Differential
Equations
& Applications
Volume 4, Number 3 (2012), 339–370 doi:10.7153/dea-04-20

OSCILLATION AND NONOSCILLATION CRITERIA FOR EVEN ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

AKIHITO SHIBUYA

(*Communicated by Norio Yoshida*)

Abstract. This paper is devoted to the study of the oscillatory and nonoscillatory behavior of even order nonlinear functional differential equations with deviating argument of the type

$$\left(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn} x^{(n)}(t)\right)^{(n)} + q(t)|x(g(t))|^{\beta}\operatorname{sgn} x(g(t)) = 0. \qquad (\mathrm{A_g})$$

1. Introduction

The objective of this paper is to study the oscillatory and nonoscillatory behavior of nonlinear functional differential equations with deviating arguments of the type

$$\left(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn} x^{(n)}(t)\right)^{(n)} + q(t)|x(g(t))|^{\beta}\operatorname{sgn} x(g(t)) = 0, \qquad (\mathrm{A_g})$$

where the following conditions are assumed to hold:

(a) α and β are positive constants;

(b) $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geqq 0$;

(c) $p(t)$ satisfies

$$\int_a^{\infty} \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty; \qquad (1.1)$$

(d) $g(t)$ is a positive continuously differentiable function on $[a,\infty)$ such that $g'(t) > 0$ and $\lim_{t\to\infty} g(t) = \infty$.

Equation $(\mathrm{A_g})$ is said to be *half-linear*, *super-half-linear* or *sub-half-linear* according to whether $\alpha = \beta$, $\alpha < \beta$ or $\alpha > \beta$.

By a solution of $(\mathrm{A_g})$ we mean a function $x : [T_x,\infty) \to \mathbb{R}$ which is n times continuously differentiable together with $p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn} x^{(n)}(t)$ and satisfies $(\mathrm{A_g})$ at all sufficiently large t. Our attention will be restricted to those solutions $x(t)$ of $(\mathrm{A_g})$ which are nontrivial in the sense that $\sup\{|x(t)| : t \geqq T\} > 0$ for any $T \geqq T_x$. A solution is said to be oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$;

Mathematics subject classification (2010): 34C10, 26A12.
Keywords and phrases: even order nonlinear differential equations, oscillation, nonoscillation, regularly varying.

Paper DEA-04-20

otherwise a solution is said to be nonoscillatory. By definition, equation ($\mathrm{A_g}$) is oscillatory if all of its solutions are oscillatory and nonoscillatory otherwise.

In Section 2 we first classify the set of all possible nonoscillatory solutions of ($\mathrm{A_g}$) into a finite number of subclasses according to their asymptotic behavior as $t \to \infty$, and then derive integral equations for the subclasses appearing in the classification list. Such integral equations will play a crucial role in the subsequent discussions. We notice that, since if $x(t)$ satisfies ($\mathrm{A_g}$), then so does $-x(t)$, it suffices for us to restrict our attention to eventually positive solutions of ($\mathrm{A_g}$).

Let P denote the set of all eventually positive solutions of ($\mathrm{A_g}$). We introduce the set of $2n$ functions $\varphi_j(t)$, $j \in \{0,1,\cdots,2n-1\}$ defined by

$$\begin{aligned} &\varphi_j(t) = (t-a)^j, \quad j \in \{0,1,\cdots,n\}; \\ &\varphi_j(t) = \int_a^t (t-s)^{n-1}\left[\frac{(s-a)^{j-n}}{p(s)}\right]^{\frac{1}{\alpha}} ds, \;\; j \in \{n+1,n+2,\cdots,2n-1\}. \end{aligned} \tag{1.2}$$

It is easy to see that these functions are particular solutions of the unperturbed ordinary differential equation

$$L_{2n}x(t) = \left(p(t)|x^{(n)}(t)|^{\alpha}\mathrm{sgn}\, x^{(n)}(t)\right)^{(n)} = 0 \tag{1.3}$$

and satisfy

$$\lim_{t\to\infty} \frac{\varphi_{j+1}(t)}{\varphi_j(t)} = \infty \quad (j = 0,\ 1,\cdots,2n-2). \tag{1.4}$$

We denote by $P(\mathrm{I}_j)$, $j \in \{0,1,\cdots,2n-1\}$, the subclass consisting of positive solutions $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{\varphi_j(t)} = \text{const} > 0, \tag{1.5}$$

and by $P(\mathrm{II}_k)$, $k \in \{1,3,\cdots,2n-1\}$, the subclass consisting of positive solutions $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{\varphi_{k-1}(t)} = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{\varphi_k(t)} = 0, \tag{1.6}$$

and use the notation

$$P(\mathrm{I}) = P(\mathrm{I}_0) \cup P(\mathrm{I}_1) \cup \cdots \cup P(\mathrm{I}_{2n-1}), \tag{1.7}$$

and

$$P(\mathrm{II}) = P(\mathrm{II}_1) \cup P(\mathrm{II}_3) \cup \cdots \cup P(\mathrm{II}_{2n-1}). \tag{1.8}$$

It can be shown that any positive solution of ($\mathrm{A_g}$) is a member of $P(\mathrm{I})$ or of $P(\mathrm{II})$, that is, $P = P(\mathrm{I}) \cup P(\mathrm{II})$. In Section 3 the integral equations derived for each subclass of $P(\mathrm{I})$ and $P(\mathrm{II})$ are solved by means of fixed point techniques to establish necessary and sufficient conditions for the existence of positive solutions belonging to $P(\mathrm{I})$ on the one hand, and sufficient conditions for the existence of solutions belonging to $P(\mathrm{II})$ on the other. Section 4 is devoted to the derivation of the comparison principles which relate the oscillation (or nonoscillation) of functional differential equation ($\mathrm{A_g}$) to that of suitably associated differential equations with or without functional arguments. In Section 5 using the comparison principles of Section 4 effectively, we establish criteria for oscillation of all solutions of equation ($\mathrm{A_g}$) for the super-half-linear and sub-half-linear cases. Extensive use is made of known oscillation results for the companion ordinary differential equation

$$\left(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t)\right)^{(n)}+q(t)|x(t)|^{\beta}\operatorname{sgn}x(t)=0. \tag{A}$$

The obtained oscillation criteria are shown to be sharp for restricted classes of the function $p(t)$ and the deviating argument $g(t)$. An example illustrating our main results will be presented in Section 6.

The oscillation theory of higher order nonlinear differential equations of Emden-Fowler type was initiated by Kiguradze [8, 9] and have had a great impact upon the development of the qualitative theory of ordinary differential equations until today. See e.g. Kiguradze and Chanturia [10]. The study of oscillation of higher order nonlinear functional differential equations with deviating arguments was attempted for the first time by Onose [28, 29]. A typical generalization of Onose's oscillation theorem can be found in [11]. Recently, wide attention of the researchers has been attracted to the investigation of oscillation (or nonoscillation) of differential equations whose principal differential operators involve nonlinear Sturm-Liouville type differential operators [12, 20, 22, 30–34]. The present work was motivated by the observation that little analysis has been made of functional differential equations involving the differential operator $(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t))^{(n)}$ from the viewpoint of oscillation.

2. Integral representation of positive solutions

We begin by examining the structure of the set of P of all possible positive solutions of equation ($\mathrm{A_g}$).

A) *Classification of positive solutions.* Let $x(t)$ be an eventually positive solution of ($\mathrm{A_g}$). We want to determine the asymptotic behavior of $x(t)$ as $t\to\infty$. For this purpose we need to know how the "quasi-derivatives"of $x(t)$:

$$L_jx(t)=x^{(j)}(t),\quad j=0,1,\cdots,n,$$
$$L_jx(t)=\left(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t)\right)^{(j-n)},\quad j=n+1,n+2,\cdots,2n$$

behave as $t\to\infty$. Note that

$$L_{j+1}x(t)=(L_jx(t))',\quad j=0,1,\cdots,n-2,n,n+1,\cdots,2n-1;$$
$$L_nx(t)=p(t)|(L_{n-1}x(t))'|^{\alpha}\operatorname{sgn}(L_{n-1}x(t))'.$$

It can be shown that all the quasi-derivatives of a positive solution $x(t)$ of ($\mathrm{A_g}$) are eventually of constant sign and that the possible combinations of their signs are determined by a simple law as stated in the following lemma.

LEMMA 2.1. *If $x(t)$ is a positive solution of* $(\mathrm{A_g})$ *on* $[T,\infty)$, $T \geqq a$, *then there exists an odd integer* $k \in \{1,3,\cdots,2n-1\}$ *and* $T \geqq a$ *such that*

$$L_j x(t) > 0,\ t \geqq T,\ 0 \leqq j \leqq k-1; \quad (-1)^{j-k} L_j x(t) > 0,\ t \geqq T,\ k \leqq j \leqq 2n-1. \tag{2.1}$$

For the proof of Lemma 2.1 see Tanigawa in [32, Lemma 2.1]. This lemma is often referred to as the generalized Kiguradze lemma.

The set of all positive solutions $x(t)$ of $(\mathrm{A_g})$ satisfying (2.1) is denoted by P_k. The above lemma asserts that P has the decomposition

$$P = P_1 \cup P_3 \cup \cdots \cup P_{2n-1}. \tag{2.2}$$

Let $x(t) \in P$. All quasi-derivatives $L_j x(t)$, $j = 0,1,\cdots,2n-1$, are eventually monotone and hence have finite or infinite limits as $t \to \infty$, that is,

$$w_j = \lim_{t\to\infty} L_j x(t) \in [0,\infty],\ j = 0,1,\cdots,2n-1. \tag{2.3}$$

Let $x(t) \in P_k$ for some $k \in \{1,3,\cdots,2n-1\}$. Then, w_k is a finite non-negative constant and the set of its asymptotic values $\{w_j\}$ falls into one of the following three cases:

$$w_1 = w_2 = \cdots = w_{k-1} = \infty,\ w_k \in (0,\infty), \qquad w_{k+1} = w_{k+2} = \cdots = w_{2n-1} = 0; \tag{2.4}$$

$$w_1 = w_2 = \cdots = w_{k-1} = \infty, \qquad w_k = w_{k+1} = \cdots = w_{2n-1} = 0; \tag{2.5}$$

$$w_1 = w_2 = \cdots = w_{k-2} = \infty,\ w_{k-1} \in (0,\infty), \qquad w_k = w_{k+1} = \cdots = w_{2n-1} = 0. \tag{2.6}$$

It is easily verified (cf. Tanigawa [32]) that (2.4), (2.5) and (2.6) are equivalent, respectively, to

(i) $\displaystyle\lim_{t\to\infty} \frac{x(t)}{\varphi_k(t)} = \text{const} > 0;$

(ii) $\displaystyle\lim_{t\to\infty} \frac{x(t)}{\varphi_k(t)} = 0,\ \ \lim_{t\to\infty} \frac{x(t)}{\varphi_{k-1}(t)} = \infty;$

and

(iii) $\displaystyle\lim_{t\to\infty} \frac{x(t)}{\varphi_{k-1}(t)} = \text{const} > 0,$

where the functions $\varphi_j(t)$, $j = 0,1,\cdots,2n-1$, are defined by (1.2). Taking into account that the above cases (i) and (iii) are of the same category, we conclude that any positive solution $x(t)$ of equation $(\mathrm{A_g})$ satisfies either (1.5) for some $j \in \{0,1,\cdots,2n-1\}$, or (1.6) for some $k \in \{1,3,\cdots,2n-1\}$, which means that it is natural to decompose the set P of all positive solutions of $(\mathrm{A_g})$ into the two classes $P(\mathrm{I})$ and $P(\mathrm{II})$ defined by (1.7) and (1.8).

We note that any function $x(t) \in P_k$ satisfies the inequality

$$c_k \varphi_{k-1}(t) \leqq x(t) \leqq C_k \varphi_k(t) \quad \text{for all large } t \tag{2.7}$$

for some positive constants c_k and C_k.

B) *Integral representation for positive solutions.* Let us now form the explicit integral equations for positive solutions of $(\mathrm{A_g})$ belonging to $P(\mathrm{I}_j)$, $j \in \{0,1,\cdots,2n-1\}$, and $P(\mathrm{II}_k)$, $k \in \{1,3,\cdots,2n-1\}$. Solving the integral equations will be the subject of the next section.

We first derive the integral representation for a solution $x(t) \in P(\mathrm{I}_j)$, $j \in \{0,1,\cdots,2n-1\}$. Suppose that $x(t) > 0$ and $x(g(t)) > 0$ on $[t_0,\infty)$, $t_0 \geqq a$. Suppose that $j \in \{n,n+1,\cdots,2n-1\}$. In this case, integrating $(\mathrm{A_g})$ first $2n-j$ times from t to ∞ and then j times from t_0 to t, we obtain the following expressions for $x(t)$.

