-up and the total blow-up are both likely to occur. In fact, for a slightly different system with usual reaction terms: $u_t = \Delta u + v^r + u^p$, $v_t = \Delta v + u^s + v^q$, Rossi-Souplet [7] and Souplet-Tayachi [10] showed that both simultaneous and non-simultaneous blow-up can occur.

There is also several studier devoted to more complicated local and nonlocal reaction terms such as in the following equation:

$$u_t = \Delta u + f(t) + u^p, \qquad v_t = \Delta v + g(t) + v^q. \tag{P1}$$

The case where $(f(t),g(t)) = (u^r(0,t), v^s(0,t))$ gives essentially a single equation which has been studied by Bebernes-Bressan-Lacey [1] and Okada-Fukuda [5]. They proved that the total blow-up occurs if $F_r(t) = \int_0^t u^r(0,\tau)\, d\tau = +\infty$, and the single point blow-up occurs if $F_r(t) < +\infty$. Zheng-Wang [11] studied the case where $(f(t), g(t)) = (v^r(0,t), u^s(0,t))$. They showed that both simultaneous and non-simultaneous blow-up can occur. Moreover, they proved that the single point and the total blow-up occur according to the boundedness of $F_s(t) = \int_0^t u^s(0,\tau)\, d\tau$ and $G_r(t) = \int_0^t v^r(0,\tau)\, d\tau$. In particular, the boundedness of $F_s(t)$ (or $G_r(t)$) easily leads to the fact that v (or u) blows up only at $x = 0$. However, since our system can not be reduced to the single equation such as in [1, 5], we can not make use of the results in [1, 5] directly. In this sense, our system is much more complicated than that of [11] and the mechanism for occurrences of the single point blow-up and the total blow-up would be more delicate than that for the system dealt by [11]. In fact, it is shown (in Section 4) that the boundedness of $F_p(t) = \int_0^t u^p(0,\tau)\, d\tau$ (or $G_q(t) = \int_0^t v^q(0,\tau)\, d\tau$) does not always imply the single point blow-up of u (or v). To overcome this difficulty, we introduce some devices which enable us to get minute estimates for the asymptotic behavior of $F_k(t) = \int_0^t u^k(0,\tau)\, d\tau$ and $G_\ell(t) = \int_0^t v^\ell(0,\tau)\, d\tau$ near the blow-up time.

For (P), Wang-Zheng [12] showed that there exist initial data such that non–simultaneous blow-up can occur if $r < q-1$ or $s < p-1$ (Theorem 2.1 in [12]). Also, the case where $q-1 \leqslant r$ and $p-1 \leqslant s$, they proved that there exists initial data such that u and v blow up on the whole domain if $r \leqslant p/(s+1-p)$ or $s \leqslant q/(r+1-q)$ (Theorem 3.3 in [12]), and u and v blow up only at $x = 0$ if $(p+1)/(s-p) < r$ and $(q+1)/(r-q) < s$ with $r = s$ (Theorem 4.1 in [12]).

One of main purposes of this paper is to clarify conditions on p,q,r,s to assure that only non-simultaneous blow-up can occur. The other one is to investigate necessary conditions to ensure that any blow-up solutions (u,v) blow up on the whole domain when $q-1 \leqslant r$ and $p-1 \leqslant s$, and to examine blow-up rates of solutions for the case of total blow-up. We give several improvements on the results of Wang-Zheng. In addition, a part of results in [11] can be improved by using our method developed here.

In section 2, we give some sufficient conditions in terms of p,q,r,s to ensure the non-simultaneous blow-up.

Section 3 is devoted to the analysis for the total blow-up mainly relying on Lemma 3.2-Lemma 3.3 which is the main device in this section.

In section 4, we develop another argument concerning the asymptotic behavior of $F_k(t) = \int_0^t u^k(0,\tau)\, d\tau$ and $G_\ell = \int_0^t v^\ell(0,\tau)\, d\tau$, from which we can derive another type of results for the total blow-up.

2. Non-simultaneous Blow-up

Let c and c' denote positive generic constants, which are not necessarily the same at different places. We shall use the notation $X \sim Y$ to mean that there exist two constants c' and c satisfying $c'Y \leqslant X \leqslant cY$ as long as X and Y are both defined, for the case where X,Y are functions of t, and $c'\Phi_0(x)Y \leqslant X \leqslant cY$ for the case where X,Y are functions of (x,t), where $\Phi_0(x)$ is some positive continuous function on B.

We introduce the notation $T_u < T_v$ (resp. $T_v < T_u$) to signify that u (resp. v) blows up at $t = T_u$ (resp. $t = T_v$) and v (resp. u) remains bounded on $[0,T_u)$ (resp. $[0,T_v)$), and $T_u = T_v$ means that u and v simultaneously blow up at $t = T_u = T_v$. $T_u \leqslant T_v$ (resp. $T_v \leqslant T_u$) represent $T_u < T_v$ (resp. $T_v < T_u$) or $T_u = T_v$.

The main result of this section is the following theorem which improves Theorem 2.1 of [12].

THEOREM 2.1. *For the problem (P), we have the following.*
(i) *If $s < p-1$ and $(q-1)\big(1+1/(p-1)\big) < r$, then only non-simultaneous blow-up ($T_u < T_v$) occurs. Furthermore, u blows up on the whole domain, and moreover u satisfies*

$$u(x,t) \sim (T_u - t)^{-\frac{1}{p-1}}. \tag{2.1}$$

(ii) *If $r < q-1$ and $(p-1)\big(1+1/(q-1)\big) < s$, then only non-simultaneous blow-up ($T_v < T_u$) occurs. Furthermore, v blows up on the whole domain, and moreover v satisfies*

$$v(x,t) \sim (T_v - t)^{-\frac{1}{q-1}}. \tag{2.2}$$

(iii) *Let $s < p-1$ and $r < q-1$. If $T_u = T_v$, then u and v blow up on the whole domain. Furthermore, u and v satisfy* (2.1) *and* (2.2) *with $T_u = T_v = T$, respectively.*

Let $\varphi(x,t)$ be a radially symmetric solution of the linear heat equation with zero-Dirichlet boundary condition satisfying $\varphi(x,0) = \varphi_0(x)$. From (1.1), we note that

$$\max_{(x,t)\in\bar{B}\times[0,\infty)} |\varphi(x,t)| \leqslant 1. \tag{2.3}$$

Throughout this paper, we denote that

$$F_k(t) = \int_0^t u^k(0,\tau)\, d\tau \quad \text{and} \quad G_\ell(t) = \int_0^t v^\ell(0,\tau)\, d\tau, \tag{2.4}$$

where $k,\ \ell > 1$.

