

## OSCILLATION OF THIRD-ORDER QUASI-LINEAR ADVANCED DIFFERENTIAL EQUATIONS

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*Abstract.* The objective of this paper is to study asymptotic properties of the third-order advanced differential equation

$$\left[ r(t) [x''(t)]^\gamma \right]' + q(t)x^\gamma[\tau(t)] = 0. \quad (E)$$

We offer new oscillation criteria that really take onto account the advanced argument. Some examples are also provided to illustrate the relevance of the main results.

### 1. Introduction

We present criteria for property (A) of the third-order advanced functional differential equations of the form

$$\left[ r(t) [x''(t)]^\gamma \right]' + q(t)x^\gamma[\tau(t)] = 0. \quad (E)$$

In the sequel we will assume :

(H<sub>1</sub>)  $r(t), q(t), \tau(t) \in C([t_0, \infty))$ ,  $r(t), q(t)$  are positive,  $\tau(t) \geq t$ ,

(H<sub>2</sub>)  $\gamma$  is a quotient of odd positive integers.

Whenever, it is assumed

$$R(t) = \int_{t_0}^t r^{-1/\gamma}(s) ds \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad (1.1)$$

By a solution of Eq. (E) we mean a function  $x(t) \in C^2([T_x, \infty))$ ,  $T_x \geq t_0$ , which has the property  $r(t)(x''(t))^\gamma \in C^1([T_x, \infty))$  and satisfies Eq. (E) on  $[T_x, \infty)$ . We consider only those solutions  $x(t)$  of (E) which satisfy  $\sup\{|x(t)| : t \geq T\} > 0$  for all  $T \geq T_x$ . We assume that (E) possesses such a solution. A solution of (E) is called oscillatory

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if it has arbitrarily large zeros on  $[T_x, \infty)$  and otherwise it is called to be nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

The problem of the oscillation of differential equations has been widely studied by many authors, who have provided many techniques especially for delay differential equations. Dong in [6] have improved and extended the Riccati transformation to obtain new oscillatory criteria. Grace et al. in [10] and the present authors in [2]-[5] have used the comparison technique in which studied equations have been compared with the oscillation of certain first order differential equation.

On the other hand, there are comparatively less methods established for the advanced differential equations. Kusano in [13], [14] has suggested to reduce the investigation of advanced equation to that of the corresponding equation without deviating argument, but in this case we do not utilize information about advanced argument.

The aim of the paper is to fill this gap in the oscillation theory. We present some new criteria that essentially utilize the value of the advanced argument, i.e. the criteria obtained involves advanced argument  $\tau(t)$  explicitly.

REMARK 1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all  $t$  large enough.

We start with the classification of the possible nonoscillatory solutions of (E).

LEMMA 1. *Let  $x(t)$  be a nonoscillatory solution of (E). Then  $x(t)$  satisfies one of the following conditions*

$$(I) \quad x(t)x'(t) < 0, \quad x(t)x''(t) > 0, \quad x(t) [r(t)[x''(t)]^\gamma]' < 0;$$

$$(II) \quad x(t)x'(t) > 0, \quad x(t)x''(t) > 0, \quad x(t) [r(t)[x''(t)]^\gamma]' < 0;$$

eventually.

*Proof.* Let  $x(t)$  be a nonoscillatory solution of Eq. (E), say  $x(t) > 0$  for  $t \geq t_0$ . It follows from (E) that  $[r(t)[x''(t)]^\gamma]' < 0$ , eventually. Thus,  $r(t)[x''(t)]^\gamma$  is decreasing and of fixed sign eventually.

If  $r(t)[x''(t)]^\gamma < 0$ , then it follows from (H1) that  $x'(t) < 0$ , which implies  $x(t) < 0$ . A contradiction and we conclude that  $r(t)[x''(t)]^\gamma > 0$ , eventually. Consequently,  $x'(t)$  is of fixed sign for all  $t$  large enough. Therefore, either Case (I) or Case (II) holds. The proof is complete.

For the partial case of (E), namely for differential equation

$$x'''(t) + p(t)x(t) = 0$$

the set of nonoscillatory solutions is not empty, i.e., there always exists (see e.g. [9], [11]) a nonoscillatory solution satisfying the Case (I) of Lemma 1. This fact led to the following definition. We say that (E) has property (A) if all its nonoscillatory solutions satisfy only case Case (I) of Lemma 1.

### 2. Main results

First, we state and prove the following useful lemma, which will be used later in the proofs of our main results.