(i) If $j \in \{n+1,n+2,\cdots,2n-1\}$, then

$$x(t) = \xi(t) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{\xi_j(s) + (-1)^{2n-j-1}\int_{t_0}^{s}\frac{(s-r)^{j-n-1}}{(j-n-1)!}\right.\right.$$
$$\left.\left.\times \int_r^{\infty}\frac{(\sigma-r)^{2n-j-1}}{(2n-j-1)!}q(\sigma)x(g(\sigma))^{\beta}\,d\sigma dr\right\}\right]^{\frac{1}{\alpha}} ds. \quad (2.8)$$

(ii) If $j = n$, then

$$x(t) = \xi(t) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{w_n\right.\right.$$
$$\left.\left.+(-1)^{n-1}\int_s^{\infty}\frac{(r-s)^{n-1}}{(n-1)!}q(r)x(g(r))^{\beta}\,dr\right\}\right]^{\frac{1}{\alpha}} ds, \quad (2.9)$$

where

$$\xi_j(t) = \sum_{l=n}^{j-1} L_l x(t_0)\frac{(t-t_0)^{l-n}}{(l-n)!} + w_j\frac{(t-t_0)^{j-n}}{(j-n)!} \quad (n+1 \leqq j \leqq 2n-1);$$

$$\xi(t) = \sum_{l=0}^{n-1} L_l x(t_0)\frac{(t-t_0)^{l}}{l!}.$$

Next, suppose that $j \in \{0,1,\cdots,n-1\}$. Then, integrating $(\mathrm{A_g})$ first $2n-j(=n+(n-j))$ times from t to ∞ and then j times from t_0 to t yields the following expressions for $x(t)$:

(i) If $j \in \{1,2,\cdots,n-1\}$, then

$$x(t) = \xi_j^*(t) + (-1)^{2n-j-1}\int_{t_0}^{t}\frac{(t-s)^{j-1}}{(j-1)!}\int_s^{\infty}\frac{(r-s)^{n-j-1}}{(n-j-1)!}$$
$$\times\left[\frac{1}{p(r)}\int_r^{\infty}\frac{(\sigma-r)^{n-1}}{(n-1)!}q(\sigma)x(g(\sigma))^{\beta}\,d\sigma\right]^{\frac{1}{\alpha}} drds, \quad (2.10)$$

where

$$\xi_j^*(t) = \sum_{l=0}^{j-1} L_l x(t_0)\frac{(t-t_0)^{l}}{l!} + w_j\frac{(t-t_0)^{j}}{j!} \quad (1 \leqq j \leqq n-1).$$

(ii) If $j=0$, then

$$x(t)=w_0+(-1)^{2n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\times\left[\frac{1}{p(s)}\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!}q(r)x(g(r))^\beta dr\right]^{\frac{1}{\alpha}}ds. \tag{2.11}$$

To obtain integral representations for members $x(t)$ of $P(\mathrm{II}_k)$, $k\in\{1,3,\cdots,2n-1\}$ we note that the asymptotic values $\{w_j\}$ of $x(t)$ satisfies (2.5), implying in particular that $w_k=0$, and integrate $(\mathrm{A_g})$ $2n-k$ times on $[t,\infty)$ and then k times on $[t_0,\infty)$. As a result we find that:
(i) if $n+1\leqq k\leqq 2n-1$, then

$$x(t)=\xi(t)+\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{\widetilde{\xi}_k(s)+\int_{t_0}^s \frac{(s-r)^{k-n-1}}{(k-n-1)!}\times\int_r^\infty \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!}q(\sigma)x(g(\sigma))^\beta d\sigma dr\right\}\right]^{\frac{1}{\alpha}}ds, \tag{2.12}$$

(ii) if $k=n$, n being odd, then

$$x(t)=\xi(t)+\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!}q(r)x(g(r))^\beta dr\right\}\right]^{\frac{1}{\alpha}}ds, \tag{2.13}$$

(iii) if $1\leqq k\leqq n-1$, then

$$x(t)=\xi_k^*(t)+\int_{t_0}^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!}\times\left[\frac{1}{p(r)}\int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!}q(\sigma)x(g(\sigma))^\beta d\sigma\right]^{\frac{1}{\alpha}}drds, \tag{2.14}$$

for $t\geqq t_0$, where the function $\xi(t)$, $\widetilde{\xi}_k(t)$ and $\xi_k^*(t)$ are defined, respectively, by

$$\xi(t)=\sum_{l=0}^{n-1}L_l x(t_0)\frac{(t-t_0)^l}{l!},$$
$$\widetilde{\xi}_k(t)=\sum_{l=n}^{k-1}L_l x(t_0)\frac{(t-t_0)^{l-n}}{(l-n)!},$$
$$\xi_k^*(t)=\sum_{l=0}^{k-1}L_l x(t_0)\frac{(t-t_0)^l}{l!}.$$

3. Existence of positive solutions

It is natural to ask whether one can actually verify the existence of positive solutions of equation $(\mathrm{A_g})$ belonging to $P(\mathrm{I})$ or to $P(\mathrm{II})$. The aim of this section is to answer this question in the affirmative by presenting necessary and sufficient conditions for $(\mathrm{A_g})$ to have members of $P(\mathrm{I}_j)$, $j \in \{0,1,\cdots,2n-1\}$, and sufficient conditions for $(\mathrm{A_g})$ to have members of $P(\mathrm{II}_k)$, $k \in \{1,3,\cdots,2n-1\}$. Integral representations (2.8)-(2.14) for positive solutions of $(\mathrm{A_g})$ obtained in the preceding section will be employed for this purpose.

THEOREM 3.1. *A necessary and sufficient condition for equation* $(\mathrm{A_g})$ *to possess a solution in* $P(\mathrm{I}_j)$, $j \in \{0,1,\cdots,2n-1\}$, *is that*

$$\int_b^\infty t^{n-j-1}\left[\frac{1}{p(t)}\int_t^\infty (s-t)^{n-1}q(s)\varphi_j(g(s))^\beta ds\right]^{\frac{1}{\alpha}} dt < \infty \quad \textit{for} \;\; j=0,1,\cdots,n-1; \tag{3.1}$$

$$\int_b^\infty t^{2n-j-1}q(t)\varphi_j(g(t))^\beta dt < \infty \quad \textit{for} \;\; j=n,n+1,\cdots,2n-1, \tag{3.2}$$

where $b \geqq a$ *is a constant such that* $g(t) \geqq a$ *for* $t \geqq b$.

Proof. (The necessity part) Assume that $x(t) \in P(\mathrm{I}_j)$ for some $j \in \{0,1,\cdots,2n-1\}$. Then, it satisfies one of (2.8), (2.9), (2.10) and (2.11) for all large t, say $t \geqq t_0$, from which it readily follows that

$$\begin{cases} \displaystyle\int_{t_0}^\infty t^{n-j-1}\left[\frac{1}{p(t)}\int_t^\infty (s-t)^{n-1}q(s)x(g(s))^\beta ds\right]^{\frac{1}{\alpha}} dt < \infty & \text{for} \;\; j=0,1,\cdots,n-1; \\ \displaystyle\int_{t_0}^\infty t^{2n-j-1}q(t)x(g(t))^\beta dt < \infty & \text{for} \;\; j=n,n+1,\cdots,2n-1. \end{cases} \tag{3.3}$$

Combining (3.3) with the obvious inequality

$$c_j\varphi_j(t) \leqq x(t) \leqq C_j\varphi_j(t), \quad t \geqq t_0,$$

where c_j and C_j are constants, we conclude that (3.1) or (3.2) holds according to whether $j \in \{0,1,\cdots,n-1\}$ or $j \in \{n,n+1,\cdots,2n-1\}$.

(The sufficiency part) We first consider the case where $j \in \{n,n+1,\cdots,2n-1\}$. Suppose that (3.2) is satisfied. Let $c>0$ be an arbitrarily fixed constant and choose $t_0 \geqq b$ so that

$$\int_{t_0}^\infty \frac{t^{2n-j-1}}{(2n-j-1)!}q(t)\varphi_j(g(t))^\beta dt \leqq A[(j-n)!]^{\frac{\beta}{\alpha}}[(n-1)!]^\beta c^{1-\frac{\beta}{\alpha}}, \tag{3.4}$$

where A is a constant such that $A = 2^{-\frac{\beta}{\alpha}}$ if j is odd and $A = 2^{-1}$ if j is even. We may assume that $\inf\limits_{t \geqq t_0} g(t) \geqq \max\{a,1\}$ and put $T = \min\{t_0, \inf\limits_{t\geqq t_0} g(t)\}$. Define the constants

k_1 and k_2 by

$$k_i = \frac{c_i}{[(j-n)!]^{\frac{1}{\alpha}}(n-1)!}, \quad i=1,2, \tag{3.5}$$

where $c_1 = c^{\frac{1}{\alpha}}$ and $c_2 = (2c)^{\frac{1}{\alpha}}$ if j is odd, and that $c_1 = (c/2)^{\frac{1}{\alpha}}$ and $c_2 = c^{\frac{1}{\alpha}}$ if j is even. We define $\widetilde{\varphi}_j(t)$ by

$$\widetilde{\varphi}_j(t) = \begin{cases} \varphi_j(t;t_0), & t \geqq t_0; \\ 0, & t \leqq t_0, \end{cases} \tag{3.6}$$

where

$$\varphi_j(t;t_0) = (t-t_0)^j, \ \ j \in \{0,1,\cdots,n\};$$
$$\varphi_j(t;t_0) = \int_{t_0}^t (t-s)^{n-1}\left[\frac{(s-t_0)^{j-n}}{p(s)}\right]^{\frac{1}{\alpha}} ds, \ \ j \in \{n+1,n+2,\cdots,2n-1\}$$

and let X denote the set

$$X = \{x \in C[T,\infty): \ k_1\widetilde{\varphi}_j(t) \leqq x(t) \leqq k_2\widetilde{\varphi}_j(t), \ t \geqq T\}. \tag{3.7}$$

Clearly, X is a closed convex subset of the Frechét space $C[T,\infty)$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Consider the integral operator $\mathscr{F}_j : X \to C[T,\infty)$ defined as follows: if $j \in \{n+1,n+2,\cdots,2n-1\}$, then

$$\begin{cases} \mathscr{F}_j x(t) = \displaystyle\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{\frac{c(s-t_0)^{j-n}}{(j-n)!} + (-1)^{2n-j-1} \times \right.\right. \\ \qquad\qquad \left.\left. \times \displaystyle\int_{t_0}^s \frac{(s-r)^{j-n-1}}{(j-n-1)!}\int_r^\infty \frac{(\sigma-r)^{2n-j-1}}{(2n-j-1)!} q(\sigma)x(g(\sigma))^\beta d\sigma dr\right\}\right]^{\frac{1}{\alpha}} ds, \ \ t \geqq t_0; \\ \mathscr{F}_j x(t) = 0, \ \ T \leqq t \leqq t_0, \end{cases} \tag{3.8}$$

and if $j=n$, then

$$\begin{cases} \mathscr{F}_n x(t) = \displaystyle\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{c + (-1)^{n-1} \times \right.\right. \\ \qquad\qquad \left.\left. \times \displaystyle\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r)x(g(r))^\beta dr\right\}\right]^{\frac{1}{\alpha}} ds, \ \ t \geqq t_0; \\ \mathscr{F}_n x(t) = 0, \ \ T \leqq t \leqq t_0. \end{cases} \tag{3.9}$$

It can be shown that $\mathscr{F}_j$ is a continuous self-map on X and sends X into a relatively compact subset of $C[T,\infty)$.