We here prepare two lemmas.

LEMMA 2.2. *Let (u,v) be a solution of (P). Then*

$$\varphi(x,t)F_p(t) \leqslant u(x,t) \leqslant G_r(t) + F_p(t) + ||u_0||_\infty, \tag{2.5}$$
$$\varphi(x,t)G_q(t) \leqslant v(x,t) \leqslant F_s(t) + G_q(t) + ||v_0||_\infty, \tag{2.6}$$

for any $(x,t) \in \bar{B}\times[0,T)$.

LEMMA 2.3. *Let (u,v) be a solution of (P). Then there exists a positive constant η such that*

$$u_t \geqslant \eta\,\varphi(v^r + u^p(0,t)), \tag{2.7}$$
$$v_t \geqslant \eta\,\varphi(u^s + v^q(0,t)), \tag{2.8}$$

for any $(x,t) \in \bar{B}\times[0,T)$, where $\varphi(x,t)$ is the same function as in Lemma 2.2.

Lemma 2.2 and Lemma 2.3 can be proved by standard arguments similar to those in the proof of Lemma 2.1 in [5] and Lemma 3.1 in [12], Lemma 2.2 in [5] and Lemma 2.1 in [12] respectively. For the sake of reader's convenience, we give their proofs.

PROOF OF LEMMA 2.2 We first show the lower estimate of u of (2.5). Set $U(x,t) = u(x,t) - \varphi(x,t)F_p(t)$. Then, by (2.3), we easily get

$$U_t - \Delta U = v^r + (1-\varphi)u^p(0,t) - (\varphi_t - \Delta\varphi)F_p(t) \geqslant 0.$$

Since $\varphi(x,t)$ satisfies the zero-Dirichlet boundary condition and $F_p(0) = 0$, we get $U(x,t) = 0$ on the boundary and $U(x,0) = u_0(x) \geqslant 0$. Using the maximum principle, we obtain $U(x,t) \geqslant 0$ for $\overline{B} \times [0,T)$ which implies $\varphi(x,t)F_p(t) \leqslant u(x,t)$.

To obtain the upper estimate of u of (2.5), we set

$$V(x,t) = G_r(t) + F_p(t) + \|u_0\|_\infty - u(x,t)$$

which satisfies

$$\begin{aligned} &V_t - \Delta V = v^r(0,t) - v^r(x,t) \geqslant 0, \\ &V(x,t)|_{\partial B} = F_r(t) + G_p(t) + \|u_0\|_\infty \geqslant 0, \\ &V(x,0) = ||u_0||_\infty - u_0(x) \geqslant 0. \end{aligned}$$

Using the maximum principle again, we obtain $V(x,t) \geqslant 0$ in $\overline{B} \times [0,T)$ which implies $u(x,t) \leqslant G_r(t) + F_p(t) + \|u_0\|_\infty$.

The estimate for v can be proved by the same arguments as above with u, $F_p(t)$ and $G_r(t)$ replaced by v, $G_q(t)$ and $F_s(t)$ respectively. □

PROOF OF LEMMA 2.3 We set

$$I(x,t) = u_t - \eta\varphi(v^r + u^p(0,t)) \quad \text{and} \quad J(x,t) = v_t - \eta\varphi(u^s + v^q(0,t)),$$

where $\eta > 0$ is a small constant which is determined later.

By simple calculations, we get

$$\begin{aligned} I_t - \Delta I - rv^{r-1}J = {} & p(1-\eta\varphi)u_t(0,t)u^{p-1}(0,t) \\ & + 2\eta r(\nabla\varphi\cdot\nabla v)v^{r-1} + \eta r(r-1)\varphi|\nabla v|^2 v^{r-2}. \end{aligned}$$

The maximum principle together with (A1) implies that $u_t(0,t) \geqslant 0$. We choose η so small that $1 - \eta\varphi(x,t) \geqslant 0$. Since both u and φ are radially symmetric and monotone decreasing with respect to $r = |x|$, we have $\nabla\varphi\cdot\nabla v \geqslant 0$. Consequently, we find that

$$I_t - \Delta I - rv^{r-1}J \geqslant 0,$$

and in the same way as above, we can also show

$$J_t - \Delta J - su^{s-1}I \geqslant 0.$$

Since $\varphi = u = v = 0$ on the boundary, we have $I(x,t) = J(x,t) = 0$ on $\partial B \times [0,T)$.

Using (A2) and (1.1), we get

$$I(x,0) \geqslant (\varepsilon-\eta)(v_0^r(x)+\varphi_0(x)u_0^p(0)),$$
$$J(x,0) \geqslant (\varepsilon-\eta)(u_0^s(x)+\varphi_0(x)v_0^q(0)).$$

We choose again η so small that $\varepsilon-\eta \geqslant 0$. Hence, I and J are nonnegative by the maximum principle, which yields (2.7) and (2.8). □

By Lemma 2.3, if u and v blow up at $t = T(u)$ and $t = T(v)$ respectively, the standard argument on ODE such as in [6] assures

$$u(0,t) \leqslant c_1(T(u)-t)^{-\frac{1}{p-1}}, \tag{2.9}$$

$$v(0,t) \leqslant c_2(T(v)-t)^{-\frac{1}{q-1}}. \tag{2.10}$$

By virtue of (A1), it is easily seen that $u(x,t) \leqslant u(0,t)$, $v(x,t) \leqslant v(0,t)$, which yield $\Delta u(0,t) \leqslant 0$, $\Delta v(0,t) \leqslant 0$. Using (P) and (2.7)-(2.8) with $x=0$, we get

$$u_t(0,t) \sim v^r(0,t)+u^p(0,t), \tag{2.11}$$
$$v_t(0,t) \sim u^s(0,t)+v^q(0,t). \tag{2.12}$$

Integration of (2.11)-(2.12) over $[0,t]$ gives

$$u(0,t) \sim G_r(t)+F_p(t), \tag{2.13}$$
$$v(0,t) \sim F_s(t)+G_q(t). \tag{2.14}$$

These facts are frequently quoted in this paper.