LEMMA 2. Assume  $x(t) > 0$ ,  $x'(t) > 0$ ,  $x''(t) > 0$ , eventually. Then for arbitrary  $k_0 \in (0, 1)$

$$x[\tau(t)] \geq k_0 \frac{\tau(t)}{t} x(t), \tag{2.1}$$

eventually.

*Proof.* It follows from the monotonicity of  $x'(t)$  that

$$x[\tau(t)] - x(t) = \int_t^{\tau(t)} x'(s) ds \geq x'(t)(\tau(t) - t).$$

That is,

$$\frac{x[\tau(t)]}{x(t)} \geq 1 + \frac{x'(t)}{x(t)}(\tau(t) - t). \tag{2.2}$$

On the other hand, since  $x(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , then for any  $k_0 \in (0, 1)$  there exists a  $t_1$  large enough, such that

$$k_0 x(t) \leq x(t) - x(t_1) = \int_{t_1}^t x'(s) ds \leq x'(t)(t - t_1) \leq x'(t)t,$$

or equivalently

$$\frac{x'(t)}{x(t)} \geq \frac{k_0}{t}. \tag{2.3}$$

Using (2.3) in (2.2), we obtain

$$\frac{x[\tau(t)]}{x(t)} \geq 1 + \frac{k_0}{t}(\tau(t) - t) \geq k_0 \frac{\tau(t)}{t}.$$

The proof is complete.

Now, we are prepared to offer our main results. For our further references we set

$$Q(t) = \int_t^\infty q(s) \left(\frac{\tau(s)}{s}\right)^\gamma ds.$$

THEOREM 1. If

$$\liminf_{t \rightarrow \infty} \frac{1}{Q(t)} \int_t^\infty R(s) Q^{1+1/\gamma}(s) ds > \frac{1}{(\gamma + 1)^{1+1/\gamma}}, \tag{2.4}$$

then (E) has property (A).

*Proof.* Assume the contrary, let  $x(t)$  be an eventually positive solution of (E) satisfying Case (II) from Lemma 1. By (2.4), it is easy to see that there exists some  $k \in (0, 1)$  such that

$$\liminf_{t \rightarrow \infty} \frac{k^{1+1/\gamma}}{Q(t)} \int_t^\infty R(s) Q^{1+1/\gamma}(s) ds > \frac{1}{(\gamma+1)^{1+1/\gamma}}. \quad (2.5)$$

We put  $k_0 = k^{1/\gamma}$ , then Lemma 2 implies

$$\left[ r(t) [x''(t)]^\gamma \right]' + kq(t) \frac{\tau^\gamma(t)}{t^\gamma} x^\gamma(t) \leq 0.$$

We define

$$w(t) = \frac{r(t) [x''(t)]^\gamma}{x^\gamma(t)} > 0. \quad (2.6)$$

Differentiating  $w(t)$ , one gets

$$w'(t) = \frac{[r(t) [x''(t)]^\gamma]'}{x^\gamma(t)} - \gamma \frac{r(t) [x''(t)]^\gamma x'(t)}{x^\gamma(t) x(t)} \leq -\frac{kq(t) \tau^\gamma(t)}{t^\gamma} - \gamma w(t) \frac{x'(t)}{x(t)}. \quad (2.7)$$

On the other hand, using the monotonicity of  $r(t) [x''(t)]^\gamma$ , we have

$$\begin{aligned} x'(t) &\geq \int_{t_1}^t [r(s) [x''(s)]^\gamma]^{1/\gamma} r^{-1/\gamma}(s) ds \\ &\geq [r(t) [x''(t)]^\gamma]^{1/\gamma} \int_{t_1}^t r^{-1/\gamma}(s) ds \\ &\geq r^{1/\gamma}(t) x''(t) kR(t), \end{aligned} \quad (2.8)$$

eventually, let say  $t \geq t_2$ . Setting the last inequality into (2.7), we obtain

$$w'(t) \leq -k \left[ q(t) \left( \frac{\tau(t)}{t} \right)^\gamma + \gamma w^{1+1/\gamma}(t) R(t) \right].$$

Integrating the last inequality from  $t (\geq t_2)$  to  $\infty$ , we get

$$w(t) \geq k \left[ Q(t) + \int_t^\infty \gamma w^{1+1/\gamma}(s) R(s) \right] ds \quad (2.9)$$

or

$$\frac{w(t)}{kQ(t)} \geq 1 + \frac{\gamma k^{1+1/\gamma}}{Q(t)} \int_t^\infty R(s) Q^{1+1/\gamma}(s) \left( \frac{w(s)}{kQ(s)} \right)^{1+1/\gamma} ds.$$