(i) $\mathscr{F}_j$ maps X into itself. Let $x(t) \in X$. Using (3.4) and (3.5), we see that if j is odd, then

$$\mathscr{F}_j x(t) \geqq \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\frac{c(s-t_0)}{(j-n)!}\right]^{\frac{1}{\alpha}} ds = k_1\widetilde{\varphi}_j(t), \ \ t \geqq t_0,$$

and

$$\mathscr{F}_j x(t) \leqq \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ \frac{c(s-t_0)}{(j-n)!} + \int_{t_0}^{s} \frac{(s-r)^{j-n-1}}{(j-n-1)!} \right.\right.$$

$$\left.\left. \times \int_{r}^{\infty} \frac{(\sigma-r)^{2n-j-1}}{(2n-j-1)!} q(\sigma)(k_2 \widetilde{\varphi}_j(g(\sigma)))^{\beta} d\sigma dr \right\} \right]^{\frac{1}{\alpha}} ds$$

$$\leqq \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ \frac{c(s-t_0)^{j-n}}{(j-n)!} \right.\right.$$

$$\left.\left. + k_2^{\beta} A[(j-n)!]^{\frac{\beta}{\alpha}} [(n-1)!]^{\beta} c^{1-\frac{\beta}{\alpha}} \int_{t_0}^{s} \frac{(s-r)^{j-n-1}}{(j-n-1)!} dr \right\} \right]^{\frac{1}{\alpha}} ds$$

$$\leqq \frac{(2c)^{\frac{1}{\alpha}}}{[(j-n)!]^{\frac{1}{\alpha}}(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} \left[\frac{(s-t_0)^{j-n}}{p(s)} \right]^{\frac{1}{\alpha}} ds = k_2 \widetilde{\varphi}_j(t), \quad t \geqq t_0,$$

while if j is even, then we obtain

$$\mathscr{F}_j x(t) \leqq \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \frac{c(s-t_0)^{j-n}}{(j-n)!} \right]^{\frac{1}{\alpha}} ds = k_2 \widetilde{\varphi}_j(t), \ \ t \geqq t_0,$$

and

$$\mathscr{F}_j x(t) \geqq \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ \frac{c(s-t_0)^{j-n}}{(j-n)!} \right.\right.$$

$$\left.\left. - k_2^{\beta} A[(j-n)!]^{\frac{\beta}{\alpha}} [(n-1)!]^{\beta} c^{1-\frac{\beta}{\alpha}} \int_{t_0}^{s} \frac{(s-r)^{j-n-1}}{(j-n-1)!} dr \right\} \right]^{\frac{1}{\alpha}} ds$$

$$\geqq \left(\frac{c}{2} \right)^{\frac{1}{\alpha}} \frac{1}{[(j-n)!]^{\frac{1}{\alpha}}(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} \left[\frac{(s-t_0)^{j-n}}{p(s)} \right]^{\frac{1}{\alpha}} ds$$

$$= k_1 \widetilde{\varphi}_j(t), \ \ t \geqq t_0.$$

This shows that $\mathscr{F}_j x(t) \in X$, implying that $\mathscr{F}(X) \subset X$.

(ii) $\mathscr{F}_j$ is a continuous mapping. Let $\{x_m(t)\}$ be a sequence in X converging to $x(t) \in X$ as $m \to \infty$ uniformly on compact subintervals of $[T,\infty)$. We need to prove that $\lim_{m\to\infty} \mathscr{F}_j x_m(t) = \mathscr{F}_j x(t)$ uniformly on any compact subinterval of $[T,\infty)$.

The proof of this convergence for the case where $j \in \{n+1, n+2, \cdots, 2n-1\}$ proceeds as follows. Define the functions $Q_m(t)$ and $Q(t)$ by

$$Q_m(t) = \frac{c(t-t_0)^{j-n}}{(j-n)!} + (-1)^{2n-j-1} \int_{t_0}^{t} \frac{(t-s)^{j-n-1}}{(j-n-1)!}$$

$$\times \int_{s}^{\infty} \frac{(r-s)^{2n-j-1}}{(2n-j-1)!} q(r) x_m(g(r))^{\beta} dr ds, \ \ t \geqq t_0, \quad (3.10)$$

and

$$Q(t) = \frac{c(t-t_0)^{j-n}}{(j-n)!} + (-1)^{2n-j-1}\int_{t_0}^{t}\frac{(t-s)^{j-n-1}}{(j-n-1)!} \times \int_{s}^{\infty}\frac{(r-s)^{2n-j-1}}{(2n-j-1)!}q(r)x(g(r))^{\beta}\,drds, \quad t \geqq t_0. \tag{3.11}$$

A simple calculation gives

$$Q_m(t) \leqq K(t-t_0)^{j-n} \quad \text{and} \quad Q(t) \leqq K(t-t_0)^{j-n}, \quad t \geqq t_0, \tag{3.12}$$

for some $K > 0$ independent of m. The inequality

$$\begin{aligned} |Q_m(t) - Q(t)| &\leqq \int_{t_0}^{t}\frac{(t-s)^{j-n-1}}{(j-n-1)!}\int_{s}^{\infty}\frac{(r-s)^{2n-j-1}}{(2n-j-1)!}q(r)|x_m(g(r))^{\beta} - x(g(r))^{\beta}|\,drds \\ &\leqq \frac{(t-t_0)^{j-n}}{(j-n)!}\int_{t_0}^{\infty}\frac{(r-t_0)^{2n-j-1}}{(2n-j-1)!}q(r)|x_m(g(r))^{\beta} - x(g(r))^{\beta}|\,dr, \end{aligned} \tag{3.13}$$

where $t \geqq t_0$, combined with the fact that

$$|x_m(g(t))^{\beta} - x(g(t))^{\beta}| \leqq 2k_2^{\beta}\widetilde{\varphi}_j(t) \quad \text{and} \quad |x_m(g(t))^{\beta} - x(g(t))^{\beta}| \to 0 \text{ as } m \to \infty$$

implies via the Lebesgue dominated convergence theorem that $Q_m(t) \to Q(t)$ as $m \to \infty$ uniformly on any compact subinterval of $[T,\infty)$. We now combine the inequality

$$\begin{aligned} |\mathscr{F}_j x_m(t) - \mathscr{F}_j x(t)| &\leqq \int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\frac{1}{p(s)^{\frac{1}{\alpha}}}|Q_m(s)^{\frac{1}{\alpha}} - Q(s)^{\frac{1}{\alpha}}|\,ds \\ &\leqq \frac{\widetilde{\varphi}_n(t)}{(n-1)!}\max_{t_0 \leqq s \leqq t}|Q_m(s)^{\frac{1}{\alpha}} - Q(s)^{\frac{1}{\alpha}}|, \; t \geqq t_0, \end{aligned} \tag{3.14}$$

with the inequalities

$$|Q_m(t)^{\frac{1}{\alpha}} - Q(t)^{\frac{1}{\alpha}}| \leqq \frac{1}{\alpha}[K(t-t_0)^{j-n}]^{\frac{1}{\alpha}-1}|Q_m(t) - Q(t)| \quad \text{for } \alpha \leqq 1 \tag{3.15}$$

and

$$|Q_m(t)^{\frac{1}{\alpha}} - Q(t)^{\frac{1}{\alpha}}| \leqq |Q_m(t) - Q(t)|^{\frac{1}{\alpha}} \quad \text{for } \alpha > 1, \tag{3.16}$$

concluding that $\mathscr{F}_j x_m(t) \to \mathscr{F}_j x(t)$ $(m \to \infty)$ uniformly on compact subintervals of $[T,\infty)$. This establishes the continuity of $\mathscr{F}_j$ for $j \in \{n+1, n+2, \cdots, 2n-1\}$. The proof of the continuity of $\mathscr{F}_j$ for $j = n$ is similar but simpler, and so it is omitted.

(iii) $\mathscr{F}_j(X)$ is relatively compact in $C[T,\infty)$. The inclusion $\mathscr{F}_j(X) \subset X$ proven in (i) implies that $\mathscr{F}_j(X)$ is locally uniformly bounded on $[T,\infty)$. The inequality

$$|(\mathscr{F}_j x(t))'| \leqq (2c)^{\frac{1}{\alpha}}\int_{t_0}^{t}\frac{(t-s)^{n-2}}{(n-2)!}\left[\frac{(s-t_0)^{j-n}}{p(s)}\right]^{\frac{1}{\alpha}}ds, \quad t \geqq t_0, \tag{3.17}$$

holding for all $x(t) \in X$ guarantees that $\mathscr{F}_j(X)$ is locally equi-continuous on $[T,\infty)$. Then, the relative compactness of $\mathscr{F}_j(X)$ follows from the Ascoli-Arzela lemma.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled. Therefore, there exists a fixed element $x(t) \in X$ of $\mathscr{F}_j$, i.e. $x(t) = \mathscr{F}_j x(t)$, $t \geqq T$, which means that $x(t)$ satisfies the integral equation,

$$x(t) = \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{\frac{c(s-t_0)^{j-n}}{(j-n)!} + (-1)^{2n-j-1}\int_{t_0}^{s}\frac{(s-r)^{j-n-1}}{(j-n-1)!}\right.\right.$$
$$\left.\left.\times \int_r^\infty \frac{(\sigma-r)^{2n-j-1}}{(2n-j-1)!} q(\sigma)x(g(\sigma))^\beta d\sigma dr\right\}\right]^{\frac{1}{\alpha}} ds, \quad (3.18)$$

if $n+1 \leqq j \leqq 2n-1$ and for $t \geqq t_0$;

$$x(t) = \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{c + (-1)^{n-1}\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!}\right.\right.$$
$$\left.\left.\times q(r)x(g(r))^\beta dr\right\}\right]^{\frac{1}{\alpha}} ds, \quad (3.19)$$

if $j = n$ and for $t \geqq t_0$. Note that (3.18) and (3.19) are special cases of (2.8) and (2.9), respectively. Differentiation of (3.18) or (3.19) shows that $x(t)$ is a positive solution of equation (A_{g}) for $t \geqq t_0$. Differentiating (3.18) or (3.19) j times and letting $t \to \infty$, we see that $\lim_{t\to\infty} L_j x(t) = c$, which is equivalent to $\lim_{t\to\infty} x(t)/\varphi_j(t) = c$. Thus the existence of a solution in $P(\mathrm{I}_j)$ has been established for the case $j \in \{n, n+1, \cdots, 2n-1\}$.

Next we consider the case where $j \in \{0,1,\cdots,n-1\}$. Suppose that (3.1) is satisfied. Let $c > 0$ be any given constant, choose $t_0 > a$ large enough so that

$$\int_{t_0}^{\infty} \frac{t^{n-j-1}}{(n-j-1)!}\left[\frac{1}{p(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s)\varphi_j(g(s))^\beta ds\right]^{\frac{1}{\alpha}} dt \leqq B(j!)^{\frac{\beta}{\alpha}} c^{1-\frac{\beta}{\alpha}}, \quad (3.20)$$

where B is a constant such that $B = 2^{-\frac{\beta}{\alpha}}$ if j is odd and $B = 2^{-1}$ if j is even, and define the constants k_1 and k_2 as follows:

$$k_1 = \frac{c}{j!} \quad \text{and} \quad k_2 = \frac{2c}{j!} \ \text{ if } j \text{ is odd}; \quad (3.21)$$

$$k_1 = \frac{c}{2j!} \quad \text{and} \quad k_2 = \frac{c}{j!} \ \text{ if } j \text{ is even}. \quad (3.22)$$

Define the set X by (3.7) with these k_1 and k_2 and consider the mapping $\mathscr{F}_j$ defined by the following formulas: if $j \in \{1,2,\cdots,n-1\}$, then

$$\begin{cases} \mathscr{F}_j x(t) = \dfrac{c(t-t_0)^j}{j!} + (-1)^{2n-j-1}\displaystyle\int_{t_0}^{t}\frac{(t-s)^{j-1}}{(j-1)!}\int_s^\infty \frac{(r-s)^{n-j-1}}{(n-j-1)!} \\ \qquad \times \left[\dfrac{1}{p(r)}\displaystyle\int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma)x(g(\sigma))^\beta d\sigma\right]^{\frac{1}{\alpha}} dr ds, \quad t \geqq t_0; \\ \mathscr{F}_j x(t) = 0, \quad T \leqq t \leqq t_0 \end{cases} \quad (3.23)$$

and if $j=0$, then

$$\begin{cases} \mathscr{F}_0 x(t) = c + (-1)^{2n-1} \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \times \\ \qquad\qquad \times \left[\dfrac{1}{p(s)} \displaystyle\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) x(g(r))^\beta dr \right]^{\frac{1}{\alpha}} ds, \ t \geqq t_0; \\ \mathscr{F}_0 x(t) = 0, \quad T \leqq t \leqq t_0. \end{cases} \tag{3.24}$$

It can be verified without difficulty that (i) $\mathscr{F}_j(X) \subset X$, (ii) $\mathscr{F}_j$ is a continuous mapping, and (iii) $\mathscr{F}_j(X)$ is relatively compact in $C[T,\infty)$. The verification is left to the reader. Consequently, by the Schauder-Tychonoff fixed point theorem $\mathscr{F}_j$ has a fixed point $x(t) \in X$, which satisfies the integral equations, for $j \in \{1,2,\cdots,n-1\}$,

$$\begin{aligned} x(t) = \frac{c(t-t_0)^j}{j!} &+ (-1)^{2n-j-1} \int_{t_0}^t \frac{(t-s)^{j-1}}{(j-1)!} \int_s^\infty \frac{(r-s)^{n-j-1}}{(n-j-1)!} \\ &\times \left[\frac{1}{p(r)} \int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma) x(g(\sigma))^\beta d\sigma \right]^{\frac{1}{\alpha}} drds \end{aligned} \tag{3.25}$$

and, for $j=0$,

$$\begin{aligned} x(t) = c + (-1)^{2n-1} &\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \\ &\times \left[\frac{1}{p(s)} \int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) x(g(r))^\beta dr \right]^{\frac{1}{\alpha}} ds. \end{aligned} \tag{3.26}$$

Note that (3.25) and (3.26) are special cases of (2.10) and (2.11), respectively. Differentiating (3.25) and (3.26), we see that $x(t)$ is a solution of $(\mathrm{A_g})$ such that $\lim_{t\to\infty} L_j x(t) = c$. Thus the existence of a positive solution of $(\mathrm{A_g})$ in $P(\mathrm{I}_j)$ has been established for the case $j \in \{0,1,\cdots,n-1\}$. This completes the proof of Theorem 3.1. $\square$

Let us now turn our attention to the class $P(\mathrm{II}) = P(\mathrm{II}_1) \cup P(\mathrm{II}_3) \cup \cdots \cup P(\mathrm{II}_{2n-1})$. Recall that $P(\mathrm{II}_k)$ is a collection of positive solutions $x(t)$ of $(\mathrm{A_g})$ satisfying (1.6). Since unlike the classes $P(\mathrm{I}_j)$ it seems difficult to characterize the membership of $P(\mathrm{II})$, we content ourselves with giving sufficient conditions for the existence of solutions belonging to each $P(\mathrm{II}_k)$, $k \in \{1,3,\cdots,2n-1\}$.

