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INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER MIXED NONLINEAR FORCED IMPULSIVE DIFFERENTIAL EQUATION WITH DAMPING TERM

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Abstract. In this paper, interval oscillation criteria are established for second order forced impulsive differential equations with mixed nonlinearities of the form

$$\begin{cases} \big(r(t)\Phi_\alpha(x'(t))\big)' + p(t)\Phi_\alpha(x'(t)) + q(t)\Phi_\alpha(x(t)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t)) = e(t), \quad t \neq \tau_k, \\ x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k), \quad k = 1,2,\dots \ . \end{cases}$$

The results obtained in this paper extend some of the existing results and are illustrated by examples.

1. Introduction

In recent years the theory of impulsive differential equations emerging as an important area of research, since such equations have applications in the control theory, physics, biology, population dynamics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equation, see for example [1, 5]. The oscillation of solutions of second order ordinary differential equations are systematically studied by several authors. In [6, 12, 13, 14], the authors studied the oscillation of solutions of second order ordinary differential equations with mixed nonlinearities. Due to difficulties caused by impulsive perturbations there is a less attention regarding the oscillation problem for impulsive differential equation [2, 7, 8, 9, 10, 15]. Motivated by the work of [2, 9], we use arithmetic-geometric mean inequality, Riccati transformation to obtain the interval oscillation criteria for second-order forced impulsive differential equation with mixed nonlinearities.Our results are extension of some known results. Examples are also given to illustrate the results.

Consider the second-order impulsive differential equation,

$$\begin{cases} \big(r(t)\Phi_\alpha(x'(t))\big)' + p(t)\Phi_\alpha(x'(t)) + q(t)\Phi_\alpha(x(t)) \\ \qquad\qquad + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x(t)) = e(t), \ \ t \neq \tau_k, \\ x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k), \quad k = 1,2,\dots, \end{cases} \tag{1.1}$$

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where $\Phi_\star(s) = |s|^{\star-1}s$, $k \in \mathbb{N}$, $t \geqslant t_0$, $\{\tau_k\}$ is the impulse moments sequence with $0 \leqslant t_0 = \tau_0 < \tau_1 < \ldots < \tau_k < \ldots \lim_{k\to\infty} \tau_k = \infty$, and

$$x(\tau_k) = x(\tau_k^-) = \lim_{t\to\tau_k^{-0}} x(t), \quad x(\tau_k{}^+) = \lim_{t\to\tau_k^{+0}} x(t),$$

$$x'(\tau_k) = x'(\tau_k{}^-) = \lim_{h\to 0^-} \frac{x(\tau_k+h)-x(\tau_k)}{h}, \quad x'(\tau_k{}^+) = \lim_{h\to 0^+} \frac{x(\tau_k+h)-x(\tau_k{}^+)}{h}.$$

Throughout this paper, we always assume the following conditions hold:

(A1) $r \in C^1([t_0,\infty),(0,\infty))$, $p,q,q_i,e \in C([t_0,\infty),\mathbb{R})$, $i = 1,2\ldots,n$;
(A2) $\beta_1 > \ldots > \beta_m > \alpha > \beta_{m+1} > \ldots > \beta_n > 0$ are constants;
(A3) $b_k \geqslant a_k > 0$, $k \in \mathbb{N}$ are constants.

Let $J \subset \mathbb{R}$ be an interval and define $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is piecewise-left-continuous and has discontinuity of first kind at $\tau_k' s\}$.

By a solution of (1.1), we mean a function $x \in PC([t_0,\infty),\mathbb{R})$ such that $x' \in PC([t_0,\infty),\mathbb{R})$ and x(t) satisfies (1.1) for $t \geqslant t_0$. A nontrivial solution is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. An equation is called oscillatory if all its solutions are oscillatory.

2. Main results

We begin with the following notation. Let $k(s) = \max\{i : t_0 < \tau_i < s\}$, let $r_j = \max\{r(t) : t \in [c_j,d_j]\}$ for $c_j < d_j$ and

$$\mathscr{F}(c_j,d_j) = \{u \in C^1\left([c_j,d_j],\mathbb{R})\right) : u(t) \not\equiv 0, u(c_j) = u(d_j) = 0\},\ j = 1,2.$$

For two constants $c,d \notin \{\tau_k\}$ with $c < d$ and a function $\phi \in C([c,d],\mathbb{R})$, we define an operator $\Omega : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\Omega_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(\tau_{k(c)+1})\theta(c) + \sum_{i=k(c)+2}^{k(d)} \phi(\tau_i)\varepsilon(\tau_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{(b_{k(c)+1})^\alpha - (a_{k(c)+1})^\alpha}{(a_{k(c)+1})^\alpha(\tau_{k(c)+1}-c)^\alpha}, \quad \varepsilon(\tau_i) = \frac{(b_i)^\alpha - (a_i)^\alpha}{(a_i)^\alpha(\tau_i - \tau_{i-1})^\alpha}.$$

Following Kong [4] and Philos [11], we introduce a class of functions: Let $D = \{(t,s) : t_0 \leqslant s \leqslant t\}$, $H_1,H_2 \in C^1(D,\mathbb{R})$. A pair of functions (H_1,H_2) is said to belong to a function class $\mathscr{H}$, if $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0$, $H_2(t,s) > 0$ for $t > s$ and there exist $h_1,h_2 \in L_{\text{loc}}(D,\mathbb{R})$ such that

$$\frac{\partial H_1(t,s)}{\partial t} = h_1(t,s)H_1(t,s), \quad \frac{\partial H_2(t,s)}{\partial s} = h_2(t,s)H_2(t,s). \tag{2.1}$$

The following preparatory lemmas will be useful to prove our theorems.

LEMMA 2.1. *Let* $\{\beta_i\}$, $i=1,2,\dots,n$, *be the* n*-tuple satisfying* $\beta_1 > \dots > \beta_m > \alpha > \beta_{m+1} > \dots > \beta_n > 0$. *Then there exist an* n*-tuple* $(\eta_1,\eta_2,\dots,\eta_n)$ *satisfying*

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \tag{2.2}$$

which also satisfies either

$$\sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1, \tag{2.3}$$

or

$$\sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.4}$$

The proof of Lemma 2.1 can be obtained easily from Lemma 1 of [14] by taking $\alpha_i = \beta_i/\alpha$.