LEMMA 2.4. *Let (u,v) be a solution of (P).*
(i) *If $T_u < T_v$, then u blows up on the whole domain. Furthermore, u satisfies*

$$u(x,t) \sim (T_u-t)^{-\frac{1}{p-1}}. \tag{2.15}$$

(ii) *If $T_v < T_u$, then v blows up on the whole domain. Furthermore, v satisfies*

$$v(x,t) \sim (T_v-t)^{-\frac{1}{q-1}}. \tag{2.16}$$

Proof. Case (i) Since u blows up and v remains bounded by the assumption, (2.5) implies $u(x,t) \sim F_p(t)$ near $t = T_u$. Hence we can show

$$F_p(t) \sim (T_u-t)^{-\frac{1}{p-1}}.$$

In fact, since $u(x,t) \sim F_p(t)$ gives $c'F_p^p(t) \leqslant u^p(0,t) \leqslant cF_p^p(t)$, integrating this on $[t_0,t]$, we get

$$c'\int_{t_0}^t F_p^p(\tau)\,d\tau \leqslant F_p(t)-F_p(t_0) \leqslant c\int_{t_0}^t F_p^p(\tau)\,d\tau.$$

Then, by Corollary 2.5 and 2.6 of [6], we can conclude $F_p(t) \sim c(T_u - t)^{-\frac{1}{p-1}}$, which implies that u blows up on the whole domain. This completes the proof of (i).

The proof of (ii) can be done similarly.

PROOF OF THEOREM 2.1 To prove (i) by contradiction, assume $T_u = T_v = T$. Since $F_s(t)$ is bounded when $s < p-1$ by (2.9), $v(0,t) \sim G_q(t)$ follows from (2.14). Hence, as in the proof of Lemma 2.4, we get

$$v(0,t) \sim G_q(t) \sim (T-t)^{-\frac{1}{q-1}}$$

which yields

$$G_r(t) \sim (T-t)^{-\frac{r+1-q}{q-1}}. \tag{2.17}$$

Using (2.9) and (2.13), we get

$$cG_r(t) \leqslant u(0,t) \leqslant (T-t)^{-\frac{1}{p-1}}. \tag{2.18}$$

On the other hand, if $(q-1)\big(1+1/(p-1)\big) < r$, then

$$(T-t)^{-\frac{1}{p-1}} < (T-t)^{-\frac{r+1-q}{q-1}}.$$

Hence, we have a contradiction.

By Lemma 2.4-(ii), if we assume $T_v < T_u$, then $G_r(t)$ satisfies (2.17) with $T = T_v$ by (2.16). From this fact and (2.18), we get a contradiction.

The proof of (ii) is done by the same arguments as that of (i) with obvious modifications.

As for (iii), recall that $F_s(t)$ and $G_r(t)$ are bounded when $s < p-1$ and $r < q-1$ by (2.9) and (2.10) respectively, then we get $u(x,t) \sim F_p(t)$ and $v(x,t) \sim G_q(t)$ by using (2.5) and (2.6), whence follows (iii). □

REMARK 2.5. Since solution (u,v) of (P1) with $(f(t),g(t)) = (v^r(0,t),u^s(0,t))$ denoted by $(P1)_{r,s}$ also satisfies properties (2.9)-(2.14), we can easily see that (i) and (ii) in Theorem 2.1 holds true also for $(P1)_{r,s}$, that is, Theorem 2.2-(i) and Theorem 2.3-(i) in [11] are improved.

The existence of simultaneous blow-up solutions is open for the cases:

(i) $s < p-1$ and $q-1 \leqslant r \leqslant (q-1)\big(1+\frac{1}{p-1}\big)$,

(ii) $r < q-1$ and $p-1 \leqslant s \leqslant (p-1)\big(1+\frac{1}{q-1}\big)$.

3. Total Blow-up

The main purpose of this section is to investigate total blow-up and blow-up rates for all simultaneous blow-up solutions. We always assume that $p-1 \leqslant s$ and $q-1 \leqslant r$. Since the contraposition of Theorem 2.1-(i) (Theorem 2.1-(ii)) in [12] implies that

$T_v \leqslant T_u$ $(T_u \leqslant T_v)$ holds if $p-1 \leqslant s$ $(q-1 \leqslant r)$, the simultaneous blow-up always occurs when both $p-1 \leqslant s$ and $q-1 \leqslant r$ are satisfied, so we here denote $T_u = T_v = T$.

The main theorem in this section is the following theorem which improves a part of Theorem 3.3 of [12].