THEOREM 3.2. (i) *Let k be an odd integer such that $0<k<n$. Equation* $(\mathrm{A_g})$ *possesses a solution of class* $P(\mathrm{II}_k)$ *if*

$$\int_b^\infty t^{n-k-1} \left[\frac{1}{p(t)} \int_t^\infty (s-t)^{n-1} q(s) \varphi_k(g(s))^\beta ds \right]^{\frac{1}{\alpha}} dt < \infty \tag{3.27}$$

and

$$\int_b^\infty t^{n-k} \left[\frac{1}{p(t)} \int_t^\infty (s-t)^{n-1} q(s) \varphi_{k-1}(g(s))^\beta ds \right]^{\frac{1}{\alpha}} dt = \infty. \tag{3.28}$$

(ii) *Let n be an odd integer and let $k=n$. Equation* $(\mathrm{A_g})$ *possesses a solution of class* $P(\mathrm{II}_k)$ *if*

$$\int_b^\infty t^{n-1}q(t)\varphi_n(g(t))^\beta dt<\infty \tag{3.29}$$

and

$$\int_b^\infty \left[\frac{1}{p(t)}\int_t^\infty (s-t)^{n-1}q(s)\varphi_{n-1}(g(s))^\beta ds\right]^{\frac{1}{\alpha}}dt=\infty. \tag{3.30}$$

(iii) *Let k be an odd interger such that $n<k<2n$. Equation* $(\mathrm{A_g})$ *possesses a solution of class* $P(\mathrm{II}_k)$ *if*

$$\int_b^\infty t^{2n-k-1}q(t)\varphi_k(g(t))^\beta dt<\infty \tag{3.31}$$

and

$$\int_b^\infty t^{2n-k}q(t)\varphi_{k-1}(g(t))^\beta dt=\infty. \tag{3.32}$$

Here $b\geqq a$ is a constant such that $g(t)\geqq a$ for $t\geqq b$.

Proof. In each case of (i), (ii) and (iii) the desired solution of $(\mathrm{A_g})$ will be constructed by means of the Schauder-Tychonoff fixed point theorem.

(i) Let k be an odd integer less than n. Suppose that (3.27) and (3.28) hold. Let $c>0$ be any fixed constant and choose $t_0\geqq b$ so that

$$\int_{t_0}^\infty \frac{t^{n-k-1}}{(n-k-1)!}\left[\frac{1}{p(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}q(s)\varphi_k(g(s))^\beta ds\right]^{\frac{1}{\alpha}}dt\leqq 2^{-\frac{\beta}{\alpha}}k!c^{1-\frac{\beta}{\alpha}}. \tag{3.33}$$

This is possible because of (3.27). We may assume that $\inf\limits_{t\geqq t_0} g(t)\geqq \max\{a,1\}$ and put $T=\min\{t_0,\inf\limits_{t\geqq t_0} g(t)\}$. Define the set X by

$$X=\{x\in C[T,\infty):\ c\widetilde{\varphi}_{k-1}(t)\leqq x(t)\leqq 2c\widetilde{\varphi}_k(t),\ \ t\geqq T\} \tag{3.34}$$

and the mapping $\mathscr{G}_k$ by

$$\begin{cases}\mathscr{G}_k x(t)=c\widetilde{\varphi}_{k-1}(t)+\displaystyle\int_{t_0}^t\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{n-k-1}}{(n-k-1)!}\times\\ \qquad\qquad\times\left[\displaystyle\frac{1}{p(r)}\int_r^\infty\frac{(\sigma-r)^{n-1}}{(n-1)!}q(\sigma)x(g(\sigma))^\beta d\sigma\right]^{\frac{1}{\alpha}}drds, \quad t\geqq t_0;\\ \mathscr{G}_k x(t)=0, \quad T\leqq t\leqq t_0.\end{cases} \tag{3.35}$$

If $x(t) \in X$, then using (3.33), we have

$$\begin{aligned}
&\int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \left[\frac{1}{p(r)} \int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma) x(g(\sigma))^\beta d\sigma \right]^{\frac{1}{\alpha}} drds \\
&\quad \leqq \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} ds \int_{t_0}^\infty \frac{s^{n-k-1}}{(n-k-1)!} \left[\frac{1}{p(s)} \int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) (2c\widetilde{\varphi}_k(g(r)))^\beta dr \right]^{\frac{1}{\alpha}} ds \\
&\quad \leqq \frac{(t-t_0)^k}{k!} \cdot (2c)^{\frac{\beta}{\alpha}} \cdot 2^{-\frac{\beta}{\alpha}} k! c^{1-\frac{\beta}{\alpha}} = c\widetilde{\varphi}_k(t), \quad t \geqq t_0,
\end{aligned}$$

which, combined with (3.35), implies that $c\widetilde{\varphi}_{k-1}(t) \leqq \mathscr{G}_k x(t) \leqq 2c\widetilde{\varphi}_k(t)$ for $t \geqq t_0$. This shows that $\mathscr{G}_k$ maps X into itself. Since it can be proved routinely that $\mathscr{G}_k$ is continuous in the topology of $C[T,\infty)$ and that $\mathscr{G}_k(X)$ is relatively compact in $C[T,\infty)$, the Schauder-Tychonoff fixed point theorem ensures the existence of a fixed point $x(t) \in X$ of $\mathscr{G}_k$, which satisfies the integral equation, for $t \geqq t_0$,

$$\begin{aligned}
x(t) = c\widetilde{\varphi}_{k-1}(t) + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \\
\times \left[\frac{1}{p(r)} \int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma) x(g(\sigma))^\beta d\sigma \right]^{\frac{1}{\alpha}} drds. \quad (3.36)
\end{aligned}$$

Note that (3.36) is a special case of (2.14). Differentiation of (3.36) shows that $x(t)$ is a solution of equation $(\mathrm{A}_{\mathrm{g}})$. Furthermore, by differentiating (3.36) $k-1$ times, we obtain, for $t \geqq t_0$,

$$\begin{aligned}
L_{k-1}x(t) = c(k-1)! + \int_{t_0}^{t} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \\
\times \left[\frac{1}{p(r)} \int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma) x(g(\sigma))^\beta d\sigma \right]^{\frac{1}{\alpha}} drds. \quad (3.37)
\end{aligned}$$

Noting that the last (repeated) integral in (3.37) is bounded from below by

$$\int_{t_0}^{t} \frac{(s-t_0)^{n-k}}{(n-k)!} \left[\frac{1}{p(s)} \int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) x(g(r))^\beta dr \right]^{\frac{1}{\alpha}} ds,$$

we find from (3.37) and (3.34) that

$$L_{k-1}x(t) \geqq c^{\frac{\beta}{\alpha}} \int_{t_0}^{t} \frac{(s-t_0)^{n-k}}{(n-k)!} \left[\frac{1}{p(s)} \int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) \widetilde{\varphi}_{k-1}(g(r))^\beta dr \right]^{\frac{1}{\alpha}} ds$$

for $t \geqq t_0$, which because of (3.28) implies that $\lim_{t\to\infty} L_{k-1}x(t) = \infty$. One more differentiation of (3.37) yields

$$L_k x(t) = \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} \left[\frac{1}{p(s)} \int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) x(g(r))^\beta dr \right]^{\frac{1}{\alpha}} ds,$$

from which it follows that $\lim_{t\to\infty} L_k x(t) = 0$. Therefore, the solution $x(t)$ belongs to $P(\mathrm{II}_k)$.

(ii) Let n be odd and let $k = n$. Given any fixed constant $c > 0$, choose $t_0 \geqq b$ so that $\inf_{t\geqq t_0} g(t) \geqq \max\{a, 1\}$ and

$$\int_{t_0}^{\infty} \frac{t^{n-1}}{(n-1)!} q(t)\varphi_n(g(t))^{\beta} dt \leqq 2^{-\beta}[(n-1)!]^{\alpha} c^{\alpha-\beta}. \tag{3.38}$$

We define the mapping $\mathscr{H}_n$ by

$$\begin{cases} \mathscr{H}_n x(t) = c\widetilde{\varphi}_{n-1}(t) + \displaystyle\int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \times \\ \qquad\qquad \times \left[\dfrac{1}{p(s)} \displaystyle\int_{s}^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} q(r)x(g(r))^{\beta} dr\right]^{\frac{1}{\alpha}} ds, \quad t \geqq t_0; \\ \mathscr{H}_n x(t) = 0, \quad T \leqq t \leqq t_0, \end{cases} \tag{3.39}$$

and let it act on the set $X \subset C[T,\infty)$ defined by (3.34) with $T = \min\{t_0, \inf_{t\geqq t_0} g(t)\}$. It can be verified that $\mathscr{H}_n$ is a continuous self-map on X which sends X into a relatively compact subset of $C[T,\infty)$, and hence the Schauder-Tychonoff fixed point theorem is applicable to $\mathscr{H}_n$. Let $x(t) \in X$ be a fixed point of $\mathscr{H}_n$. Then, it satisfies the integral equation, for $t \geqq t_0$,

$$x(t) = c\widetilde{\varphi}_{n-1}(t) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)} \int_{s}^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} q(r)x(g(r))^{\beta} dr\right]^{\frac{1}{\alpha}} ds, \tag{3.40}$$

which is a special case of (2.13). It is easy to check that $x(t)$ is a solution of equation $(\mathrm{A}_{\mathrm{g}})$ and satisfies $\lim_{t\to\infty} L_{n-1}x(t) = \infty$ and $\lim_{t\to\infty} L_n x(t) = 0$. This guarantees that $x(t) \in P(\mathrm{II}_n)$ as desired.

(iii) Let k be an odd integer such that $n < k < 2n$. For any given $c > 0$ one can choose $t_0 \geqq b$ so that $\inf_{t\geqq t_0} g(t) \geqq \max\{a, 1\}$ and

$$\int_{t_0}^{\infty} \frac{t^{2n-k-1}}{(2n-k-1)!} q(t)\varphi_k(g(t))^{\beta} dt \leqq 2^{-\beta}(k-n)![(n-1)!]^{\alpha} c^{\alpha-\beta}. \tag{3.41}$$

Let X be defined by (3.34) with $T = \min\{t_0, \inf_{t\geqq t_0} g(t)\}$ and define the mapping $\mathscr{I}_k$ by

$$\begin{cases} \mathscr{I}_k x(t) = c\widetilde{\varphi}_{k-1}(t) + \displaystyle\int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\dfrac{1}{p(s)} \times \right. \\ \quad \left. \times \displaystyle\int_{t_0}^{s} \frac{(s-r)^{k-n-1}}{(k-n-1)!} \int_{r}^{\infty} \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} q(\sigma)x(g(\sigma))^{\beta} d\sigma dr\right]^{\frac{1}{\alpha}} ds, \quad t \geqq t_0; \\ \mathscr{I}_k x(t) = 0, \quad T \leqq t \leqq t_0. \end{cases} \tag{3.42}$$

Since it can be proved routinely that $\mathscr{I}_k$ is continuous and sends X into a relatively compact subset of X, by the Schauder-Tychonoff theorem there exists $x(t) \in X$ such that $x(t) = \mathscr{I}_k x(t)$ for $t \geqq t_0$. i.e., for $t \geqq t_0$,

$$x(t) = c\widetilde{\varphi}_{k-1}(t) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \right. \\ \left. \times \int_{t_0}^{s} \frac{(s-r)^{k-n-1}}{(k-n-1)!} \int_{r}^{\infty} \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} q(\sigma) x(g(\sigma))^{\beta} d\sigma dr \right]^{\frac{1}{\alpha}} ds. \quad (3.43)$$

This is a special case of (2.12). Differentiation of (3.43) shows that $x(t)$ is a solution of equation ($\mathrm{A_g}$). One easily sees that $x(t)$ satisfies $\lim_{t\to\infty} L_{k-1}x(t) = \infty$ and $\lim_{t\to\infty} L_k x(t) = 0$, and so $x(t)$ must be member of $P(\mathrm{II}_k)$. This completes the proof of Theorem 3.2. □

REMARK 3.1. One may ask if it is possible to obtain necessary and sufficient conditions for ($\mathrm{A_g}$) to have positive solutions belonging to class $P(\mathrm{II}_k)$. This question is extremely difficult even for ordinary differential equations of the form (A). To the best of our knowledge Kusano and Naito [16] is the only paper that gives an affirmative answer to the question for the special case of (A) with $\alpha = 1 > \beta$ and $p(t) \equiv 1$.