REMARK 2.1. For a given set of exponents $\{\beta_i\}$ satisfying $\beta_1 > \dots > \beta_m > \alpha > \beta_{m+1} > \dots > \beta_n > 0$, Lemma2.1 ensures the existence of an n-tuple $(\eta_1,\eta_2,...,\eta_n)$ such that either (2.2) and (2.3) hold or (2.2) and (2.4) hold. When $n=2$ and $\beta_1 > \alpha > \beta_2 > 0$, in the first case, we have

$$\eta_1 = \frac{\alpha - \beta_2(1-\zeta)}{\beta_1 - \beta_2}, \quad \eta_2 = \frac{\beta_1(1-\zeta) - \alpha}{\beta_1 - \beta_2},$$

where ζ can be any positive number satisfying $0 < \zeta < (\beta_1 - \alpha)/\beta_1$. This will ensure that $0 < \eta_1, \eta_2 < 1$ and conditions (2.2) and (2.3) are satisfied. In the second case, we simply solve (2.2) and (2.4) and obtain

$$\eta_1 = \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \quad \eta_2 = \frac{\beta_1 - \alpha}{\beta_1 - \beta_2}.$$

The Lemma below can be found in [3].

LEMMA 2.2. *Let* $A, B \in \mathbb{R}$ *and* $\gamma > 0$ *be a constant, then*

$$A\Phi_\gamma(A) + \gamma B\Phi_\gamma(B) \geqslant (\gamma+1)A\Phi_\gamma(B), \tag{2.5}$$

where inequality holds if and only if $A=B$.

THEOREM 2.1. *Suppose that for any* $T>0$, *there exist* $c_j, d_j \notin \{\tau_k\}$, $j=1,2$ *such that* $c_1 < d_1 \leqslant c_2 < d_2$, *and*

$$\begin{gathered} q(t), q_i(t) \geqslant 0, \quad t \in [c_1,d_1] \cup [c_2,d_2],\ i=1,2\dots,n; \\ (-1)^j e(t) \leqslant 0, \quad t \in [c_j,d_j], j=1,2. \end{gathered} \tag{2.6}$$

Let $\{\eta_i\}$, $i=1,2,\dots,n$, *be an* n *tuple satisfying* (2.2) *and* (2.3). *If there exist* $u \in \mathscr{F}(c_j,d_j)$ *and* $\rho(t) \in C^1([c_1,d_1] \cup [c_2,d_2], (0,\infty))$ *such that*

$$\int_{c_j}^{d_j} \rho(t)\Big[Q(t)|u(t)|^{\alpha+1} - \frac{r(t)}{(\alpha+1)^{\alpha+1}}\Big|(\alpha+1)u'(t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)|u(t)|\Big|^{\alpha+1}\Big]dt > r_j M_j \Omega_{c_j}^{d_j}\big[|u(t)|^{\alpha+1}\big], \tag{2.7}$$

for $j=1,2$, *where* M_j *is maximum value of* $\rho(t)$ *in* $[c_j,d_j]$ *and*

$$Q(t) = q(t)+\eta_0{}^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i},\ \eta_0 = 1-\sum_{i=1}^{n}\eta_i, \tag{2.8}$$

then (1.1) *is oscillatory.*

Proof. Let us suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)>0$ for $t\in[c_1,d_1]$. Define

$$w(t) = \rho(t)\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}, \quad t\in[c_1,d_1]. \tag{2.9}$$

Then for $t\in[c_1,d_1]$ and $t\neq\tau_k$, we have

$$w'(t) = \Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)w(t) + \rho(t)\Big[-q(t)-\sum_{i=1}^{n}q_i(t)\Phi_{\beta_i-\alpha}(x(t))-\frac{|e(t)|}{\Phi_\alpha(x(t))}\Big] - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}. \tag{2.10}$$

Recall the arithmetic-geometric mean inequality,

$$\sum_{i=0}^{n}\eta_i v_i \geqslant \prod_{i=0}^{n} v_i^{\eta_i}, \quad v_i\geqslant 0.$$

Take

$$v_0 = \eta_0^{-1}\tfrac{|e(t)|}{\Phi_\alpha(x(t))} \ \text{ and } \ v_i = \eta_i^{-1}q_i(t)\Phi_{\beta_i-\alpha}(x(t)),\ i=1,2,\dots,n$$

and applying (2.2) and (2.3), we get

$$-\sum_{i=1}^{n}q_i(t)\Phi_{\beta_i-\alpha}(x(t))-\frac{|e(t)|}{\Phi_\alpha(x(t))} \leqslant -\eta_0{}^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}. \tag{2.11}$$

Now, equation (2.10) becomes

$$w'(t)\leqslant\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)w(t)-\rho(t)Q(t)-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}, \tag{2.12}$$

where

$$Q(t) = q(t) + \eta_0{}^{-\eta_0}|e(t)|^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i}(q_i(t))^{\eta_i}.$$

For $t = \tau_k$, $k = 1,2,\ldots$, from (2.9), we have

$$w(\tau_k{}^+) = \frac{b_k{}^\alpha}{a_k{}^\alpha} w(\tau_k). \tag{2.13}$$

If $k(c_1) < k(d_1)$,then there are all impulsive moments in $[c_1,d_1]$; $\tau_{k(c_1)+1}$, $\tau_{k(c_1)+2}$, $\ldots$, $\tau_{k(d_1)}$. Multiplying both sides of (2.12) by $|u(t)|^{\alpha+1}$, where $u(t) \in \mathscr{F}(c_1,d_1)$ and integrating over $[c_1,d_1]$,then using integration by parts and the fact that $u(c_1) = u(d_1)$, we obtain

$$\begin{aligned}
&\int_{c_1}^{d_1} \rho(t)Q(t)|u(t)|^{\alpha+1}dt - \sum_{i=k(c_1)+1}^{k(d_1)} |u(\tau_i)|^{\alpha+1}[w(\tau_i^+) - w(\tau_i)] \\
&\quad \leqslant \Big(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \ldots + \int_{\tau_{k(d_1)}}^{d_1}\Big)\Big[(\alpha+1)\Phi_\alpha(u(t))u'(t)w(t) \\
&\qquad + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)|u(t)|^{\alpha+1}w(t) - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}|u(t)|^{\alpha+1}\Big]dt \\
&\quad \leqslant \Big(\int_{c_1}^{\tau_{k(c_1)+1}} + \int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}} + \ldots + \int_{\tau_{k(d_1)}}^{d_1}\Big)\Big[\Big|(\alpha+1)\Phi_\alpha(u(t))u'(t) \\
&\qquad + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)|u(t)|^{\alpha+1}\Big||w(t)| - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}|u(t)|^{\alpha+1}\Big]dt.
\end{aligned} \tag{2.14}$$