THEOREM 3.1. *Let (u,v) be a solution of (P). Then we have the following.*
(i) *If*

$$q-1<r \quad and \quad (p-1)\Big(1+\frac{1}{q-1}\Big)<s \leqslant \frac{q}{r+1-q},$$

or

$$q-1=r \quad and \quad (p-1)\Big(1+\frac{1}{q-1}\Big)<s,$$

then u and v blow up on the whole domain. Moreover, u satisfies

$$u(x,t) \geqslant \begin{cases} c'(T-t)^{-\frac{r+1-q}{q-1}}, & q-1<r, \\ c'\log(T-t)^{-1}, & r=q-1 \end{cases} \tag{3.1}$$

on any compact subsets of B, and v satisfies

$$v(x,t) \sim G_q(t) \sim (T-t)^{-\frac{1}{q-1}}.$$

(ii) *If*

$$p-1<s \quad and \quad (q-1)\Big(1+\frac{1}{p-1}\Big)<r \leqslant \frac{p}{s+1-p},$$

or

$$p-1=s \quad and \quad (q-1)\Big(1+\frac{1}{p-1}\Big)<r,$$

then u and v blow up on the whole domain. Moreover, u satisfies

$$v(x,t) \geqslant \begin{cases} c(T-t)^{-\frac{s+1-p}{p-1}}, \; p-1<s, \\ c\log(T-t)^{-1}, \; s=p-1 \end{cases} \tag{3.2}$$

on any compact subsets of B, and u satisfies

$$u(x,t) \sim F_p(t) \sim (T-t)^{-\frac{1}{p-1}}.$$

We shall make use of the following lemma due to Wang-Zheng [12]. This lemma gives the blow-up rates of solutions of (P1).

LEMMA 3.2. *Let (u,v) be a solution of (P) on $[0,T]$.*
(i) *If $p-1<s$, then there exists some positive constant C such that*

$$u^{\alpha}(0,t) \leqslant Cv(0,t), \;\; t \in (0,T),$$

where $\alpha = \min(s+1-p,\ \frac{1+s}{1+r},\ \frac{s}{q})$.

(ii) *If* $q-1<r$*, then there exists some positive constant* C *such that*

$$v^{\beta}(0,t) \leqslant Cu(0,t),\ \ t \in (0,T),$$

where $\beta = \min(r+1-q,\ \frac{1+r}{1+s},\ \frac{r}{p})$.

In order to show total blow-up, we shall make use of the blow-up of $F_p(t)$ and $\int_0^t F_p^s(\tau)\,d\tau$, or $G_q(t)$ and $\int_0^t G_q^r(\tau)\,d\tau$. To do this, we first recall the relation given in the left hand side of (2.5) and (2.6), i.e.,

$$u(x,t) \geqslant \varphi(x,t)F_p(t), \tag{3.3}$$

$$v(x,t) \geqslant \varphi(x,t)G_q(t). \tag{3.4}$$

Substituting (3.3)-(3.4) in (2.7)-(2.8), we get

$$u_t(x,t) \geqslant \eta\varphi(x,t)v^r(x,t) \geqslant \eta\varphi^{1+r}(x,T)G_q^r(t), \tag{3.5}$$

$$v_t(x,t) \geqslant \eta\varphi(x,t)u^s(x,t) \geqslant \eta\varphi^{1+s}(x,T)F_p^s(t). \tag{3.6}$$

Next we integrate (3.5) and (3.6) over $[0,t]$ to get

$$u(x,t) \geqslant \eta\varphi^{1+r}(x,T)\int_0^t G_q^r(\tau)\,d\tau, \tag{3.7}$$

$$v(x,t) \geqslant \eta\varphi^{1+s}(x,T)\int_0^t F_p^s(\tau)\,d\tau. \tag{3.8}$$

Now it is evident from (3.3) and (3.8) that blow-up of $F_p(t)$ and $\int_0^t F_p^s(\tau)\,d\tau$ lead to total blow-up of u and v. Similarly, blow-up of $G_q(t)$ and $\int_0^t G_q^r(\tau)\,d\tau$ lead to total blow-up of u and v by (3.4) and (3.7). Therefore, we have the following lemma which is the key in this section.

LEMMA 3.3. *Let* (u,v) *be a solution of (P) on* $[0,T]$.

(i) *If* $\int_0^t F_p^s(\tau)\,d\tau = +\infty$*, then* u *and* v *blow up on the whole domain.*

(ii) *If* $\int_0^t G_q^r(\tau)\,d\tau = +\infty$*, then* u *and* v *blow up on the whole domain.*

By making use of Lemma 3.2, we get the following blow-up rates of solutions.

LEMMA 3.4. *Let*

$$p-1 < s \leqslant (p-1)\Big(1+\frac{1}{q-1}\Big) \quad \text{and} \quad q-1 \leqslant r < \frac{p}{s+1-p}.$$

(i) *If* $p-1<s<(p-1)\big(1+1/(q-1)\big)$*, then there exists a positive constant* c *such that*

$$v(0,t) \geqslant c(T-t)^{-\frac{s+1-p}{p-1}}. \tag{3.9}$$

Furthermore, if $s=(p-1)\big(1+1/(q-1)\big)$*, then*

$$v(x,t)\sim G_q(t)\sim (T-t)^{-\frac{1}{q-1}}. \tag{3.10}$$

(ii) *If* $(p+1)\big(1+1/q\big)\leqslant s<(p-1)\big(1+1/(q-1)\big)$*, then there exists a positive constant* c *such that*

$$G_q(t)\geqslant\begin{cases} c(T-t)^{-\frac{q}{p-1}\left\{s-(p-1)\left(1+\frac{1}{q}\right)\right\}}, & (p-1)\big(1+\frac{1}{q}\big)<s<(p-1)\big(1+\frac{1}{q-1}\big),\\ c\log(T-t)^{-1}, & s=(p-1)\big(1+\frac{1}{q}\big). \end{cases} \tag{3.11}$$

Proof. Case (i). By assumption, α in Lemma 3.2-(i) is given by $\alpha=s+1-p$. Then we have $u^s(0,t)\leqslant cv^{\frac{s}{\alpha}}(0,t)$, which together with (2.12) yields

$$v_t(0,t)\leqslant cv^{\frac{s}{\alpha}}(0,t). \tag{3.12}$$

Hence, by Corollary 2.5 of [6], we obtain (3.9).

Since $u^s(0,t)\leqslant cv^q(0,t)$ if $s=(p-1)\big(1+1/(q-1)\big)$, we get $v_t(0,t)\leqslant cv^q(0,t)$ by using (2.12). Then, this fact and (2.12) conduce to

$$v_t(0,t)\sim v^q(0,t),$$

from which we obtain $v(0,t)\sim(T-t)^{-\frac{1}{q-1}}$ and $v(0,t)\sim G_q(t)$. Hence, in of the fact that $v(x,t)\leqslant v(0,t)$ and (3.4), we can derive (3.10).