4. Comparison theorems

When it is not easy to acquire information about the oscillation (or nonoscillation) of a given differential equation directly, it would be natural to compare the equation in question with another differential equation, more or less of similar type, whose oscillatory (or nonoscillatory) behavior is already known or can be analyzed with relative ease. In this section we will establish comparison principles which correlate the oscillation (or nonoscillation) of the equation

$$\left(p(t)|u^{(n)}(t)|^{\alpha} \operatorname{sgn} u^{(n)}(t)\right)^{(n)} + F(t, u(h(t))) = 0, \quad (4.1)$$

with that of the equation

$$\left(p(t)|v^{(n)}(t)| \operatorname{sgn} v^{(n)}(t)\right)^{(n)} + G(t, v(k(t))) = 0 \quad (4.2)$$

or

$$\left(p(t)|y^{(n)}(t)| \operatorname{sgn} y^{(n)}(t)\right)^{(n)} + \frac{l'(t)}{h'(h^{-1}(l(t)))} F(h^{-1}(l(t)), y(l(t))) = 0. \quad (4.3)$$

With regard to the above equations the following conditions are always assumed to hold:

(a) $h(t)$, $k(t)$ and $l(t)$ are continuously differentiable functions on $[a,\infty)$ such that
$h'(t)>0$, $k'(t)>0$, $l'(t)>0$, $\lim_{t\to\infty} h(t)=\lim_{t\to\infty} k(t)=\lim_{t\to\infty} l(t)=\infty$;

(b) $F(t,u)$ and $G(t,u)$ are continuous nonnegative functions on $[a,\infty)\times\mathbb{R}$ which are non-decreasing in u and such that $uF(t,u)\geqq 0$ and $uG(t,u)\geqq 0$ for any fixed $t\geqq a$.

THEOREM 4.1. *Assume that*

$$h(t)\geqq k(t), \quad t\geqq a, \tag{4.4}$$

and

$$F(t,u)\mathrm{sgn}\, u\geqq G(t,u)\mathrm{sgn}\, u, \quad (t,u)\in[a,\infty)\times\mathbb{R}. \tag{4.5}$$

If equation (4.1) *has a nonoscillatory solution, then so does equation* (4.2)*, or equivalently, if equation* (4.2) *is oscillatory, then so is* (4.1)*.*

THEOREM 4.2. *Assume that $l(t)\geqq h(t)$ for $t\geqq a$. If equation* (4.3) *is oscillatory, then so is equation* (4.1)*.*

We remark that a prototype of comparison principle of type Theorem 4.1 was given by Chanturia [1] and Kusano and Naito [17]. A comparison principle of type Theorem 4.2 was first obtained Mahfoud [21] and extended by Kusano and Naito [17]. Our results generalize earlier comparison results [17, 20–22, 30, 34] for nonlinear differential equations involving the operator $(p(t)|u^{(n)}(t)|^{\alpha}\mathrm{sgn}\, u^{(n)}(t))^{(n)}$, $n\geqq 2$. The proof of Theorems 4.1 and 4.2 is based on the following lemma describing the relation between (4.1) and the differential inequality

$$\left(p(t)|z^{(n)}(t)|^{\alpha}\mathrm{sgn}\, z^{(n)}(t)\right)^{(n)}+F(t,z(h(t)))\leqq 0. \tag{4.6}$$

LEMMA 4.1. *If these exists an eventually positive function satisfying* (4.6)*, then equation* (4.1) *possesses a positive solution.*

Proof. Let $z(t)$ be a positive function satisfying (4.6). By Lemma 2.1 there exists $k\in\{1,3,\cdots,2n-1\}$ such that

$$\begin{cases} L_j z(t)>0, & t\geqq t_0 \quad \text{for} \quad 0\leqq j\leqq k-1;\\ (-1)^{j-k}L_j z(t)>0, & t\geqq t_0 \quad \text{for} \quad k\leqq j\leqq 2n-1 \end{cases} \tag{4.7}$$

for sufficiently large $t_0>a$. We may assume $h(t)\geqq\max\{a,1\}$ for $t\geqq t_0$ and put $T=\min\{t_0,\inf_{t\geqq t_0} h(t)\}$. Note that $w_k=\lim_{t\to\infty} L_k z(t)\geqq 0$ is finite. Define the set U by

$$U=\{u\in C[T,\infty):\ 0\leqq u(t)\leqq z(t),\ t\geqq T\}, \tag{4.8}$$

which is closed convex in $C[T,\infty)$. Suppose that $n<k\leqq 2n-1$. Integrating (4.6) $2n-k$ times on $[t,\infty)$ gives

$$\left(p(t)|z^{(n)}(t)|^{\alpha}\operatorname{sgn} z^{(n)}(t)\right)^{(k-n)} \geqq w_k+\int_t^{\infty}\frac{(s-t)^{2n-k-1}}{(2n-k-1)!}F(s,z(h(s)))ds, \quad t\geqq t_0,$$

which integrated further $k=(k-n)+n$ times on $[t_0,t]$ yields, for $t\geqq t_0$,

$$z(t)\geqq z(t_0)+\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{w_k\frac{(s-t_0)^{k-n}}{(k-n)!}+\int_{t_0}^{s}\frac{(s-r)^{k-n-1}}{(k-n-1)!}\right.\right.$$
$$\left.\left.\times\int_r^{\infty}\frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!}F(\sigma,z(h(\sigma)))d\sigma dr\right\}\right]^{\frac{1}{\alpha}}ds. \quad (4.9)$$

We now consider the mapping $\Phi: X\to C[T,\infty)$ defined by

$$\begin{cases}\Phi u(t)=z(t_0)+\displaystyle\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{w_k\frac{(s-t_0)^{k-n}}{(k-n)!}+\int_{t_0}^{s}\frac{(s-r)^{k-n-1}}{(k-n-1)!}\right.\right.\\ \qquad\qquad\left.\left.\times\displaystyle\int_r^{\infty}\frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!}F(\sigma,u(h(\sigma)))d\sigma dr\right\}\right]^{\frac{1}{\alpha}}ds,\ t\geqq t_0;\\ \Phi u(t)=z(t), \quad T\leqq t\leqq t_0.\end{cases} \quad (4.10)$$

It is a matter of routine computation to verify that (i) $\Phi(U)\subset U$, (ii) Φ is continuous, and (iii) $\Phi(U)$ is relatively compact. Therefore, Φ has a fixed point $u(t)\in U$ which satisfies the integral equation, for $t\geqq t_0$,

$$u(t)=z(t_0)+\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{w_k\frac{(s-t_0)^{k-n}}{(k-n)!}+\int_{t_0}^{s}\frac{(s-r)^{k-n-1}}{(k-n-1)!}\right.\right.$$
$$\left.\left.\times\int_r^{\infty}\frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!}F(\sigma,u(h(\sigma)))d\sigma dr\right\}\right]^{\frac{1}{\alpha}}ds. \quad (4.11)$$

From which it follows that $u(t)$ is a solution of the differential equation (4.1). A similar arguments applies to the remaining cases of k. In fact, if $k=n$, which is possible for odd n, then, integrating (4.6) n times on $[t,\infty)$ and then n times on $[t_0,t]$, we have for $t\geqq t_0$,

$$z(t)\geqq z(t_0)+\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[\frac{1}{p(s)}\left\{w_n+\int_s^{\infty}\frac{(r-s)^{n-1}}{(n-1)!}F(r,z(h(r)))dr\right\}\right]^{\frac{1}{\alpha}}ds, \quad (4.12)$$

and if $1\leqq k<n$, then integrating (4.6) first $2n-k(=n+(n-k))$ times on $[t,\infty)$ and then k times on $[t_0,t]$, we obtain for $t\geqq t_0$,

$$z(t)\geqq z(t_0)+\int_{t_0}^{t}\frac{(t-s)^{k-1}}{(k-1)!}\int_s^{\infty}\frac{(r-s)^{n-k-1}}{(n-k-1)!}$$
$$\times\left[\frac{1}{p(r)}\int_r^{\infty}\frac{(\sigma-r)^{n-1}}{(n-1)!}F(\sigma,z(h(\sigma)))d\sigma\right]^{\frac{1}{\alpha}}drds. \quad (4.13)$$

Applying the same fixed-point argument which has solved (4.11) on the basis of (4.9), we conclude from (4.12) or (4.13) that there exist a positive solution $u(t) \leqq z(t)$ of the integral equation for $t \geqq t_0$ $(k = n)$,

$$u(t) = z(t_0) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \times \left[\frac{1}{p(s)}\left\{w_n + \int_s^\infty \frac{(s-r)^{n-1}}{(n-1)!} F(r, u(h(r)))dr\right\}\right]^{\frac{1}{\alpha}} ds; \quad (4.14)$$

or for $t \geqq t_0$ $(1 \leqq k < n)$,

$$u(t) = z(t_0) + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \times \left[\frac{1}{p(r)} \int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} F(\sigma, u(h(\sigma)))d\sigma\right]^{\frac{1}{\alpha}} drds, \quad (4.15)$$

respectively. It is clear that each of the above $u(t)$ is a positive solution of equation (4.1). This completes the proof of Lemma 4.1. □

Proof of Theorem 4.1. Suppose that (4.1) has a nonoscillatory solution $u(t)$ which may be assumed to be positive for all large t. By Lemma 2.1 $u(t)$ is monotone increasing, so that because of (4.4) and (4.5) there exists $t_0 > a$ so large that $u(h(t)) \geqq u(k(t))$ and $F(t, u(h(t))) \geqq G(t, u(k(t)))$ for $t \geqq t_0$. It follows that $u(t)$ satisfies the differential inequality

$$\left(p(t)|u^{(n)}(t)|^\alpha \operatorname{sgn} u^{(n)}(t)\right)^{(n)} + G(t, u(k(t))) \leqq 0, \quad t \geqq t_0, \quad (4.16)$$

and application of Lemma 4.1 then ensures that the existence of a positive solution of the differential equation (4.2). This completes the proof. □

Proof of Theorem 4.2. It is sufficient to prove that if (4.1) has a positive solution, then so does (4.3). Let $u(t)$ be a positive solution of (4.1). By Lemma 2.1 there exists $k \in \{1, 3, \cdots, 2n-1\}$ such that

$$L_j u(t) > 0, \quad 0 \leqq j \leqq k-1, \quad (-1)^{j-k} L_j u(t) > 0, \quad k \leqq j \leqq 2n-1, \quad (4.17)$$

for all large t. Let w_k denote the k-th asymptotic value of $u(t)$, i.e. $w_k = \lim_{t\to\infty} L_k u(t) \geqq 0$, which is finite.