Use Lemma 2.2 with

$$\gamma = \frac{1}{\alpha} \quad \text{and} \quad A = \alpha^{\frac{\alpha}{\alpha+1}} \frac{|u|^\alpha|w|}{(\rho r)^{\frac{1}{\alpha+1}}},$$

$$B = \Big(\frac{\alpha\rho r}{(\alpha+1)^{(\alpha+1)}|u|^{\alpha(\alpha+1)}}\Big)^{\frac{\alpha}{\alpha+1}} \Big|(\alpha+1)\Phi_\alpha(u)u' + \Big(\frac{\rho'}{\rho} - \frac{p}{r}\Big)|u|^{\alpha+1}\Big|^\alpha,$$

we have

$$\begin{aligned}
&\Big|(\alpha+1)\Phi_\alpha(u)u' + \Big(\frac{\rho'}{\rho} - \frac{p}{r}\Big)|u|^{\alpha+1}\Big||w| - \frac{\alpha}{(\rho r)^{\frac{1}{\alpha}}}|w|^{\frac{\alpha+1}{\alpha}}|u|^{\alpha+1} \\
&\qquad\qquad \leqslant (\alpha+1)^{-(\alpha+1)}\Big|(\alpha+1)u' + \Big(\frac{\rho'}{\rho} - \frac{p}{r}\Big)|u|\Big|^{\alpha+1},
\end{aligned}$$

where dependent variable t suppressed for clarity. By (2.13) and (2.14), we have

$$\int_{c_1}^{d_1} \rho(t)Q(t)|u(t)|^{\alpha+1}dt - \sum_{i=k(c_1)+1}^{k(d_1)} |u(\tau_i)|^{\alpha+1}\Big[\frac{b_i{}^\alpha - a_i{}^\alpha}{a_i{}^\alpha}\Big]w(\tau_i)$$

$$\leqslant (\alpha+1)^{-(\alpha+1)} \int_{c_1}^{d_1} \rho(t) r(t) \Big| (\alpha+1) u'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} \Big) |u(t)| \Big|^{\alpha+1} dt.$$

It follows that

$$\int_{c_1}^{d_1} \rho(t) \Big[Q(t) |u(t)|^{\alpha+1} - (\alpha+1)^{-(\alpha+1)} r(t) \times \Big| (\alpha+1) u'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} \Big) |u(t)| \Big|^{\alpha+1} \Big] dt \leqslant \sum_{i=k(c_1)+1}^{k(d_1)} |u(\tau_i)|^{\alpha+1} \Big[\frac{b_i{}^{\alpha} - a_i{}^{\alpha}}{a_i{}^{\alpha}} \Big] w(\tau_i). \tag{2.15}$$

On the other hand, for $t \in (c_1, \tau_{k(c_1)+1}]$, from (1.1), it is clear that

$$\big(r(t) \Phi_\alpha(x'(t)) \big)' + p(t) \Phi_\alpha(x'(t)) = e(t) - q(t) \Phi_\alpha(x(t)) - \sum_{i=1}^{n} q_i(t) \Phi_{\beta_i}(x(t)) \leqslant 0$$

i.e.

$$\big[\Phi_\alpha(x'(t)) \big]' + \Big(\frac{r'(t) + p(t)}{r(t)} \Big) \Phi_\alpha(x'(t)) \leqslant 0$$

which implies that

$$\Phi_\alpha(x'(t)) \exp \Big(\int_{c_1}^{t} \frac{r'(s) + p(s)}{r(s)} ds \Big)$$

is non-increasing on $(c_1, \tau_{k(c_1)+1}]$. So for any $t \in (c_1, \tau_{k(c_1)+1}]$, we have $x(t) - x(c_1) = x'(\xi)(t - c_1), \xi \in (c_1, t)$, which implies from $x(c_1) > 0$ that $x(t) > x'(\xi)(t - c_1), \xi \in (c_1, t)$. According to stipulation of α, we have

$$(x(t))^\alpha > (x'(\xi))^\alpha (t - c_1)^\alpha = |x'(\xi)|^{\alpha-1} (x'(\xi)) (t - c_1)^\alpha. \tag{2.16}$$

Since $\Phi_\alpha(x'(t)) \exp \big(\int_{c_1}^{t} \frac{r'(s)+p(s)}{r(s)} ds \big)$ is non-increasing, then

$$|x'(\xi)|^{\alpha-1} x'(\xi) \exp \Big(\int_{c_1}^{\xi} \frac{r'(s) + p(s)}{r(s)} ds \Big) \geqslant |x'(t)|^{\alpha-1} x'(t) \exp \Big(\int_{c_1}^{t} \frac{r'(s) + p(s)}{r(s)} ds \Big), \tag{2.17}$$

where $\xi \in (c_1, t)$. By (2.16) and (2.17), we obtain

$$(x(t))^\alpha \geqslant \frac{|x'(t)|^{\alpha-1} x'(t) \exp \Big(\int_{c_1}^{t} \frac{r'(s)+p(s)}{r(s)} ds \Big)}{\exp \Big(\int_{c_1}^{\xi} \frac{r'(s)+p(s)}{r(s)} ds \Big)} (t - c_1)^\alpha \geqslant |x'(t)|^{\alpha-1} x'(t) (t - c_1)^\alpha$$

for some $\xi \in (c_1, t)$. That is, $\Phi_\alpha(x(t)) \geqslant \Phi_\alpha(x'(t))(t-c_1)^\alpha$. It follows

$$\frac{\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))} \leqslant \frac{1}{(t-c_1)^\alpha}.$$

Letting $t \to \tau_{k(c_1)+1}^-$, from (2.9) we have

$$w(\tau_{k(c_1)+1}) = \rho(\tau_{k(c_1)+1}) \frac{r(\tau_{k(c_1)+1})\Phi_\alpha(x'(\tau_{k(c_1)+1}))}{\Phi_\alpha(x(\tau_{k(c_1)+1}))} \leqslant \frac{r_1 M_1}{(\tau_{k(c_1)+1} - c_1)^\alpha}. \tag{2.18}$$