Case (ii). The property (3.11) is derived from (3.9) via integration. □

Exchanging the roles of u and v in Lemma 3.4, we have the following lemma:

LEMMA 3.5. *Let* p*,* q*,* r*, and* s *satisfy*

$$q-1<r\leqslant(q-1)(1+\frac{1}{p-1})\quad and\quad p-1\leqslant s<\frac{q}{r+1-q}.$$

(i) *If* $r<(q-1)\big(1+1/(p-1)\big)$*, then there exists a positive constant* c *such that*

$$u(0,t)\geqslant c(T-t)^{-\frac{r+1-q}{q-1}}. \tag{3.13}$$

Furthermore, if $r=(q-1)\big(1+1/(p-1)\big)$*, then*

$$u(x,t)\sim F_p(t)\sim(T-t)^{-\frac{1}{p-1}}. \tag{3.14}$$

(ii) *If* $(q-1)\big(1+1/p\big)\leqslant r<(q-1)\big(1+1/(p-1)\big)$*, then there exists a positive constant* c *such that*

$$F_p(t)\geqslant\begin{cases} c(T-t)^{-\frac{p}{q-1}\left\{r-(q-1)(1+\frac{1}{p})\right\}}, & (q-1)\big(1+\frac{1}{p}\big)<r<(q-1)\big(1+\frac{1}{p-1}\big),\\ c\log(T-t)^{-1}, & r=(q-1)\big(1+\frac{1}{p}\big). \end{cases} \tag{3.15}$$

These lemmas give the following theorem which improves a part of Theorem 3.1 of [12].

THEOREM 3.6. *Let* p, q, r, *and* s *satisfy:* $q-1<r\leqslant(q-1)\big(1+1/(p-1)\big)$ *and* $p-1<s\leqslant(p-1)\big(1+1/(q-1)\big)$.
(i) *If*

$$(q-1)\Big(1+\frac{1}{p}\Big)\leqslant r<(q-1)\Big(1+\frac{1}{p-1}\Big),$$
$$(p-1)\Big(1+\frac{1}{q}\Big)\leqslant s<(p-1)\Big(1+\frac{1}{q-1}\Big),$$

then u *and* v *blow up on the whole domain. Furthermore,* u *and* v *are estimated from below such as* (3.15) *and* (3.11)*, respectively.*
(ii) *If* $r=(q-1)\big(1+1/(p-1)\big)$ *or* $s=(p-1)\big(1+1/(q-1)\big)$, *then* u *and* v *blow up on the whole domain. Furthermore, when* $r=(q-1)\big(1+1/(p-1)\big)$, u *satisfies* (3.14) *and* v *is estimated from below such as* (3.9)*, when* $s=(p-1)\big(1+1/(q-1)\big)$, v *satisfies* (3.10) *and* u *is estimated from below such as* (3.13)*.*

Proof. The first assertion (i) is derived directly from Lemma 3.4-(ii) and Lemma 3.5-(ii). The second assertion (ii) can be proved by substituting (3.10) and (3.14) in (3.7) and (3.8) respectively. □

Using again Lemma 3.2, we have the following lemma:

LEMMA 3.7. *If* $q-1<r$ *and* $(p-1)\big(1+1/(q-1)\big)<s\leqslant q/(r+1-q)$, *or* $q-1=r$ *and* $(p-1)\big(1+1/(q-1)\big)<s$, *then*

$$v(x,t)\sim G_q(t)\sim(T-t)^{-\frac{1}{q-1}}. \tag{3.16}$$

Proof. Since $\alpha=\frac{s}{q}$ in Lemma 3.2-(i) by the assumption, we have $u^s(0,t)\leqslant cv^q(0,t)$, from which we obtain $v_t(0,t)\sim v^q(0,t)$. Hence, (3.14) can be derived from arguments similar to those for (3.10). □

Exchanging the roles of u and v in Lemma 3.7, we have the following lemma:

LEMMA 3.8. *If* $p-1<s$ *and* $(q-1)\big(1+1/(p-1)\big)<r\leqslant p/(s+1-p)$, *or* $p-1=s$ *and* $(q-1)\big(1+1/(p-1)\big)<r$, *then*

$$u(x,t)\sim F_p(t)\sim(T-t)^{-\frac{1}{p-1}}. \tag{3.17}$$

From Lemma 3.7 and Lemma 3.8, we can prove Theorem 3.1 which improves a part of Theorem 3.3 of [12].

PROOF OF THEOREM 3.1 To show (i), it suffices to prove that u blows up on the whole domain, since it is obvious that v blows up on the whole domain by (3.16). Substituting (3.16) in (3.7), we obtain (3.1), which means that u blows up on the whole domain.

The proof of (ii) can be done similarly. □

REMARK 3.9. For the blow-up rate of u (resp. v) from above, when s (resp. r) is restricted to $s<(q-1)/(r+1-q)$ (resp. $r<(p-1)/(s+1-p)$), then $u(x,t)$ (resp. $v(x,t)$) is estimated from above by same order to the right hand side of (3.1) (resp. (3.2)) (see Lemma 4.2-(ii) and Lemma 4.4-(ii)).

4. Total Blow-up II

In this section, we discuss the total blow-up for the case where (p,q,r,s) lie in the region where results in §3 can not cover. In the previous section, we dealt with the case where $F_p(t)$ and $\int_0^t F_p^s(\tau)\,d\tau$ (or $G_q(t)$ and $\int_0^t G_q^r(\tau)\,d\tau$) blow up at $t=T$. However we here deal with the case where $F_p(t)$ or $G_q(t)$ might be bounded.

Throughout this section, without loss of generality, we let $p\leqslant q$.

The main result in this section is stated as follows.