Let $n < k \leqq 2n-1$. Then, repeated integration of (4.1) gives

$$u(t) = u(t_0) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)}\left\{w_k \frac{(s-t_0)^{k-n}}{(k-n)!} + \int_{t_0}^{s} \frac{(s-r)^{k-n-1}}{(k-n-1)!} \times \int_r^\infty \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} F(\sigma, u(h(\sigma)))d\sigma dr\right\}\right]^{\frac{1}{\alpha}} ds, \quad (4.18)$$

for sufficiently large t, say $t \geqq t_0$. To proceed further we transform the last inner integral by the change of variables $\sigma = h^{-1}(l(\rho))$. Using the inequalities $h^{-1}(l(\rho)) \geqq \rho$ and $r \geqq l^{-1}(h(r))$, we find that

$$\begin{aligned}
&\int_r^\infty \frac{(\sigma - r)^{2n-k-1}}{(2n-k-1)!} F(\sigma, u(h(\sigma)))d\sigma \\
&\quad \geqq \int_{l^{-1}(h(r))}^\infty \frac{(\rho - r)^{2n-k-1}}{(2n-k-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} d\rho \\
&\quad \geqq \int_r^\infty \frac{(\rho - r)^{2n-k-1}}{(2n-k-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} d\rho. \qquad (4.19)
\end{aligned}$$

Combining this with (4.18), we obtain for $t \geqq t_0$,

$$\begin{aligned}
u(t) \geqq u(t_0) &+ \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ w_k \frac{(s-t_0)^{k-n}}{(k-n)!} + \int_{t_0}^s \frac{(s-r)^{k-n-1}}{(k-n-1)!} \right.\right. \\
&\left.\left. \times \int_r^\infty \frac{(\rho - r)^{2n-k-1}}{(2n-k-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} d\rho dr \right\} \right]^{\frac{1}{\alpha}} ds. \qquad (4.20)
\end{aligned}$$

We may assume that t_0 is chosen so large that $l(t) \geqq \max\{a, 1\}$ for $t \geqq t_0$.
Put $T = \min\{t_0, \inf_{t \geqq t_0} l(t)\}$. Define the set Y and the mapping $\Psi : Y \to C[T, \infty)$ by

$$Y = \{y \in C[T, \infty) : \ 0 \leqq y(t) \leqq u(t), \ t \geqq T\} \qquad (4.21)$$

and

$$\begin{cases}
\Psi y(t) = u(t_0) + \displaystyle\int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ w_k \frac{(s-t_0)^{k-n}}{(k-n)!} + \int_{t_0}^s \frac{(s-r)^{k-n-1}}{(k-n-1)!} \right.\right. \\
\qquad \left.\left. \times \displaystyle\int_r^\infty \frac{(\rho - r)^{2n-k-1}}{(2n-k-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} d\rho dr \right\} \right]^{\frac{1}{\alpha}} ds, \ t \geqq t_0 \\
\Psi y(t) = u(t), \quad T \leqq t \leqq t_0.
\end{cases} \qquad (4.22)$$

There is no difficulty in showing that Ψ is a continuous self-map on Y such that $\Psi(Y)$ is relatively compact in $C[T, \infty)$, and hence the Schauder-Tychonoff theorem implies the existence of $y(t) \in Y$ such that $y(t) = \Psi y(t)$ for $t \geqq T$, which provides a positive solution of equation (4.3) for $t \geqq t_0$.

The remaining cases where $k = n$ and $1 \leqq k < n$ can be dealt with in a similar manner. In fact, it suffices to notice that if $k = n$, n being odd, then for $t \geqq t_0$ the repeated integration of (4.1) gives

$$u(t) = u(t_0) + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ w_n + \int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} F(r, u(h(r)))dr \right\} \right]^{\frac{1}{\alpha}} ds,$$

then, using (4.19) in the above equality, we obtain

$$u(t) \geqq u(t_0) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\frac{1}{p(s)} \left\{ w_n \right.\right.$$
$$\left.\left. + \int_{s}^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} F(h^{-1}(l(r)), u(l(r))) \frac{l'(r)}{h'(h^{-1}(l(r)))} dr \right\} \right]^{\frac{1}{\alpha}} ds, \quad (4.23)$$

and that if $1 \leqq k < n$, then the repeated integrations of (4.1) give

$$u(t) = u(t_0) + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!}$$
$$\times \left[\frac{1}{p(r)} \left\{ \int_{r}^{\infty} \frac{(\rho-r)^{n-1}}{(n-1)!} F(\sigma, u(h(\sigma))) d\sigma \right\} \right]^{\frac{1}{\alpha}} dr ds, \quad t \geqq t_0.$$

then, using (4.19) in the above equality, we get for $t \geqq t_0$ that

$$u(t) \geqq u(t_0) + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!}$$
$$\times \left[\frac{1}{p(r)} \left\{ \int_{r}^{\infty} \frac{(\rho-r)^{n-1}}{(n-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} d\rho \right\} \right]^{\frac{1}{\alpha}} dr ds. \quad (4.24)$$

As is easily seen the fixed point argument which was used for the case $n < k \leqq 2n-1$ can also be applied to the case where $k = n$ (or $1 \leqq k < n$) to establish the existence of a positive solution $y(t)$ for equation (4.3) on the basis of the inequality (4.23) (or (4.24)). The details may be omitted. This completes the proof of Theorem 4.2. □

5. Oscillation theorems

This section is concerned with oscillation criteria for equation ($\mathrm{A_g}$), that is, conditions (preferably sharp) under which all of its solutions are oscillatory. With respect to the nonlinearity of ($\mathrm{A_g}$) we require that ($\mathrm{A_g}$) is either sub-half-linear ($\alpha > \beta$) or super-half-linear ($\alpha < \beta$). In establishing the desired criteria a central role is played by the comparison principles of Section 4 combined with the known result (see the two Propositions below) on the oscillation of ordinary differential equations of the form (A).

PROPOSITION 5.1. *Let* $\alpha \geqq 1 > \beta$. *All solutions of equation* (A) *are oscillatory if and only if*

$$\int_{a}^{\infty} q(t) \varphi_{2n-1}(t)^{\beta} dt = \infty. \quad (5.1)$$

PROPOSITION 5.2. *Let* $\alpha \leqq 1 < \beta$. *All solutions of equation* (A) *are oscillatory if and only if either*

$$\int_a^\infty t^{n-1} q(t) dt = \infty \tag{5.2}$$

or else

$$\int_a^\infty t^{n-1} q(t) dt < \infty \tag{5.3}$$

and

$$\int_a^\infty t^{n-1} \left[\frac{1}{p(t)} \int_t^\infty (s-t)^{n-1} q(s) ds \right]^{\frac{1}{\alpha}} dt = \infty. \tag{5.4}$$

For the proof of these Propositions see Tanigawa [32]. We first present sufficient conditions for oscillation of equation $(\mathrm{A_g})$.

THEOREM 5.1. *Let* $\alpha \geqq 1 > \beta$. *Suppose that there exists a continuously differentiable function* $h : [a, \infty) \to (0, \infty)$ *such that* $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, *and*

$$h(t) \leqq \min\{t, g(t)\} \quad \text{for all large } t. \tag{5.5}$$

All solutions of equation $(\mathrm{A_g})$ *are oscillatory if*

$$\int_b^\infty q(t) \varphi_{2n-1}(h(t))^\beta dt = \infty, \tag{5.6}$$

where $b \geqq a$ *is such that* $h(t) \geqq a$ *for* $t \geqq b$.

Proof. Consider the two differential equations

$$\left(p(t)|z^{(n)}(t)|^\alpha \operatorname{sgn} z^{(n)}(t)\right)^{(n)} + q(t)|z(h(t))|^\beta \operatorname{sgn} z(h(t)) = 0, \tag{5.7}$$

and

$$\left(p(t)|y^{(n)}(t)|^\alpha \operatorname{sgn} y^{(n)}(t)\right)^{(n)} + \frac{q(h^{-1}(t))}{h'(h^{-1}(t))} |y(t)|^\beta \operatorname{sgn} y(t) = 0. \tag{5.8}$$

Since (5.6) implies

$$\int_{h(b)}^\infty \frac{q(h^{-1}(t))}{h'(h^{-1}(t))} \varphi_{2n-1}(t)^\beta dt = \int_b^\infty q(t) \varphi_{2n-1}(h(t))^\beta dt = \infty,$$

from Proposition 5.1 it follows that equation (5.8) is oscillatory. We now compare (5.8) with (5.7), concluding from Theorem 4.2 that equation (5.7) is oscillatory. Finally, comparison of (5.7) with equation $(\mathrm{A_g})$ on the basis of Theorem 4.1 shows that, because of the assumed inequality $g(t) \geqq h(t)$, equation $(\mathrm{A_g})$ is oscillatory, which means that all of its solutions are oscillatory. This completes the proof. □

THEOREM 5.2. *Let* $\alpha \leqq 1 < \beta$. *Assume that there exists a continuously differentiable function* $h : [a,\infty) \to (0,\infty)$ *such that* $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, *and* (5.5) *holds. If*

$$\int_b^\infty (h(t))^{n-1} q(t)dt = \infty \tag{5.9}$$

or

$$\int_b^\infty (h(t))^{n-1} q(t)dt < \infty, \quad \int_b^\infty t^{n-1}\left[\frac{1}{p(t)}\int_{h^{-1}(t)}^\infty (h(s)-t)^{n-1} q(s)ds\right]^{\frac{1}{\alpha}} dt = \infty, \tag{5.10}$$

then all solutions of (A_g) *are oscillatory. Here* $b \geqq a$ *is such that* $h(t) \geqq a$ *for* $t \geqq b$.

Proof. We also consider the two differential equations (5.7) and (5.8) as handled in the proof of Theorem 5.1. Since, by conditions (5.9) and (5.10),

$$\int_b^\infty t^{n-1}\frac{q(h^{-1}(t))}{h'(h^{-1}(t))}dt = \int_{h(b)}^\infty (h(t))^{n-1} q(t)dt = \infty$$

and

$$\int_b^\infty t^{n-1}\left[\frac{1}{p(t)}\int_t^\infty (s-t)^{n-1}\frac{q(h^{-1}(s))}{h'(h^{-1}(s))}ds\right]^{\frac{1}{\alpha}} dt$$

$$= \int_b^\infty t^{n-1}\left[\frac{1}{p(t)}\int_{h^{-1}(t)}^\infty (h(s)-t)^{n-1} q(s)ds\right]^{\frac{1}{\alpha}} dt = \infty,$$

we find that equation (5.8) is oscillatory by Proposition 5.2. Moreover, by the comparison principle (Theorem 4.2), (5.7) is also oscillatory. Here, applying another comparison principle (Theorem 4.1) to compare (5.7) with (A_g), we conclude that (A_g) is oscillatory. This completes the proof. □

It remains to consider the possibility of establishing necessary and sufficient conditions for oscillation of equation (A_g) which is either super-half-linear or sub-half-linear. This seems to be a difficult task for equation (A_g) with general positive continuous function $p(t)$ satisfying (1.1), and so we focus our attention on the case where $p(t)$ is a regularly varying function in the sense of Karamata [7] and show that for such a restricted class of equations of the form (A_g) sharp oscillation criteria can be obtained for both super-half-linear and sub-half-linear cases.

DEFINITION 5.1. A measurable function $f : (0,\infty) \to (0,\infty)$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for any } \lambda > 0, \tag{5.11}$$

or equivalently, if $f(t)$ is expressed in the form

$$f(t) = c(t)\exp\left\{\int_{t_0}^t \frac{\delta(s)}{s}ds\right\}, \quad t \geqq t_0,$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. If in particular $\rho = 0$, we use the notation SV for $\mathrm{RV}(0)$, referring to a member of SV as a *slowly varying function*. If $f(t) \in \mathrm{RV}(\rho)$, then $f(t) = t^\rho g(t)$ for some $g(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. Typical examples of slowly varying functions are: positive functions tending to positive constants as $t \to \infty$,

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ (m_i \in \mathbb{R}), \quad \text{and} \quad \exp\left\{ \prod_{i=1}^{N} (\log_i t)^{n_i} \right\} \ (n_i \in (0,1)),$$

where $\log_i t$ denotes the i-th iteration of the logarithm.

For an almost complete exposition of theory and applications of regular variation we refer to Bingham et al. [2]. A comprehensive survey of results up to 2000 on asymptotic analysis of second order linear and nonlinear ordinary differential equations can be found in Marić [23].

We begin with a necessary and sufficient condition for oscillation of the sub-half-linear equation $(\mathrm{A_g})$.

THEOREM 5.3. *Let $\alpha \geqq 1 > \beta$. Assume that $p(t)$ is a regularly varying function of index ρ satisfying* (1.1) *and that*

$$\limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{5.12}$$

Then, all solutions of $(\mathrm{A_g})$ *are oscillatory if and only if*

$$\int_b^\infty q(t)\varphi_{2n-1}(g(t))^\beta \, dt = \infty, \tag{5.13}$$

where $b \geqq a$ is such that $g(t) \geqq a$ for $t \geqq b$.

Proof. (The “only if”part) If (5.13) does not hold, then by Theorem 3.1 equation $(\mathrm{A_g})$ has a positive solution of class $P(\mathrm{I}_{2n-1})$. Note that the regularity of $p(t)$ is unnecessary here.