Similarly we can prove that on (τ_{i-1}, τ_i),

$$w(\tau_i) \leqslant \frac{r_1 M_1}{(\tau_i - \tau_{i-1})^\alpha} \quad \text{for } i = k(c_1)+2, \ldots, k(d_1). \tag{2.19}$$

Using (2.18), (2.19) and (A3), we obtain

$$\begin{aligned}
&\sum_{i=k(c_1)+1}^{k(d_1)} \Big(\frac{b_i{}^\alpha - a_i{}^\alpha}{a_i{}^\alpha}\Big)|u(\tau_i)|^{\alpha+1} w(\tau_i) \\
&= \left(\frac{b_{k(c_1)+1}^\alpha - a_{k(c_1)+1}^\alpha}{a_{k(c_1)+1}^\alpha}\right)|u(\tau_{k(c_1)+1})|^{\alpha+1} w(\tau_{k(c_1)+1}) \\
&\qquad + \sum_{i=k(c_1)+2}^{k(d_1)} \Big(\frac{b_i{}^\alpha - a_i{}^\alpha}{a_i{}^\alpha}\Big)|u(\tau_i)|^{\alpha+1} w(\tau_i) \\
&\leqslant \left(\frac{b_{k(c_1)+1}^\alpha - a_{k(c_1)+1}^\alpha}{a_{k(c_1)+1}^\alpha}\right)|u(\tau_{k(c_1)+1})|^{\alpha+1} \frac{r_1 M_1}{(\tau_{k(c_1)+1} - c_1)^\alpha} \\
&\qquad + \sum_{i=k(c_1)+2}^{k(d_1)} \Big(\frac{b_i{}^\alpha - a_i{}^\alpha}{a_i{}^\alpha}\Big)|u(\tau_i)|^{\alpha+1} \frac{r_1 M_1}{(\tau_i - \tau_{i-1})^\alpha} \\
&= r_1 M_1 \Omega_{c_1}^{d_1}[|u(t)|^{\alpha+1}].
\end{aligned}$$

Thus, from (2.15) we have

$$\begin{aligned}
&\int_{c_1}^{d_1} \rho(t) \bigg[Q(t)|u(t)|^{\alpha+1} \\
&\qquad - \frac{r(t)}{(\alpha+1)^{\alpha+1}} \Big| (\alpha+1)u'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\Big)|u(t)| \Big|^{\alpha+1} \bigg] dt \\
&\qquad\qquad \leqslant r_1 M_1 \Omega_{c_1}^{d_1}\left[|u(t)|^{\alpha+1}\right],
\end{aligned}$$

which contradicts (2.7).

If $k(c_1) = k(d_1)$ then $\Omega_{c_1}^{d_1}[|u(t)|^{\alpha+1}] = 0$ and there is no impulsive moments in $[c_1, d_1]$. Similar to the proof of (2.15), we obtain

$$\int_{c_1}^{d_1} \rho(t) \Bigg[Q(t)|u(t)|^{\alpha+1} - \frac{r(t)}{(\alpha+1)^{\alpha+1}} \Big| (\alpha+1)u'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} \Big) |u(t)| \Big|^{\alpha+1} \Bigg] dt \leqslant 0.$$

It is again a contradiction with (2.7). This completes the proof when $x(t)$ is positive. The proof when $x(t)$ is eventually negative is analogous by repeating a similar argument on the interval $[c_2, d_2]$. □

The next theorem is for the case $e(t) = 0$.

THEOREM 2.2. *Suppose that for any $T > 0$, there exist $c_1, d_1 \notin \{\tau_k\}$, such that $c_1 < d_1$, and $q(t), q_i(t) \geqslant 0$, $t \in [c_1, d_1]$, $i = 1, 2 \ldots, n$. Let $\{\eta_i\}$, $i = 1, 2, \ldots, n$, be an n tuple satisfying (2.2) and (2.3). If there exist $u \in \mathscr{F}(c_1, d_1)$ and $\rho(t) \in C^1([c_1, d_1], (0, \infty))$ such that*

$$\int_{c_1}^{d_1} \rho(t) \Bigg[\overline{Q}(t)|u(t)|^{\alpha+1} - \frac{r(t)}{(\alpha+1)^{\alpha+1}} \Big| (\alpha+1)u'(t) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} \Big) |u(t)| \Big|^{\alpha+1} \Bigg] dt > r_1 M_1 \Omega_{c_1}^{d_1} \big[|u(t)|^{\alpha+1} \big], \tag{2.20}$$

where M_1 is maximum value of $\rho(t)$ in $[c_1, d_1]$ and

$$\overline{Q}(t) = q(t) + \prod_{i=1}^{n} \eta_i^{-\eta_i} (q_i(t))^{\eta_i}, \tag{2.21}$$

then (1.1) with $e(t) = 0$ is oscillatory.

Proof. The proof of the above theorem is immediate by putting $e(t) = 0$ and $\eta_0 = 0$ in the proof of Theorem 2.1. □

THEOREM 2.3. *Suppose that for any $T > 0$, there exist $c_j, d_j, \delta_j \notin \{\tau_k\}$, $j = 1, 2$ such that $c_1 < \delta_1 < d_1 \leqslant c_2 < \delta_2 < d_2$, and (2.6) holds. Let $\{\eta_i\}$, $i = 1, 2, \ldots, n$, be an n tuple satisfying (2.2) and (2.3). If there exist $(H_1, H_2) \in \mathscr{H}$ and $\rho(t) \in C^1([c_1, d_1] \cup [c_2, d_2], (0, \infty))$ such that*

$$\frac{1}{H_1(\delta_j, c_j)} \int_{c_j}^{\delta_j} H_1(t, c_j)\rho(t) \Big[Q(t) - \frac{r(t)}{(\alpha+1)^{\alpha+1}} \Big| h_1(t, c_j) + \Big(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)} \Big) \Big|^{\alpha+1} \Big] dt + \frac{1}{H_2(d_j, \delta_j)} \int_{\delta_j}^{d_j} H_2(d_j, t)\rho(t) \Big[Q(t) - \frac{r(t)}{(\alpha+1)^{\alpha+1}}$$