Since  $w(t) > kQ(t)$ , then

$$\inf_{t \geq t_1} \frac{w(t)}{kQ(t)} = \lambda \geq 1.$$

Thus

$$\frac{w(t)}{kQ(t)} \geq 1 + \frac{\gamma(k\lambda)^{1+1/\gamma}}{Q(t)} \int_t^\infty R(s) Q^{1+1/\gamma}(s) ds. \quad (2.10)$$

From (2.5), we see that there exists some positive  $\eta$ , such that

$$\frac{k^{1+1/\gamma}}{Q(t)} \int_t^\infty R(s)Q^{1+1/\gamma}(s) ds > \eta > (\gamma + 1)^{-\frac{\gamma+1}{\gamma}}. \tag{2.11}$$

Combining (2.10) together with (2.11), we have

$$\frac{w(t)}{kQ(t)} \geq 1 + \gamma\lambda^{1+1/\gamma}\eta.$$

Therefore

$$\lambda \geq 1 + \gamma\lambda^{1+1/\gamma}\eta > 1 + \gamma\lambda^{1+1/\gamma}(\gamma + 1)^{-\frac{\gamma+1}{\gamma}}$$

or equivalently

$$0 > \frac{1}{\gamma + 1} + \frac{\gamma}{\gamma + 1} \left( \frac{\lambda}{\gamma + 1} \right)^{1+1/\gamma} - \frac{\lambda}{\gamma + 1}.$$

This contradicts the fact, that the function

$$f(\alpha) = \frac{1}{\gamma + 1} + \frac{\gamma}{\gamma + 1} \alpha^{1+1/\gamma} - \alpha$$

is positive for all  $\alpha > 0$ .

Now we present some useful corollaries.

COROLLARY 1. *If*

$$\int_{t_0}^\infty q(s) \left( \frac{\tau(s)}{s} \right)^\gamma ds = \infty, \tag{2.12}$$

*then (E) has property (A).*

*Proof.* The proof follows immediately from (2.9).

COROLLARY 2. *If*

$$\int_{t_0}^\infty R(s)Q(s)^{1+1/\gamma} ds = \infty, \tag{2.13}$$

*then (E) has property (A).*

*Proof.* It follows from (2.9) and  $w(t) > kQ(t)$  that

$$w(t_1) \geq k \left[ Q(t_1) + k^{1+1/\gamma} \int_{t_1}^\infty \gamma Q^{1+1/\gamma}(s)R(s) \right] ds$$

which contradicts our assumption.

THEOREM 2. *Assume that (E) has property (A). If, moreover,*

$$\int_{t_0}^\infty \int_v^\infty r^{-1/\gamma}(u) \left[ \int_u^\infty q(s) ds \right]^{1/\gamma} dudv = \infty, \tag{2.14}$$

*then every nonoscillatory solution  $x(t)$  of (E) tends to zero as  $t \rightarrow \infty$ .*

*Proof.* Property (A) of (E) implies that an eventually positive solution  $x(t)$  of (E) satisfies Case (I) of Lemma 1. Then there exists a finite  $\lim_{t \rightarrow \infty} x(t) = \ell$ . We claim that  $\ell = 0$ . Assume that  $\ell > 0$ .

Integrating (E) from  $t$  to  $\infty$  and using  $x[\tau(t)] > \ell$ , we obtain

$$r(t)(x''(t))^\gamma \geq \int_t^\infty q(s)x^\gamma[\tau(s)]ds \geq \ell^\gamma \int_t^\infty q(s)ds,$$

which implies

$$x''(t) \geq \frac{\ell}{r^{1/\gamma}(t)} \left[ \int_t^\infty q(s)ds \right]^{1/\gamma}.$$

Integrating the last inequality from  $t$  to  $\infty$ , we get

$$-x'(t) \geq \ell \int_t^\infty \frac{1}{r^{1/\gamma}(u)} \left[ \int_u^\infty q(s)ds \right]^{1/\gamma} du.$$

Now integrating from  $t_1$  to  $\infty$ , we arrive at

$$x(t_1) \geq \ell \int_{t_1}^\infty \int_v^\infty \frac{1}{r^{1/\gamma}(u)} \left[ \int_u^\infty q(s)ds \right]^{1/\gamma} dudv.$$

A contradiction with (2.14) and so we have verified that  $\lim_{t \rightarrow \infty} x(t) = 0$ .