(The “if”part) Suppose that (5.13) holds. Note that since $p(t)$ satisfies (1.1) its regularity index ρ must not greater than α. Let $k \in (0,1)$ be any fixed constant. Choose $b > a$ so that $kb \geqq a$. Since $p(t)$ satisfies $\lim_{t\to\infty} p(kt)/p(t) = k^\rho$ (cf. (5.11)), there exists a positive constant l such that

$$l p(kt) \leqq p(t), \quad t \geqq b. \tag{5.14}$$

Using (5.14) we compute:

$$\begin{aligned}
\varphi_{2n-1}(kt;kb) &= \int_{kb}^{kt} (kt-s)^{n-1}\left[\frac{(s-kb)^{n-1}}{p(s)}\right]^{\frac{1}{\alpha}} ds \\
&= k^{n+\frac{1}{\alpha}(n-1)} \int_b^t (t-s)^{n-1}\left[\frac{(s-b)^{n-1}}{p(ks)}\right]^{\frac{1}{\alpha}} ds \\
&\geqq k^{n+\frac{1}{\alpha}(n-1)} l^{\frac{1}{\alpha}} \int_b^t (t-s)^{n-1}\left[\frac{(s-b)^{n-1}}{p(s)}\right]^{\frac{1}{\alpha}} ds \\
&= k^{n+\frac{1}{\alpha}(n-1)} l^{\frac{1}{\alpha}} \varphi_{2n-1}(t;b), \quad t \geqq b,
\end{aligned}$$

from which it follows that for any $k \in (0,1)$,

$$\int^{\infty} q(t)\varphi_{2n-1}(g(t))^{\beta} dt = \infty \implies \int^{\infty} q(t)\varphi_{2n-1}(kg(t))^{\beta} dt = \infty. \tag{5.15}$$

By (5.12) there exists a constant $c > 1$ such that

$$g(t) \leqq ct \quad \text{for all large } t. \tag{5.16}$$

Now consider the ordinary differential equation

$$\left(p(t)|y^{(n)}(t)|^{\alpha} \operatorname{sgn} y^{(n)}(t)\right)^{(n)} + \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))}|y(t)|^{\beta} \operatorname{sgn} y(t) = 0. \tag{5.17}$$

Since (5.15) (with $k = 1/c$) implies

$$\int^{\infty} (\varphi_{2n-1}(t))^{\beta} \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))} dt = \int^{\infty} q(t)\left(\varphi_{2n-1}\left(\frac{g(t)}{c}\right)\right)^{\beta} dt = \infty, \tag{5.18}$$

equation (5.17) is oscillatory by Proposition 5.1. The comparison theorem (Theorem 4.1) then implies the oscillation of

$$\left(p(t)|u^{(n)}(t)|^{\alpha} \operatorname{sgn} u^{(n)}(t)\right)^{(n)} + \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))}|u(ct)|^{\beta} \operatorname{sgn} u(ct) = 0 \tag{5.19}$$

which, compared with (A_g) via another comparison principle (Theorem 4.2), guarantees that (A_g) is oscillatory. This completes the proof. □

The following theorem provides a necessary and sufficient condition for the super-half-linear equation (A_g) to be oscillatory.

THEOREM 5.4. *Let $\alpha \leqq 1 < \beta$. Assume that $p(t)$ is a regularly varying function of index ρ and*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0. \tag{5.20}$$

Then, all solutions of equation (A_g) are oscillatory if and only if either (5.2) holds or else (5.3) and (5.4) hold.

Proof. (The "only if"part) Suppose that ($\mathrm{A_g}$) is oscillatory. If (5.4) would fail to hold, that is,

$$\int_a^\infty t^{n-1}\left[\frac{1}{p(t)}\int_t^\infty (s-t)^{n-1}q(s)ds\right]^{\frac{1}{\alpha}}dt<\infty,$$

then by Theorem 3.1 equation ($\mathrm{A_g}$) would possess a positive solution of class $\mathrm{P(I_0)}$, a contradiction. Note that the regularity of $p(t)$ is not needed here.

(The "if"part) Because of (5.20) there exists a positive constant $c<1$ such that

$$g(t)\geqq ct \quad \text{for all sufficiently large } t. \tag{5.21}$$

With this c form the ordinary differential equation

$$\left(p(t)|z^{(n)}(t)|^\alpha \operatorname{sgn} z^{(n)}(t)\right)^{(n)}+\frac{1}{c}q\left(\frac{t}{c}\right)|z(t)|^\beta \operatorname{sgn} z(t)=0. \tag{5.22}$$

Choose $b\geqq a$ so that $g(t)\geqq a$ for $t\geqq b$. Since $p(t)$ is regularly varying, there exists a constant $L>0$ such that

$$p(t)\leqq Lp\left(\frac{t}{c}\right),\quad t\geqq b. \tag{5.23}$$

We notice that if (5.2) holds, then

$$\int_a^\infty t^{n-1}\frac{1}{c}q\left(\frac{t}{c}\right)dt=c^{n-1}\int_{\frac{a}{c}}^\infty t^{n-1}q(t)dt=\infty \tag{5.24}$$

and if (5.3) and (5.4) hold, then using (5.23), we easily see that

$$\begin{aligned}
&\int_b^t s^{n-1}\left[\frac{1}{p(s)}\int_s^\infty (r-s)^{n-1}\frac{1}{c}q\left(\frac{r}{c}\right)dr\right]^{\frac{1}{\alpha}}ds\\
&=\int_b^t s^{n-1}\left[\frac{1}{p(s)}\int_{\frac{s}{c}}^\infty (cr-s)^{n-1}q(r)dr\right]^{\frac{1}{\alpha}}ds\\
&=c^{(n-1)\left(1+\frac{1}{\alpha}\right)}\int_b^t\left(\frac{s}{c}\right)^{n-1}\left[\frac{1}{p(s)}\int_{\frac{s}{c}}^\infty\left(r-\frac{s}{c}\right)^{n-1}q(r)dr\right]^{\frac{1}{\alpha}}ds\\
&\geqq c^{(n-1)\left(1+\frac{1}{\alpha}\right)}L^{\frac{1}{\alpha}}\int_b^t\left(\frac{s}{c}\right)^{n-1}\left[\frac{1}{p\left(\frac{s}{c}\right)}\int_{\frac{s}{c}}^\infty\left(r-\frac{s}{c}\right)^{n-1}q(r)dr\right]^{\frac{1}{\alpha}}ds\to\infty,
\end{aligned} \tag{5.25}$$

as $t\to\infty$. Therefore, equation (5.22) is oscillatory by Proposition 5.2. With the help of Theorem 4.2 the oscillation of (5.22) implies that of the equation

$$\left(p(t)|u^{(n)}(t)|^\alpha \operatorname{sgn} u^{(n)}(t)\right)^{(n)}+q(t)|u(ct)|^\beta \operatorname{sgn} u(ct)=0. \tag{5.26}$$

We now apply Theorem 4.1 to compare (5.26) with our equation (A_g). Because of (5.21) this comparison is possible and leads us to the conclusion that (A_g) is oscillatory. This completes the proof. □

Finally we are interested in the situation in which the oscillation of the functional differential equation (A_g) is equivalent to that of the companion ordinary differential equation (A) in the sense that the oscillation of (A) implies that of (A_g) and vice versa. A partial answer to this question will be given below.

THEOREM 5.5. *Let either $\alpha \geqq 1 > \beta$ or $\alpha \leqq 1 < \beta$. Suppose that $p(t)$ is a regularly varying function satisfying* (1.1)*. Suppose moreover that $g(t)$ satisfies* (5.12) *and* (5.20)*, that is,*

$$0 < \liminf_{t\to\infty} \frac{g(t)}{t}, \quad \limsup_{t\to\infty} \frac{g(t)}{t} < \infty.$$

Then, the oscillation of equation (A_g) *is equivalent to that of equation* (A).

Proof. (The case where $\alpha \geqq 1 > \beta$.) We claim that the following equivalence holds under the assumption that $p(t)$ is regularly varying:

$$\int^{\infty} q(t)\varphi_{2n-1}(t)dt = \infty \quad \Longleftrightarrow \quad \int^{\infty} q(t)\varphi_{2n-1}(ct)dt = \infty \quad \text{for all } c > 0. \tag{5.27}$$

In fact, let b be such that $cb \geqq a$. From the regularity of $p(t)$ for any $c > 0$ there are two positive constants l and L such that

$$lp(ct) \leqq p(t) \leqq Lp(ct), \quad t \geqq b. \tag{5.28}$$

It is easy to see that

$$\begin{aligned}
\varphi_{2n-1}(ct;cb) &= \int_{cb}^{ct} (ct-s)^{n-1}\left[\frac{(s-cb)^{n-1}}{p(s)}\right]^{\frac{1}{\alpha}} ds \\
&= c^{n+\frac{1}{\alpha}(n-1)} \int_{b}^{t} (t-s)^{n-1}\left[\frac{(s-b)^{n-1}}{p(cs)}\right]^{\frac{1}{\alpha}} ds \\
&\geqq c^{n+\frac{1}{\alpha}(n-1)} l^{\frac{1}{\alpha}} \int_{b}^{t} (t-s)^{n-1}\left[\frac{(s-b)^{n-1}}{p(s)}\right]^{\frac{1}{\alpha}} ds \\
&= c^{n+\frac{1}{\alpha}(n-1)} l^{\frac{1}{\alpha}} \varphi_{2n-1}(t;b), \quad t \geqq b,
\end{aligned}$$

where the left-hand side of (5.28) has been used. Likewise, using the right-hand side of (5.28), we have

$$\varphi_{2n-1}(ct;cb) \leqq c^{n+\frac{1}{\alpha}(n-1)} L^{\frac{1}{\alpha}} \varphi_{2n-1}(t;b), \quad t \geqq b.$$

It is clear that (5.27) is an immediate consequence of the above observation.

By hypothesis there exist positive constants c_1 and c_2 such that

$$c_1 t \leqq g(t) \leqq c_2 t \quad \text{for } t \geqq t_0. \tag{5.29}$$

We may assume that $c_1 < 1$ and $c_2 > 1$.

Suppose now that equation (A) is oscillatory. Since (5.1) holds by Proposition 5.1, we see from (5.27) that

$$\int^\infty q(t)\varphi_{2n-1}(c_1 t)^\beta dt = \infty,$$

which implies in view of Theorem 5.3 that all solutions of the equation

$$\left(p(t)|x^{(n)}(t)|^\alpha \operatorname{sgn} x^{(n)}(t)\right)^{(n)} + q(t)|x(c_1 t)|^\beta \operatorname{sgn} x(c_1 t) = 0, \tag{5.30}$$

are oscillatory. Since $g(t) \geqq c_1 t$ for large t, the oscillation of (5.30) ensures that of equation (A_g) by Theorem 4.1.

Conversely suppose that equation (A_g) is oscillatory. Then, (5.13) holds by Theorem 5.3. Since $g(t) \leqq c_2 t$ for large t, this implies

$$\int^\infty q(t)\varphi_{2n-1}(c_2 t)^\beta dt = \infty, \tag{5.31}$$

from which (5.1) follows immediately. (cf. (5.27)) This means that equation (A) is oscillatory. This completes the proof.

(The case where $\alpha \leqq 1 < \beta$.) We first remark that if $p(t)$ is regularly varying, then the integral

$$\int^\infty t^{n-1}\left[\frac{1}{p(t)}\int_{\frac{t}{c}}^\infty (cs-t)^{n-1} q(s)ds\right]^{\frac{1}{\alpha}} dt \tag{5.32}$$

either converge for all $c > 0$ or diverges for all $c > 0$. In fact, let any $c > 0$ be given and choose b so that $cb \geqq a$. Noting that

$$\int_{\frac{t}{c}}^\infty (cs-t)^{n-1} q(s)ds = c^{n-1}\int_{\frac{t}{c}}^\infty \left(s-\frac{t}{c}\right)^{n-1} q(s)ds,$$

and using the inequality $lp(t) \leqq p(t/c) \leqq Lp(t)$, $t \geqq b$, $l > 0$ and $L > 0$ being constants, which is implied by the regularity of $p(t)$, we see that there exist positive constants m and M such that

$$\begin{aligned}\frac{m}{p(\frac{t}{c})}\int_{\frac{t}{c}}^\infty \left(s-\frac{t}{c}\right)^{n-1} q(s)ds &\leqq \frac{1}{p(t)}\int_{\frac{t}{c}}^\infty (cs-t)^{n-1} q(s)ds \\ &\leqq \frac{M}{p(\frac{t}{c})}\int_{\frac{t}{c}}^\infty \left(s-\frac{t}{c}\right)^{n-1} q(s)ds\end{aligned}$$

for $t \geqq b$. This shows that for any $c > 0$ the convergence or divergence of (5.32) is identical with that of the integral

$$\int^\infty t^{n-1}\left[\frac{1}{p(t)}\int_t^\infty (s-t)^{n-1} q(s)ds\right]^{\frac{1}{\alpha}} dt.$$

Suppose now that equation (A) is oscillatory. By Proposition 5.2 we may assume that (5.4) holds without loss of generality. Hence from the above remark (5.32) holds with $c = c_1 < 1$, which implies by Theorem 5.2 (with $h(t) = c_1 t$) that the equation

$$\left(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t)\right)^{(n)} + q(t)|x(c_1 t)|^{\beta}\operatorname{sgn}x(c_1 t) = 0$$

is oscillatory. Since $c_1 t < g(t)$, Theorem 4.1 ensures that equation (A_g) is oscillatory.