$$\times \Big|h_2(d_j,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt > \Lambda(H_1,H_2;c_j,d_j), \quad (2.22)$$

where

$$\Lambda(H_1,H_2;c_j,d_j)=\frac{r_jM_j}{H_1(\delta_j,c_j)}\Omega_{c_j}^{\delta_j}[H_1(.,c_j)]+\frac{r_jM_j}{H_2(d_j,\delta_j)}\Omega_{\delta_j}^{d_j}[H_2(d_j,.)], \quad (2.23)$$

$$Q(t)=q(t)+\eta_0{}^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i} \quad \text{and} \quad \eta_0=1-\sum_{i=1}^{n}\eta_i,$$

then (1.1) *is oscillatory.*

Proof. Without loss of generality, we suppose that the solution $x(t)$ of (1.1) is eventually positive. Proceed as in the proof of Theorem 2.1,we get (2.12) and (2.13). Notice that whether there are or not impulsive moments in $[c_1,\delta_1]$ and $[\delta_1,d_1]$, we must consider the following 4 cases, namely, $k(c_1)<k(\delta_1)<k(d_1)$; $k(c_1)=k(\delta_1)<k(d_1)$; $k(c_1)<k(\delta_1)=k(d_1)$ and $k(c_1)=k(\delta_1)=k(d_1)$.

CASE 1. If $k(c_1)<k(\delta_1)<k(d_1)$, then there are impulsive moments $\tau_{k(c_1)+1}$, $\tau_{k(c_1)+2},\ldots,$ $\tau_{k(\delta_1)}$ in $[c_1,\delta_1]$ and $\tau_{k(\delta_1)+1},\tau_{k(\delta_1)+2},\ldots,\tau_{k(d_1)}$ in $[\delta_1,d_1]$ respectively. Multiplying both sides of inequality (2.12) by $H_1(t,c_1)$, then integrating it from c_1 to δ_1, we have

$$\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)Q(t)dt \leqslant -\int_{c_1}^{\delta_1}H_1(t,c_1)w'(t)dt+\int_{c_1}^{\delta_1}\Big[\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)w(t) - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}\Big]H_1(t,c_1).$$

Applying integration by parts on first integral of RHS inequality we get,

$$\begin{aligned}\int_{c_1}^{\delta_1}&\rho(t)H_1(t,c_1)Q(t)dt\\ &\leqslant \sum_{i=k(c_1)+1}^{k(\delta_1)}H_1(\tau_i,c_1)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i)-H_1(\delta_1,c_1)w(\delta_1)\\ &\quad+\Big(\int_{c_1}^{\tau_{k(c_1)+1}}+\int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}}+\ldots+\int_{\tau_{k(\delta_1)}}^{\delta_1}\Big)\Big[h_1(t,c_1)w(t)\\ &\qquad+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)w(t)-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}\Big]H_1(t,c_1)dt\\ &\leqslant \sum_{i=k(c_1)+1}^{k(\delta_1)}H_1(\tau_i,c_1)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i)-H_1(\delta_1,c_1)w(\delta_1)\\ &\quad+\Big(\int_{c_1}^{\tau_{k(c_1)+1}}+\int_{\tau_{k(c_1)+1}}^{\tau_{k(c_1)+2}}+\ldots+\int_{\tau_{k(\delta_1)}}^{\delta_1}\Big)\end{aligned}$$

$$\times \Big[\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big||w(t)| - \frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}\Big]H_1(t,c_1)dt. \quad (2.24)$$

Now using Lemma 2.2 with the notations $\gamma=\dfrac{1}{\alpha}, A=\alpha^{\frac{\alpha}{\alpha+1}}\dfrac{|w|}{(\rho r)^{\frac{1}{\alpha+1}}}$,

$$B=\Big(\frac{\alpha\rho r}{(\alpha+1)^{(\alpha+1)}}\Big)^{\frac{\alpha}{\alpha+1}}\Big|h_1(t,c_1)+\Big(\frac{\rho'}{\rho}-\frac{p}{r}\Big)\Big|^{\alpha},$$

we have

$$\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)Q(t)dt \leqslant \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i,c_1)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i)-H_1(\delta_1,c_1)w(\delta_1) + \int_{c_1}^{\delta_1}\Big[(\alpha+1)^{-(\alpha+1)}\rho(t)r(t)\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]H_1(t,c_1)dt,$$

that is

$$\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt \leqslant \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i,c_1)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i)-H_1(\delta_1,c_1)w(\delta_1). \quad (2.25)$$

Next, multiplying both sides of inequality (2.12) by $H_2(d_1,t)$ and using similar analysis to the above, we obtain

$$\int_{\delta_1}^{d_1}\rho(t)H_2(d_1,t)\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\Big|h_2(d_1,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt \leqslant \sum_{i=k(\delta_1)+1}^{k(d_1)} H_2(d_1,\tau_i)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i)+H_2(d_1,\delta_1)w(\delta_1). \quad (2.26)$$

Dividing (2.25) and (2.26) by $H_1(\delta_1,c_1)$ and $H_2(d_1,\delta_1)$ respectively, then adding them, we get

$$\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t) \times\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt+\frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1}\rho(t)H_2(d_1,t) \times\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\Big|h_2(d_1,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt$$

$$\leqslant \frac{1}{H_1(\delta_1,c_1)} \sum_{i=k(c_1)+1}^{k(\delta_1)} H_1(\tau_i,c_1)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i) + \frac{1}{H_2(d_1,\delta_1)} \sum_{i=k(\delta_1)+1}^{k(d_1)} H_2(d_1,\tau_i)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i). \quad (2.27)$$

In view of (2.18) and (2.19) we obtain

$$\begin{aligned}
&\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\\
&\quad\times\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt+\frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1}\rho(t)H_2(d_1,t)\\
&\quad\times\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\Big|h_2(d_1,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt\\
&\leqslant\frac{r_1M_1}{H_1(\delta_1,c_1)}\Omega_{c_1}^{\delta_1}[H_1(.,c_1)]+\frac{r_1M_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}[H_2(d_j,.)]\\
&=\Lambda(H_1,H_2;c_1,d_1),
\end{aligned}$$

which contradicts (2.22).