THEOREM 4.1. *For the problem (P), we have the following.*

(i) *Let* $q-1\leqslant r\leqslant (q-1)\big(1+1/(p-1)\big)$ *and* $p-1\leqslant s<(p-1)(1+1/q)$. *If* $r\leqslant \big(1+1/p\big)(p-1)/(s+1-p)$, *then all blow-up solutions blow up on the whole domain. In particular, if*

$$\lim_{t\nearrow T} G_q(t)<+\infty \ \textit{(resp. } \lim_{t\nearrow T} G_q(t)=+\infty\textit{)},$$

then u *and* v *satisfy*

$$u(x,t)\sim F_p(t)\sim (T-t)^{-\frac{1}{p-1}},$$

$$v(x,t)\sim F_s(t)\sim \begin{cases} (T-t)^{-\frac{s+1-p}{p-1}}, & p-1<s<(p-1)(1+\frac{1}{q}),\\ \log(T-t)^{-1}, & s=p-1.\end{cases}$$

$$\left(\begin{array}{c} \textit{resp. } v(x,t)\sim G_q(t)\sim (T-t)^{-\frac{1}{q-1}},\quad cG_r(t)\leqslant u(x,t)\leqslant c'(T-t)^{-\frac{1}{p-1}} \\ \textit{where } G_r(t) \textit{ satisfies}\quad G_r(t)\sim \begin{cases} (T-t)^{-\frac{r+1-q}{q-1}}, & q-1<r,\\ \log(T-t)^{-1}, & q-1=r.\end{cases}\end{array}\right)$$

(ii) *If* $p-1\leqslant s\leqslant (p-1)/\big(1+1/(q-1)\big)$ *and* $q-1\leqslant r<(q-1)\big(1+1/p\big)$, *then all blow-up solutions blow up on the whole domain. In particular, if*

$$\lim_{t\nearrow T} F_p(t)<+\infty \ \textit{(resp. } \lim_{t\nearrow T} F_p(t)=+\infty\textit{)},$$

then u *and* v *satisfy*

$$v(x,t)\sim G_q(t)\sim (T-t)^{-\frac{1}{q-1}},$$

$$u(x,t)\sim G_r(t)\sim \begin{cases} (T-t)^{-\frac{r+1-q}{q-1}}, & q-1<r<(q-1)(1+\frac{1}{p}),\\ \log(T-t)^{-1}, & r=q-1.\end{cases}$$

$$\left(\begin{array}{c} resp.\ u(x,t) \sim F_p(t) \sim (T-t)^{-\frac{1}{p-1}}, \quad cF_s(t) \leqslant v(x,t) \leqslant c'(T-t)^{-\frac{1}{q-1}} \\ where\ F_s(t)\ satisfies \quad F_s(t) \sim \begin{cases} (T-t)^{-\frac{s+1-p}{p-1}}, \ p-1<s, \\ \log(T-t)^{-1}, \ \ p-1=s. \end{cases} \end{array} \right)$$

We first analyze the case where $G_q(t)$ or $G_r(t)$ might be bounded.

LEMMA 4.2. *Let* $p-1 \leqslant s < (p-1)\big(1+1/q\big)$.

(i) *If* $(q-1)\big(1+1/(p-1)\big) < r$, *then* $\lim_{t \nearrow T} G_q(t) < +\infty$.

(ii) *If*

$$p-1<s \quad and \quad (q-1)\Big(1+\frac{1}{p-1}\Big) < r < \frac{p-1}{s+1-p},$$

or

$$p-1=s \quad and \quad (q-1)\Big(1+\frac{1}{p-1}\Big) < r,$$

then $\lim_{t \nearrow T} G_r(t) < +\infty$. *Moreover, u and v satisfy*

$$u(x,t) \sim F_p(t) \sim (T-t)^{-\frac{1}{p-1}}, \tag{4.1}$$

$$v(x,t) \sim F_s(t) \sim \begin{cases} (T-t)^{-\frac{s+1-p}{p-1}}, \ p-1<s<(p-1)(1+\frac{1}{q}), \\ \log(T-t)^{-1}, \ \ s=p-1. \end{cases} \tag{4.2}$$

(iii) *Let* $q-1 \leqslant r \leqslant (q-1)\big(1+1/(p-1)\big)$.

(a) *If* $\lim_{t \nearrow T} G_q(t) = +\infty$, *then*

$$v(x,t) \sim G_q(t) \sim (T-t)^{-\frac{1}{q-1}}, \qquad cG_r(t) \leqslant u(x,t) \leqslant c'(T-t)^{-\frac{1}{p-1}}, \tag{4.3}$$

where $G_r(t)$ satisfies

$$G_r(t) \sim \begin{cases} (T-t)^{-\frac{r+1-q}{q-1}}, \ q-1<r, \\ \log(T-t)^{-1}, \ \ q-1=r. \end{cases}$$

(b) *If* $\lim_{t \nearrow T} G_q(t) < +\infty$ *and* $r \leqslant (p-1)/(s+1-p)$, *then* (4.1) *and* (4.2) *hold.*

Proof. Case (i) By (2.14), we have

$$G_q(t) \sim \int_0^t F_s^q(\tau)\, d\tau + \int_0^t G_q^q(\tau)\, d\tau.$$

The first term of the right hand side of the above is bounded using (2.9) when $p-1 \leqslant s < (p-1)\big(1+1/q\big)$, which yields

$$G_q(t) \sim \int_0^t G_q^q(\tau)\, d\tau. \tag{4.4}$$

Let $G_q(t)$ blow up as t goes to T. We solve (4.4) and get

$$\int_0^t G_q^q(\tau)\, d\tau \sim (T-t)^{-\frac{1}{q-1}}$$

which implies $G_q(t) \sim (T-t)^{-\frac{1}{q-1}}$. Using (2.10) and (3.4), we have

$$v(x,t) \sim G_q(t) \sim (T-t)^{-\frac{1}{q-1}}. \tag{4.5}$$

By (4.5), we can get the growth order of $G_r(t)$ and use (3.7) which deduce that

$$u(x,t) \geqslant c\varphi^{1+r}(x,T)G_r(t) \sim \begin{cases} (T-t)^{-\frac{r+1-q}{q-1}}, \; q-1<r, \\ \log(T-t)^{-1}, \;\; q-1=r. \end{cases} \tag{4.6}$$

From (2.9) and (4.6), it follows that

$$c'(T-t)^{-\frac{r+1-q}{q-1}} \leqslant u(x,t) \leqslant c(T-t)^{-\frac{1}{p-1}}.$$

On the other hand, $(T-t)^{-\frac{1}{p-1}} < (T-t)^{-\frac{r+1-q}{q-1}}$ since $(q-1)\big(1+1/(p-1)\big) < r$, which leads to a contradiction. Hence, we obtain the boundedness of $G_q(t)$.