Conversely, suppose that equation (A_g) is oscillatory. Since $g(t) \leqq c_2 t$, the equation

$$\left(p(t)|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t)\right)^{(n)} + q(t)|x(c_2 t)|^{\beta}\operatorname{sgn}x(c_2 t) = 0$$

is necessarily oscillatory. We may assume that $\int_a^{\infty} t^{n-1} q(t)dt < \infty$, in this case (5.1) must hold by Theorem 5.4. This implies the oscillation of equation (A) (cf. Proposition 5.2.) This completes the proof. $\square$

6. Example

We present an example which illustrate our main results developed in the previous sections. Consider the equation

$$(t^{-\gamma}|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t))^{(n)} + t^{-\delta}|x(t^{\lambda})|^{\beta}\operatorname{sgn}x(t^{\lambda}) = 0, \quad t \geqq 1, \tag{6.1}$$

where γ, δ and λ are constants such that $\gamma > \alpha$ and $\lambda > 0$. We let γ and λ be fixed and regard δ as a varying parameter. It is clear that the regularly varying function $p(t) = t^{-\gamma}$ with $\gamma > \alpha$ satisfies condition (1.1). The particular solutions $\varphi_j(t)$, $j \in \{0,1,\cdots,2n-1\}$, of the ordinary differential equation

$$(t^{-\gamma}|x^{(n)}(t)|^{\alpha}\operatorname{sgn}x^{(n)}(t))^{(n)} + t^{-\delta}|x(t)|^{\beta}\operatorname{sgn}x(t) = 0 \tag{6.2}$$

defined by (1.2) satisfy

$$\varphi_j(t) \sim t^j, \quad j \in \{0,1,\cdots,n\}, \quad \varphi_j(t) \sim t^{n+\frac{\gamma+j-n}{\alpha}}, \quad j \in \{n+1,n+2,\cdots,2n-1\}, \tag{6.3}$$

where the symbol $\sim$ denotes the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \quad \iff \quad \lim_{t\to\infty}\frac{g(t)}{f(t)} = 1.$$

A simple calculation shows that condition (3.1) holds if and only if

$$\delta > n + \alpha(n-j) + \gamma + \beta\lambda j, \quad j \in \{0,1,\cdots,n\}, \tag{6.4}$$

and that (3.2) holds if and only if

$$\delta > 2n - j + \left(n + \frac{j-n+\gamma}{\alpha}\right)\beta\lambda, \quad j \in \{n+1,n+2,\cdots,2n-1\}. \tag{6.5}$$

Thus, from Theorem 3.1 applied to equation (6.1) it follows that if δ satisfies (6.4) or (6.5) then (6.1) posseses a positive solution belonging to class $P(\mathrm{I}_j)$, $j \in \{0,1,\cdots,2n-1\}$.

Let k be an integer in $\{1,3,\cdots,2n-1\}$. In order to apply Theorem 3.2 to (6.1) we need to express the conditions (3.27)-(3.32) as the relations between k and the parameters defining equation (6.1). This can be done without difficulty as follows:

$$(3.27) \iff \delta > n+\alpha(n-k)+\gamma+\beta\lambda k \quad \text{for } 0<k<n;$$

$$(3.28) \iff \delta \leqq n+\alpha(n-k)+\gamma+\beta\lambda k+\alpha-\beta\lambda \quad \text{for } 0<k<n;$$

$$(3.29) \iff \delta > n+\beta\lambda n = (1+\beta\lambda)n \quad \text{for } k=n;$$

$$(3.30) \iff \delta \leqq (1+\beta\lambda)n+\gamma+\alpha-\beta\lambda \quad \text{for } k=n;$$

$$(3.31) \iff \delta > 2n-k+\beta\lambda\left(n+\frac{k-n+\gamma}{\alpha}\right) \quad \text{for } n<k\leqq 2n-1;$$

$$(3.32) \iff \delta \leqq 2n-k+\beta\lambda\left(n+\frac{k-n+\gamma}{\alpha}\right)+1-\frac{\beta\lambda}{\alpha} \quad \text{for } n<k\leqq 2n-1.$$

From Theorem 3.2 we then conclude that if $\{(3.27),(3.28)\}$, $\{(3.29),(3.30)\}$ or $\{(3.31),(3.32)\}$ holds, then (6.1) possesses a positive solution of class $P(\mathrm{II}_k)$ for the corresponding value of odd k.

Next, we refer to oscillation criteria for (6.1). A simple calculation shows that

$$(5.1) \iff \delta \leqq 1+\beta\left(n+\frac{n-1+\gamma}{\alpha}\right);$$

$$(5.2) \iff \delta \leqq n;$$

$$(5.3) \iff \delta > n;$$

$$(5.4) \iff \delta \leqq (\alpha+1)n+\gamma;$$

$$(5.9) \text{ with } h(t)=t^\lambda \iff \delta \leqq \lambda(n-1)+1;$$

$$(5.10) \text{ with } h(t)=t^\lambda \iff \lambda(n-1)+1<\delta\leqq\lambda(\alpha+1)n+\gamma\lambda;$$

$$(5.13) \iff \delta \leqq 1+\beta\lambda\left(n+\frac{n-1+\gamma}{\alpha}\right).$$

Assume that $\alpha \geqq 1 > \beta$. If $0<\lambda\leqq 1$, then from Theorem 5.3 we see that all solutions of (6.1) are oscillatory if and only if

$$\delta \leqq 1+\beta\lambda\left(n+\frac{n-1+\gamma}{\alpha}\right).$$

If $\lambda > 1$, then from Theorem 5.1 we find that all solutions of (6.1) are oscillatory if

$$\delta \leqq 1+\beta\left(n+\frac{n-1+\gamma}{\alpha}\right).$$

Assume that $\alpha \leqq 1 < \beta$. In the case where $0 < \lambda \leqq 1$, it follows from Theorem 5.2 that all solution (6.1) are oscillatory if either $\delta \leqq \lambda(n-1)+1$ holds or else $\lambda(n-1)+1 < \delta \leqq \lambda(\alpha+1)n+\gamma\lambda$ hold. In the case where $\lambda > 1$, we see from Theorems 5.2 and 5.4 that all solutions of (6.1) are oscillatory if and only if either $\delta \leqq n$ holds or else $n < \delta \leqq (\alpha+1)n+\gamma$ hold.

REFERENCES

[1] T. A. CHANTURIA, *Monotone and oscillatory solutions of higher-order ordinary differential equations*, (Russian) Ann. Polon. Math., **37**, 1 (1980), 93–111.

[2] N. H. BINGHAM, C. M. GOLDIE, J. L. TEUGELS, *Regular Variation*, Encyclopedia of Mathematics and its Applications **27**, Cambridge Univ. Press, 1987.

[3] H. C. HOWARD, V. MARIĆ, *Regularity and nonoscillation of solutions of second order linear differential equations*, Bell. T. CXIV de Acad. Serbe Sci. et Arts, Classe Sci. mat. nat. Sci. math., **22** (1997), 85–98.

[4] H. C. HOWARD, V. MARIĆ, Z. RADAŠIN, *Asymptotics of nonoscillatory solutions of second order linear differential equations*, Zbornik Rad. Prirod. Mat. Fak. Univ. Novi Sad, Ser. Mat., **20** (1990), 107–116.

[5] J. JAROŠ, T. KUSANO, *Remarks on the existence of regularly varying solutions for second order linear differential equations*, Publ. Inst. Math. (Beograd), **72** (2002), 113–118.

[6] J. JAROŠ, T. KUSANO, T. TANIGAWA, *Nonoscillation theory for second order half-linear differential equations in the framework of regular variation*, Results Math., **43** (2003), 129–149.

[7] J. KARAMATA, *Sur un mode de croissance régulière des fonctions*, Math. (Cluj), **4** (1930), 38–53.

[8] I. T. KIGURADZE, *On the oscillation of solutions of the equation* $d^m u/dt^m + a(t)\,|u|^n \operatorname{sign} u = 0$, Mat. Sb., **65** (1964), 172–187.

[9] I. T. KIGURADZE, *On the question of variability of solutions of nonlinear differential equations*, Differencial'nye Uravnenija, **1** (1965), 995–1006.

[10] I. T. KIGURADZE, T. A. CHANTURIA, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Translated from the 1985 Russian original, Mathematics and its Applications (Soviet Series), **89**, Kluwer Academic Publishers Group, Dordrecht, xiv+331 p., 1993.

[11] Y. KITAMURA, T. KUSANO, *Nonlinear oscillation of higher-order functional-differential equations with deviating arguments*. J. Math. Anal. Appl., **77**, 1 (1980), 100–119.

[12] T. KUSANO, B. S. LALLI, *On oscillation of half-linear functional differential equations with deviating arguments*, Hiroshima Math. J., **24** (1994), 549–563.

[13] T. KUSANO, V. MARIĆ, *On a class of functional differential equations having slowly varying solutions*, Publ. Inst. Math. (Beograd), **80**, (94) (2006), 207–217.

[14] T. KUSANO, V. MARIĆ, *Slowly varying solutions of functional differential equations with retarded and advanced arguments*, Georgian Math. J., **14**, (2) (2007), 301–314.

[15] T. KUSANO, V. MARIĆ, *Regularly varying solutions to functional differential equations with deviating arguments*, Bull. T. CXXXIV de Acad. Serbe Sci. Arts, Classe Sci. Nat., Sci. Math., **32** (2007), 105–128.

[16] T. KUSANO, M. NAITO, *Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order*, Hiroshima Math. J., **18** (1988), 361–372.

[17] T. KUSANO, M. NAITO, *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. Japan, **33** (1981), 509–532.

[18] T. KUSANO, H. ONOSE, *Oscillation of solutions of nonlinear differential delay equations of arbitrary order*, Hiroshima Math. J., **2** (1972), 1–13.

[19] T. KUSANO, H. ONOSE, *Oscillation theorems for delay equations of arbitrary order*, Hiroshima Math. J., **2** (1972), 263–270.

[20] T. KUSANO, J. WANG, *Oscillation properties of half-linear functional differential equations of the second order*, Hiroshima Math. J., **25** (1995), 371–385.

[21] W. E. MAHFOUD, *Comparison theorems for delay differential equations*, Pacific J. Math., **83** (1979), 187–197.

[22] J. MANOJLOVIĆ, T. TANIGAWA, *Oscillation and nonoscillation theorems for a class of even-order quasilinear functional differential equations*, J. Inequal. Appl., (2006) Art. ID 42120, 22 pp.

[23] V. MARIĆ, *Regular Variation and Differential Equations*, Lecture Notes in Mathematics, Vol. 1726, Springer-Verlag, Berlin-Heidelberg, 2000.

[24] V. MARIĆ, M. TOMIĆ, *Asymptotic properties of solutions of the equation* $y'' = f(x)\varphi(y)$, Math. Z., **149** (1976), pp. 261–266.

[25] V. MARIĆ, M. TOMIĆ, *A trichotomy of solutions of second order linear differential equations*, Zbornik Rad. Prirod.-Mat. Fak. Univ. Novi Sad, Ser. Mat., **14** (1984), 1–11.

[26] V. MARIĆ, M. TOMIĆ, *A classification of solutions of second order linear differential equations by means of regularly varying functions*, Publ. Inst. Math. (Beograd), **48**, (62) (1990), 199–207.

[27] V. MARIĆ, M. TOMIĆ, *Slowly varying solutions of second order linear differential equations*, Publ. Inst. Math. (Beograd), **58**, (72) (1995), 129–136.

[28] H. ONOSE, *Some oscillation criteria for* n*-th order nonlinear delay-differential equations*, Hiroshima Math. J., **1** (1971), 171–176.

[29] H. ONOSE, *Oscillation and asymptotic behavior of solutions of retarded differential equations of arbitrary order*, Hiroshima Math. J., **3** (1973), 333–360.

[30] T. TANIGAWA, *Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations*, Hiroshima Math. J., **23** (2003), 297–316.

[31] T. TANIGAWA, *Oscillation theorems for a class of even order quasilinear differential equations*, Mem. Differential Equations Math. Phys., **37** (2006), 137–152.

[32] T. TANIGAWA, *Oscillation theorems for differential equations involving even order nonlinear Sturm-Liouville operator*, Georgian Math. J., **14** (2007), 737–768.

[33] T. TANIGAWA, F. WU, *On the existence of positive solutions for a class of even order quasilinear differential equations*, Adv. Math. Sci. Appl., **14** (2004), 75–85.

[34] J. WANG, *Oscillation and nonoscillation theorems for a class of second order quasilinear functional differential equations*, Hiroshima Math. J., **27** (1997), 449–466.

(Received August 3, 2011)

Akihito Shibuya
Department of Mathematics
Graduate School of Science and Technology
Kumamoto University
2–39–1, Kurokami, Kumamoto
860–8555, Japan
e-mail: akihito.shibuya@gmail.com