CASE 2. If $k(c_1)=k(\delta_1)<k(d_1)$, there is no impulsive moment in $[c_1,\delta_1]$,then (2.24) is replaced by

$$\begin{aligned}
&\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)Q(t)dt\\
&\quad\leqslant -H_1(\delta_1,c_1)w(\delta_1)+\int_{c_1}^{\delta_1}\Big[\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\times\Big||w(t)|\\
&\qquad-\frac{\alpha}{(\rho(t)r(t))^{\frac{1}{\alpha}}}|w(t)|^{\frac{\alpha+1}{\alpha}}\Big]H_1(t,c_1)dt.
\end{aligned}$$

Using Lemma 2.2, we have

$$\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt \leqslant -H_1(\delta_1,c_1)w(\delta_1). \quad (2.28)$$

Dividing (2.26) and (2.28) by $H_2(d_1,\delta_1)$ and $H_1(\delta_1,c_1)$ respectively, then adding them, we get

$$\begin{aligned}
&\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1}\rho(t)H_1(t,c_1)\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\\
&\quad\times\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt+\frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1}\rho(t)H_2(d_1,t)
\end{aligned}$$

$$\times\Big[Q(t)-(\alpha+1)^{-(\alpha+1)}r(t)\Big|h_2(d_1,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt$$
$$\leqslant\frac{1}{H_2(d_1,\delta_1)}\sum_{i=k(\delta_1)+1}^{k(d_1)}H_2(d_1,\tau_i)\Big(\frac{b_i{}^{\alpha}-a_i{}^{\alpha}}{a_i{}^{\alpha}}\Big)w(\tau_i)$$
$$\leqslant\frac{r_1M_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}[H_2(d_1,.)]\leqslant\Lambda(H_1,H_2;c_1,d_1).$$

which is a contradiction.By a similar argument, we can prove the other two cases. Hence the proof is complete. □

The next Theorem is for the case $e(t) = 0$.

THEOREM 2.4. *Suppose that for any* $T>0$, *there exist* $c_1,d_1\notin\{\tau_k\}$, *such that* $c_1<d_1$, *and* $q(t),q_i(t)\geqslant 0$, $t\in[c_1,d_1]$ $i=1,2\ldots,n$. *Let* $\{\eta_i\}$, $i=1,2,\ldots,n$, *be an n tuple satisfying* (2.2) *and* (2.3). *If there exist* $(H_1,H_2)\in\mathscr{H}$ *and* $\rho(t)\in C^1([c_1,d_1],(0,\infty))$ *such that*

$$\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1}H_1(t,c_1)\rho(t)\Big[\overline{Q}(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt$$
$$+\frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1}H_2(d_1,t)\rho(t)\Big[\overline{Q}(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}$$
$$\times\Big|h_2(d_1,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt>\Lambda(H_1,H_2;c_1,d_1),\quad(2.29)$$

where Λ *is defined as in Theorem* 2.1 *and* $\overline{Q}(t)=q(t)+\prod_{i=1}^{n}\eta_i^{-\eta_i}(q_i(t))^{\eta_i}$, *then equation* (1.1) *with* $e(t)=0$ *is oscillatory.*

Proof. The proof of the above theorem is immediate by putting $e(t)=0$ and $\eta_0=0$ in the proof of Theorem 2.3. □

REMARK 2.2. When $\alpha=1$ and $\rho(t)=1$, Theorem 2.1 and Theorem 2.3 reduces to Theorem 2.6 and Theorem 2.3 of [15].

REMARK 2.3. When $p(t)=0$ and $\rho(t)=1$, Theorem 2.1 reduces to Theorem 2.1 of [10].

REMARK 2.4. When $p(t)=0$, Theorem 2.1 and Theorem 2.3 reduces to Theorem 2.1 and Theorem 2.3 of [2] with $\alpha=p/q$, where p and q are odds.

3. Examples

In this section, we give some examples to illustrate our results.

EXAMPLE 3.1. Consider the impulsive differential equation

$$\left(|x'(t)|^{-\frac{3}{4}}x'(t)\right)' + (\cos 2t)\left(|x'(t)|^{-\frac{3}{4}}x'(t)\right) + ({v_1}^2\cos t)|x(t)|^{-\frac{2}{3}}x(t) + ({v_2}^{\frac{12}{5}}\cos t)|x(t)|^{-\frac{4}{5}}x(t) = \sin 2t, \quad t \neq 2k\pi - \frac{\pi}{4}, \tag{3.1}$$

$$x({\tau_k}^+) = a_k x(\tau_k), \quad x'({\tau_k}^+) = b_k x'(\tau_k), \quad \tau_k = 2k\pi - \frac{\pi}{4},$$

where $t \geqslant t_0 > 0$, v_i, $i = 1,2$ are positive constants, $b_k \geqslant a_k, k \in \mathbb{N}$. We see that:

$$r(t) = 1,\ p(t) = \cos 2t,\ q(t) = 0,\ q_1(t) = {v_1}^2\cos t,\ q_2(t) = {v_2}^{\frac{12}{5}}\cos t,\ e(t) = \sin 2t,$$
$$\alpha = 1/4,\ \beta_1 = 1/3,\ \beta_2 = 1/5.$$

For any $T \geqslant 0$,we can choose n large enough such that $T < c_1 = 2n\pi - \frac{\pi}{2}, d_1 = c_2 = 2n\pi, d_2 = 2n\pi + \frac{\pi}{2}, n = 1,2,\ldots$. Then, (2.6) in Theorem 2.1 is satisfied. Now choose $\eta_0 = 1/12$, $\eta_1 = 1/2$, $\eta_2 = 1/8$, therefore

$$Q(t) = (2.504)v_1v_2(\cos t)^{\frac{11}{12}}|\sin 2t|^{\frac{1}{12}}.$$

Let $u(t) = \sin 2t$ and $\rho(t) = 1$. Then by using the mathematical software Maple 6, we obtain

$$\begin{aligned}
&\int_{c_1}^{d_1} \rho(t)\left[Q(t)|u(t)|^{\alpha+1} - \frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|(\alpha+1)u'(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)|u(t)|\right|^{\alpha+1}\right]dt \\
&\quad = \int_{2n\pi-\pi/2}^{2n\pi}\Bigg[(2.504)v_1v_2(\cos t)^{\frac{11}{12}}|\sin 2t|^{4/3} \\
&\qquad\qquad - (4/5)^{5/4}\Big|[(5/2) - |\sin 2t|]\cos 2t\Big|^{5/4}\Bigg]dt \\
&\quad = (2.504)v_1v_2(0.62947) - (0.757)(7.28316) \\
&\quad = (1.5762)v_1v_2 - 5.51335
\end{aligned}$$