Case (ii) Since the boundedness of $G_q(t)$ is assured by the assumption and (i), we have

$$v(0,t) \sim F_s(t) \tag{4.7}$$

by (2.14). Using (2.9) and (4.7), we get

$$G_r(t) \leqslant \begin{cases} Const., & r < \frac{p-1}{s+1-p}, \\ c\log(T-t)^{-1}, & r = \frac{p-1}{s+1-p}, \end{cases} \tag{4.8}$$

when $p-1<s$. Then the first half part of (ii) is derived. To obtain the boundedness of $G_r(t)$ for the case where $p-1=s$, it suffices to note that

$$v(0,t) \sim F_s(t) \leqslant c\log(T-t)^{-1}.$$

Since $G_r(t)$ is bounded, we have $u(0,t) \sim F_p(t)$ by (2.13). Hence we get (4.1). Furthermore, we have

$$\int_0^t F_p^s(\tau)\, d\tau \sim F_s(t) \sim \begin{cases} (T-t)^{-\frac{s+1-p}{p-1}}, \; p-1<s, \\ \log(T-t)^{-1}, \;\; p-1=s. \end{cases}$$

Hence, (3.8) and (4.7) imply (4.2).

Case (iii) The assertion (a) can be easily verified by the first half part of the proof for (i). As for the assertion (b), we notice that

$$p \leqslant q \quad \text{implies} \quad q \leqslant (q-1)\Big(1+\frac{1}{p-1}\Big).$$

For the case where $r \leqslant q$, the boundedness of $G_r(t)$ follows from the boundedness of $G_q(t)$. Hence, the statement of (b) for the case where $r \leqslant q$ is proved by the same arguments as those for (ii). For the case where $q < r < (p-1)/(s+1-p)$, since $G_r(t)$ is bounded by (4.8), we can get (4.1) -(4.2) from arguments similar to those for (ii). As for the case where $q < r = (p-1)/(s+1-p)$, by the boundedness of $G_q(t)$, (2.14) and (4.8), we have

$$G_r(t) \sim \int_0^t F_s^r(\tau)\, d\tau \leqslant c\log(T-t)^{-1}.$$

From this and (2.13), it follows that $F_p(t) \sim \int_0^t F_p^p(\tau)\, d\tau$ which yields

$$F_p(t) \sim (T-t)^{-\frac{1}{p-1}},$$

so we get (4.1). Since

$$G_r(t) \leqslant c\log(T-t)^{-1} \leqslant F_s(t) \sim \int_0^t F_p^s(\tau)\, d\tau$$

by (4.1) and (4.8). Hence, using (2.14) and (3.8), we obtain (4.2). □

REMARK 4.3. If $p = q$, we can prove (iii)-(b) without the assumption $r \leqslant \frac{p-1}{s+1-p}$, since $q = (q-1)\big(1+1/(p-1)\big)$ assures $r \leqslant q$.

Exchanging the roles of u and v, the following lemma holds.

LEMMA 4.4. *Let* $q-1 \leqslant r < (q-1)(1+1/p)$.
(i) *If* $(p-1)\big(1+1/(q-1)\big) < s$, *then* $\lim\limits_{t \nearrow T} F_p(t) < +\infty$.

(ii) *If*

$$q-1 < r \quad \textit{and} \quad (p-1)\Big(1+\frac{1}{q-1}\Big) < s < \frac{q-1}{r+1-q},$$

or

$$q-1 = r \quad \textit{and} \quad (p-1)\Big(1+\frac{1}{q-1}\Big) < s,$$

then $\lim\limits_{t \nearrow T} F_s(t) < +\infty$. *Moreover, u and v satisfy*

$$v(x,t) \sim G_q(t) \sim (T-t)^{-\frac{1}{q-1}}, \tag{4.9}$$

$$u(x,t) \sim G_r(t) \sim \begin{cases} (T-t)^{-\frac{r+1-q}{q-1}}, \, q-1 < r < (q-1)\big(1+\frac{1}{p}\big), \\ \log(T-t)^{-1}, \ \ r = q-1. \end{cases} \tag{4.10}$$

(iii) *Let* $p-1 \leqslant s \leqslant (p-1)\big(1+1/(q-1)\big)$.

(a) *If* $\lim\limits_{t\nearrow T} F_p(t) = +\infty$, *then*

$$u(x,t) \sim F_p(t) \sim (T-t)^{-\frac{1}{p-1}}, \qquad cF_s(t) \leqslant v(x,t) \leqslant c'(T-t)^{-\frac{1}{q-1}} \tag{4.11}$$

where $F_s(t)$ *satisfies*

$$F_s(t) \sim \begin{cases} (T-t)^{-\frac{s+1-p}{p-1}}, \, p-1 < s, \\ \log(T-t)^{-1}, \ \ p-1 = s. \end{cases}$$

(b) *If* $\lim\limits_{t\nearrow T} F_p(t) < +\infty$, *then* (4.9) *and* (4.10) *hold.*

Proof. Lemma 4.4 can be proved by much the same arguments as in the proof of Lemma 4.2. In particular, since $p \leqslant q$ leads to $s \leqslant (p-1)\big(1+1/(q-1)\big) < p$, the verification of (iii)-(b) is easier than the case for Lemma 4.2. □

The functions $G_r(t)$, $F_p(t)$ and $F_s(t)$ blow up when

$$s < (p-1)\Big(1+\frac{1}{q}\Big) \quad \text{and} \quad \frac{p-1}{s+1-p} < r \leqslant (q-1)\Big(1+\frac{1}{p-1}\Big),$$

by (3.9), Lemma 3.5-(ii) and (3.13), respectively. As for $G_q(t)$, however, it is not known whether it blows up or not. If $G_q(t)$ blows up in this region, then all blow-up solutions (u,v) blow up on the whole domain such that (4.3) is satisfied. On the other hand, by using Lemma 3.2 in [12], we can easily prove the existence of the initial data so that $G_q(t)$ remains bounded. For such a case, the following lemma gives blow-up rates of solutions.