EXAMPLE 1. Consider the third order nonlinear differential equation

$$\left( t(x''(t))^3 \right)' + \frac{\beta}{t^6} x^3(\lambda t) = 0, \quad \beta > 0, \quad \lambda \geq 1, \quad t \geq 1. \quad (E_2)$$

Here  $q(t) = \beta/t^6$  and  $\tau(t) = \lambda t$ , so that

$$Q(t) = \int_t^\infty q(s) \left( \frac{\tau(s)}{s} \right)^3 ds = \frac{\lambda^3 \beta}{5t^5}$$

and (2.4) reduces to

$$\lambda \beta^{1/3} > \frac{2}{3} \left( \frac{5}{4} \right)^{4/3},$$

which, by Theorem 1 guarantees property (A) for (E<sub>2</sub>). On the other hand,

$$\int_{t_0}^\infty \int_v^\infty r^{-1/3}(u) \left[ \int_u^\infty q(s)ds \right]^{1/3} dudv = \frac{\beta}{5^{1/3}} \int_{t_0}^\infty \frac{1}{v} dv = \infty,$$

i.e. (2.14) holds, so every nonoscillatory solution  $x(t)$  of (E<sub>2</sub>) tends to zero as  $t \rightarrow \infty$ .

EXAMPLE 2. Consider the third order differential equation

$$\left( t(x''(t))^3 \right)' + \frac{\beta}{t^9} x^3(t^2) = 0, \quad \beta > 0, \quad t \geq 1. \quad (E_3)$$

Now  $q(t) = \beta/t^9$  and  $\tau(t) = t^2$ , consequently

$$Q(t) = \int_t^\infty q(s) \left(\frac{\tau(s)}{s}\right)^3 ds = \frac{\beta}{5t^5}$$

and (2.4) takes the form

$$\beta > \left(\frac{2}{3}\right)^3 \left(\frac{5}{4}\right)^4,$$

which, in view of Theorem 1 insures property (A) of  $(E_3)$ .

The previous two examples reveal that our criterion for property (A) essentially utilize the greatness of the advanced argument. In the second example the advanced argument is considerably greater than in the first one and this fact permits to essentially reduce the function  $q(t)$ .

Let  $\{A_n(t)\}_{n=0}^\infty$  be a sequence of continuous functions defined as follows.

$$A_0(t) = kQ(t), \quad k \in (0, 1)$$

and

$$A_{n+1}(t) = A_0(t) + \gamma k \int_t^\infty A_n^{1+1/\gamma}(s)R(s) ds, \quad n = 0, 1, \dots \tag{2.15}$$

Then we have the following result.

**THEOREM 3.** *Assume that there exists some  $A_n(t)$  such that*

$$\int_{t_0}^\infty q(t) \left(\frac{\tau(t)}{t}\right)^\gamma \left( e^{k\gamma \int_{t_0}^t A_n^{1/\gamma}(s)R(s) ds} \right) dt = \infty, \tag{2.16}$$

for some  $k \in (0, 1)$ . Then (E) has property (A).

*Proof.* Assume that  $x(t)$  is an eventually positive solution of (E) satisfying Case (II) from Lemma 1. It follows from the proof of Theorem 1 that (2.9) holds for every  $k \in (0, 1)$ .

By (2.9) and definition of  $A_0(t)$  it is clear that  $w(t) \geq A_0(t)$ . On the other hand,

$$\begin{aligned} A_1(t) &= A_0(t) + \gamma k \int_t^\infty A_0^{1+1/\gamma}(s)R(s) ds \\ &\leq A_0(t) + \gamma k \int_t^\infty w^{1+1/\gamma}(s)R(s) ds \leq w(t). \end{aligned}$$

By induction, it is easy to see that the sequence  $\{A_n(t)\}_{n=0}^\infty$  is nondecreasing and  $w(t) \geq A_n(t)$ . Thus the sequence  $\{A_n(t)\}_{n=0}^\infty$  converges to  $A(t)$ . By Lebesgue monotone convergence theorem and letting  $n \rightarrow \infty$  in (2.15), we get

$$A(t) = A_0(t) + \gamma k \int_t^\infty A^{1+1/\gamma}(s)R(s) ds,$$

which in view of  $A(t) \geq A_n(t)$  implies

$$\begin{aligned} A'(t) &= -kq(t) \left( \frac{\tau(t)}{t} \right)^\gamma - \gamma k A^{1+1/\gamma}(t) R(t) \\ &\leq -kq(t) \left( \frac{\tau(t)}{t} \right)^\gamma - \gamma k A(t) A_n^{1/\gamma}(t) R(t), \end{aligned}$$

eventually, let us say  $t \geq t_1$ . Therefore,

$$\left[ A(t) \left( e^{\int_{t_1}^t k\gamma A_n^{1/\gamma}(s) R(s) ds} \right) \right]' \leq -kq(t) \left( \frac{\tau(t)}{t} \right)^\gamma \left( e^{\int_{t_1}^t k\gamma A_n^{1/\gamma}(s) R(s) ds} \right).$$