and

$$\begin{aligned}
&\int_{c_2}^{d_2} \rho(t)\left[Q(t)|u(t)|^{\alpha+1} - \frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|(\alpha+1)u'(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{p(t)}{r(t)}\right)|u(t)|\right|^{\alpha+1}\right]dt \\
&\quad = \int_{2n\pi-\pi/2}^{2n\pi}\Bigg[(2.504)v_1v_2(\cos t)^{\frac{11}{12}}|\sin 2t|^{4/3} \\
&\qquad\qquad - (4/5)^{5/4}\Big|[(5/2) - |\sin 2t|]\cos 2t\Big|^{5/4}\Bigg]dt \\
&\quad = (2.504)v_1v_2(0.62947) - (0.757)(7.28316)
\end{aligned}$$

$$= (1.5762)v_1v_2 - 5.51335.$$

Since $k(c_1) = n-1$, $k(d_1) = n$, $r_1 = 1$ and $k(c_2) = k(d_2)$, we obtain

$$\begin{aligned} r_1M_1\Omega_{c_1}^{d_1}[|u|^{\alpha+1}] &= |\sin 2(\tau_n)|^{5/4}(4/\pi)^{1/4}\left[\frac{(b_n)^{1/4}-(a_n)^{1/4}}{(a_n)^{1/4}}\right] \\ &= |\sin(4n\pi - \pi/2)|^{5/4}(4/\pi)^{1/4}\left[\frac{(b_n)^{1/4}-(a_n)^{1/4}}{(a_n)^{1/4}}\right] \\ &= (1.06222)\left[\frac{(b_n)^{1/4}-(a_n)^{1/4}}{(a_n)^{1/4}}\right] \end{aligned}$$

and $r_2M_2\Omega_{c_2}^{d_2}[|u|^{\alpha+1}] = 0$. So, if we choose the constants v_1, v_2 large enough such that

$$1.5762v_1v_2 > 5.51335 + (1.06222)\left[\frac{(b_n)^{1/4}-(a_n)^{1/4}}{(a_n)^{1/4}}\right],$$
$$1.5762v_1v_2 > 5.51335,$$

then by Theorem 2.1 equation (3.1) is oscillatory.

EXAMPLE 3.2. Consider the impulsive differential equation

$$\begin{aligned} &\left(|x'(t)|^{-\frac{2}{3}}x'(t)\right)' + (\sin t)\left(|x'(t)|^{-\frac{2}{3}}x'(t)\right) + (\mu_1 \sin t)|x(t)|^{-\frac{1}{6}}x(t) \\ &\qquad + (\mu_2 \cos t)|x(t)|^{-\frac{5}{6}}x(t) = -\cos 2t, \quad t \neq \tau_k, \quad (3.2) \\ &x(\tau_k{}^+) = a_k x(\tau_k), \quad x'(\tau_k{}^+) = b_k x'(\tau_k), \end{aligned}$$

where $t \geqslant t_0 > 0, b_k \geqslant a_k, k \in \mathbb{N}$, We see that

$$r(t) = 1,\ p(t) = \sin t,\ q(t) = 0,\ q_1(t) = \mu_1 \sin t,\ q_2(t) = \mu_2 \cos t,\ e(t) = -\cos 2t,$$
$$\alpha = 1/3,\ \beta_1 = 5/6,\ \beta_2 = 1/6.$$

The numbers $\tau_{2n} = 2n\pi + \frac{\pi}{6}, \tau_{2n+1} = 2n\pi + \frac{\pi}{3}$, $n = 0,1,2,\ldots$, μ_1, μ_2 are positive constants. For any $T \geqslant 0$, we can choose n large enough such that

$$\begin{aligned} T < c_1 = 2n\pi < \delta_1 = 2n\pi + \frac{\pi}{8} < d_1 = 2n\pi + \frac{\pi}{4} \\ = c_2 < \delta_2 = 2n\pi + \frac{3\pi}{8} < d_2 = 2n\pi + \frac{\pi}{2}. \end{aligned}$$

Let $\eta_0 = 1/5$, $\eta_1 = 3/10$, $\eta_2 = 1/2$, then

$$Q(t) = (2.8001)|-\cos 2t|^{1/5}\mu_1^{3/10}\mu_2^{1/2} sin^{3/10}\cos^{1/2} t.$$

If we choose $H_1(t,s) = H_2(t,s) = (t-s)^2$ and $\rho(t) = 1$ then $h_1(t,s) = -h_2(t,s) = \frac{2}{t-s}$. Then by using the mathematical software Maple 6, the left hand side of the inequality (2.22) with $j = 1$ is

$$\frac{1}{H_1(\delta_1,c_1)}\int_{c_1}^{\delta_1} H_1(t,c_1)\rho(t)\Big[Q(t) - \frac{r(t)}{(\alpha+1)^{\alpha+1}}\Big|h_1(t,c_1)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt + \frac{1}{H_2(d_1,\delta_1)}\int_{\delta_1}^{d_1} H_2(d_1,t)\rho(t)\Big[Q(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}} \times\Big|h_2(d_1,t)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt$$

$$= \frac{64}{\pi^2}\int_{2n\pi}^{2n\pi+\frac{\pi}{8}}(t-2n\pi)^2\Big[(2.8001)|-\cos 2t|^{1/5}\mu_1^{3/10}\mu_2^{1/2} sin^{3/10}\cos^{1/2} t - (3/4)^{4/3}\Big|\frac{2}{t-2n\pi}-\sin t\Big|^{4/3}\Big]dt$$

$$+\frac{64}{\pi^2}\int_{2n\pi+\frac{\pi}{8}}^{2n\pi+\frac{\pi}{4}}(2n\pi+\frac{\pi}{4}-t)^2\Big[(2.8001)|-\cos 2t|^{1/5}\mu_1^{3/10}\mu_2^{1/2} sin^{3/10}\cos^{1/2} t - (3/4)^{4/3}\Big|\frac{-2}{2n\pi+\frac{\pi}{4}-t}-\sin t\Big|^{4/3}\Big]dt$$

$$= (0.23459)\mu_1^{3/10}\mu_2^{1/2} - (1.34283) + (0.240716)\mu_1^{3/10}\mu_2^{1/2} - (1.518068) \approx (0.475307)\mu_1^{3/10}\mu_2^{1/2} - (2.860903).$$