LEMMA 4.5. *We suppose that* p,q,r,s *satisfy*

$$\frac{p-1}{s+p-1} < r \leqslant \Big(1+\frac{1}{p}\Big)\frac{p-1}{s+1-p} \quad \textit{and} \quad p-1 < s < (p-1)\Big(1+\frac{1}{q}\Big),$$

and $\lim\limits_{t\nearrow T} G_q(t) < \infty$. *Then* u *and* v *blow up on the whole domain. Moreover,* u *and* v *satisfy* (4.1)-(4.2).

Proof. Using (2.9) and (3.9), we get

$$F_s(t) \leqslant c(T-t)^{-\frac{s+1-p}{p-1}} \leqslant v(0,t).$$

Combining (2.14) with the boundedness of $G_q(t)$, we get

$$v(0,t) \sim F_s(t) \sim (T-t)^{-\frac{s+1-p}{p-1}}$$

which yields

$$G_r(t) \sim \int_0^t F_s^r(\tau)\,d\tau \sim (T-t)^{-\frac{s+1-p}{p-1}\left(r-\frac{p-1}{s+1-p}\right)}, \quad \frac{p-1}{s+1-p} < r. \tag{4.12}$$

On the other hand, (2.13) implies

$$F_p(t) \sim \int_0^t G_r^p(\tau)\,d\tau + \int_0^t F_p^p(\tau)\,d\tau.$$

Using (4.12), we obtain

$$\int_0^t G_r^p(\tau)\,d\tau \sim \begin{cases} Const., & r < (1+\frac{1}{p})\frac{p-1}{s+1-p}, \\ \log(T-t)^{-1}, & r = (1+\frac{1}{p})\frac{p-1}{s+1-p}. \end{cases}$$

Hence, if

$$r < \left(1+\frac{1}{p}\right)\frac{p-1}{s+1-p},$$

then it follows that

$$F_p(t) \sim \int_0^t F_p^p(\tau)\,d\tau \sim (T-t)^{-\frac{1}{p-1}}. \tag{4.13}$$

By (2.9), (4.13) and (3.3), we get

$$u(x,t) \sim F_p(t) \sim (T-t)^{-\frac{1}{p-1}} \tag{4.14}$$

which yields that $F_s(t) \sim \int_0^t F_p^s(\tau)\,d\tau$. Here using (3.8) and the boundedness of $G_q(t)$, we get

$$v(x,t) \sim F_s(t) \sim (T-t)^{-\frac{s+1-p}{p-1}}. \tag{4.15}$$

Hence, the assertion can be proved by (4.14) and (4.15).

As for the case where

$$r = \left(1+\frac{1}{p}\right)\frac{p-1}{s+1-p},$$

we have

$$F_p(t) \sim \log(T-t)^{-1} + \int_0^t F_p^p(\tau)\,d\tau.$$

If

$$\int_0^t F_p^p(\tau)\,d\tau \leqslant \log(T-t)^{-1},$$

then

$$F_s(t) < F_p(t) \sim \log(T-t)^{-1},$$

since $s \leqslant (p-1)(1+\frac{1}{q-1}) < p$. From this fact and (4.12), we get

$$G_r(t) \sim \int_0^t F_s^r(\tau)\,d\tau < \infty$$

which contradicts (4.12). Hence, it follows that $\log(T-t)^{-1} < \int_0^t F_p^p(\tau)\, d\tau$, which implies

$$F_p(t) \sim \int_0^t F_p^p(\tau)\, d\tau.$$

Since $F_p(t)$ blows up when $t \to T$, we can repeat the same arguments as for the case where $r < (1+\frac{1}{p})\frac{p-1}{s+1-p}$. □

REMARK 4.6. It is clear that Lemmas 4.2, 4.4 and 4.5 hold true also for $(P1)_{r,s}$, and these lemmas are quite useful in analyzing the blow-up phenomena for $(P1)_{r,s}$. For example, in Theorem 4.2 and 4.4 of [11], it is shown that under suitable assumptions on p,q,r,s, it is always possible to construct examples of solutions such that one component of (u,v) blow up at a single point and the blow-up of the other component is total blow-up. By applying lemmas above, we can show that under almost the same (a little bit stranger) assumptions on p,q,r,s as those of Theorems 4.2 and 4.4 of [11], the behavior of all blow-up solutions of $(P1)_{r,s}$ are the same one mentioned above (i.e., the single point blow-up and the total blow-up appear at the same time).

REMARK 4.7. As for Lemma 4.1-(iii)-(b) and Lemma 4.5 (resp. Lemma 4.3-(iii)-(b)), the existence of the initial data (u_0, v_0) so that

$$\lim_{t \nearrow T} G_q(t) < +\infty \ \text{(resp. } \lim_{t \nearrow T} F_p(t) < +\infty)$$

can be proved by using Lemma 3.2 in [12]. However, the existence of the initial data (u_0, v_0) so that $\lim_{t \nearrow T} G_q(t) = +\infty$ (resp. $\lim_{t \nearrow T} F_p(t) = +\infty$) is still open.

Proof of Theorem 4.1 Lemma 4.2-(iii), Lemma 4.4-(iii) and Lemma 4.5 assure that u and v blow up on the whole domain. Hence, these lemmas assure the results of Theorem 4.1. □

Acknowledgements. The author wishes to express his cordial thanks to Professor Mitsuharu Ôtani for his valuable criticism and unceasing encouragement.

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(Received August 25, 2011)

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