Note that there is no impulsive moment in (c_1,δ_1) and $\tau_{2n}\in(\delta_1,d_1)$. Also $k(\delta_1)=2n-1, k(d_1)=2n$. Take $r_1=1=M_1=1$.Hence the right side of the inequality (2.22) with $j=1$ is

$$\begin{aligned}\Lambda(H_1,H_2;c_1,d_1) &= \frac{r_1}{H_2(d_1,\delta_1)}\Omega_{\delta_1}^{d_1}[H_2(d_1,.)]\\ &= \frac{64}{\pi^2}H_2(d_1,\tau_{2n})\theta(\delta_1)\\ &= \frac{64}{\pi^2}(d_1-\tau_{2n})^2\Big[\frac{(b_{k(\delta_1)+1})^{1/3}-(a_{k(\delta_1)+1})^{1/3}}{(a_{k(\delta_1)+1})^{1/3}(\tau_{k(\delta_1)+1}-\delta_1)^{1/3}}\Big]\\ &= \frac{64}{\pi^2}\frac{\pi^2}{144}\Big[\frac{(b_{2n})^{1/3}-(a_{2n})^{1/3}}{(a_{2n})^{1/3}(\pi/24)^{1/3}}\Big]\\ &= (0.87524)\Big[\frac{(b_{2n})^{1/3}-(a_{2n})^{1/3}}{(a_{2n})^{1/3}}\Big]\end{aligned}$$

Thus (2.22) is satisfied with $j=1$ if

$$(0.475307)\mu_1^{3/10}\mu_2^{1/2} > (2.860903) + (0.87524)\Big[\frac{(b_{2n})^{1/3}-(a_{2n})^{1/3}}{(a_{2n})^{1/3}}\Big]$$

In a similar way, the left hand side of the inequality (2.22) with $j=2$ is

$$\frac{1}{H_1(\delta_2,c_2)}\int_{c_2}^{\delta_2} H_1(t,c_2)\rho(t)\Big[Q(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\Big|h_1(t,c_2)+\Big(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\Big)\Big|^{\alpha+1}\Big]dt$$

$$+\frac{1}{H_2(d_2,\delta_2)}\int_{\delta_2}^{d_1} H_2(d_2,t)\rho(t)$$
$$\times\left[Q(t)-\frac{r(t)}{(\alpha+1)^{\alpha+1}}\left|h_2(d_2,t)+\left(\frac{\rho'(t)}{\rho(t)}-\frac{p(t)}{r(t)}\right)\right|^{\alpha+1}\right]dt$$
$$=\frac{64}{\pi^2}\int_{2n\pi+\pi/4}^{2n\pi+\frac{3\pi}{8}}(t-2n\pi-\pi/4)^2\Big[(2.8001)|-\cos 2t|^{1/5}\mu_1^{3/10}\mu_2^{1/2}sin^{3/10}\cos^{1/2}t$$
$$-(3/4)^{4/3}\left|\frac{2}{t-2n\pi-\pi/4}-\sin t\right|^{4/3}\Big]dt$$
$$+\frac{64}{\pi^2}\int_{2n\pi+\frac{3\pi}{8}}^{2n\pi+\frac{\pi}{2}}(2n\pi+\frac{\pi}{2}-t)^2\Big[(2.8001)|-\cos 2t|^{1/5}\mu_1^{3/10}\mu_2^{1/2}sin^{3/10}\cos^{1/2}t$$
$$-(3/4)^{4/3}\left|\frac{-2}{2n\pi+\frac{\pi}{2}-t}-\sin t\right|^{4/3}\Big]dt$$
$$=(0.211854)\mu_1^{3/10}\mu_2^{1/2}-(1.210017)+(0.184395)\mu_1^{3/10}\mu_2^{1/2}-(1.631927)$$
$$\approx(0.396249)\mu_1^{3/10}\mu_2^{1/2}-(2.841945).$$

Note that $\tau_{2n+1}\in(c_2,\delta_2)$ and there is no impulsive moment in (δ_2,d_2). Also $k(c_2)=2n$, $k(\delta_2)=2n+1$. Hence the right side of the inequality (2.22) with $j=2$ is

$$\begin{aligned}\Lambda(H_1,H_2;c_1,d_1)&=\frac{r_2M_2}{H_1(\delta_2,c_2)}\Omega_{c_2}^{\delta_2}[H_1(.,c_2)]\\&=\frac{64}{\pi^2}H_1(\tau_{2n+1},c_2)\theta(c_2)\\&=\frac{64}{\pi^2}(\tau_{2n+1},c_2)^2\left[\frac{(b_{k(c_2)+1})^{1/3}-(a_{k(c_2)+1})^{1/3}}{(a_{k(c_2)+1})^{1/3}(\tau_{k(c_2)+1}-c_2)^{1/3}}\right]\\&=\frac{64}{\pi^2}\frac{\pi^2}{144}\left[\frac{(b_{2n+1})^{1/3}-(a_{2n+1})^{1/3}}{(a_{2n+1})^{1/3}(\pi/12)^{1/3}}\right]\\&=(0.69468)\left[\frac{(b_{2n+1})^{1/3}-(a_{2n+1})^{1/3}}{(a_{2n+1})^{1/3}}\right]\end{aligned}$$

Thus (2.22) is satisfied with $j=2$ if

$$(0.396249)\mu_1^{3/10}\mu_2^{1/2}>(2.841945)+(0.69468)\left[\frac{(b_{2n+1})^{1/3}-(a_{2n})^{1/3}}{(a_{2n})^{1/3}}\right].$$

So,if we choose the constants μ_1,μ_2 large enough such that

$$\mu_1^{3/10}\mu_2^{1/2}>(6.019063)+(1.841423)\left[\frac{(b_{2n+1})^{1/3}-(a_{2n})^{1/3}}{(a_{2n})^{1/3}}\right],$$

$$\mu_1^{3/10}\mu_2^{1/2}>(7.172119)+(1.75314)\left[\frac{(b_{2n+1})^{1/3}-(a_{2n})^{1/3}}{(a_{2n})^{1/3}}\right],$$

then by Theorem 2.3, equation (3.2) is oscillatory.

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