An integration from  $t_1$  to  $t$ , yields

$$\begin{aligned} 0 &\leq A(t) \left( e^{\int_{t_1}^t k\gamma A_n^{1/\gamma}(s) R(s) ds} \right) \\ &\leq A(t_1) - k \int_{t_1}^t q(u) \left( \frac{\tau(u)}{u} \right)^\gamma \left( e^{\int_{t_1}^u k\gamma A_n^{1/\gamma}(s) R(s) ds} \right) du \rightarrow -\infty \end{aligned}$$

as  $t \rightarrow \infty$ . A contradiction. The proof is complete.

**THEOREM 4.** *Assume that there exists some  $A_n(t)$  such that*

$$\limsup_{t \rightarrow \infty} \left[ \int_{t_1}^t (R(s) - R(t_1)) ds \right]^\gamma A_n(t) > 1, \quad (2.17)$$

for some  $k \in (0, 1)$ . Then (E) has property (A).

*Proof.* Assume that  $x(t)$  is an eventually positive solution of (E) satisfying Case (II) from Lemma 1. It follows from (2.8) that

$$x(t) \geq [r(t)[x''(t)]^\gamma]^{1/\gamma} \int_{t_1}^t \int_{t_1}^u r^{-1/\gamma}(s) ds du. \quad (2.18)$$

On the other hand, combining (2.6) together with (2.18), we get

$$\frac{1}{w(t)} = \frac{1}{r(t)} \left( \frac{x(t)}{x''(t)} \right)^\gamma \geq \left[ \int_{t_1}^t (R(s) - R(t_1)) ds \right]^\gamma,$$

or equivalently

$$1 \geq \left[ \int_{t_1}^t (R(s) - R(t_1)) ds \right]^\gamma w(t) \geq \left[ \int_{t_1}^t (R(s) - R(t_1)) ds \right]^\gamma A_n(t),$$

which letting  $\limsup$  on the both sides contradicts to (2.17).

Since the sequence  $\{A_n(t)\}_{n=0}^\infty$  is increasing, the greater  $n$  in (2.16) and (2.17), the better criteria we obtain. Letting  $n = 0$  and  $n = 1$  in Theorem 4, we have



COROLLARY 3. Assume that

$$\limsup_{t \rightarrow \infty} \left[ \int_{t_1}^t (R(s) - R(t_1)) ds \right]^\gamma \int_t^\infty q(s) \left( \frac{\tau(s)}{s} \right)^\gamma ds > 1. \tag{2.19}$$

Then (E) has property (A).

COROLLARY 4. Assume that for some  $k \in (0, 1)$ ,

$$\limsup_{t \rightarrow \infty} \left[ \int_{t_1}^t (R(s) - R(t_1)) ds \right]^\gamma \left[ Q(t) + \gamma k \int_t^\infty Q^{1+1/\gamma}(s) R(s) \right] ds > 1.$$

Then (E) has property (A).

EXAMPLE 3. Consider the third order nonlinear differential equation

$$\left( t^2 (x''(t))^3 \right)' + \frac{\beta}{t^5} x^3(\lambda t) = 0, \quad \beta > 0, \quad \lambda \geq 1, \quad t \geq 1. \tag{E_3}$$

A simple calculation leads to

$$Q(t) = \int_t^\infty q(s) \left( \frac{\tau(s)}{s} \right)^3 ds = \frac{\lambda^3 \beta}{4t^4}.$$

Then by criteria presented in Corollaries 3 and 4, property (A) of (E<sub>3</sub>) is guaranteed provided that

$$\frac{9^3}{4^4} \beta \lambda^3 > 1$$

or

$$\frac{9^3}{4^4} \beta \lambda^3 + \frac{9^4}{4^{16/3}} \beta^{4/3} \lambda^4 > 1,$$

respectively. Of course, the second criterion is better since it is obtained for  $n = 1$  in (2.17), while the first one for  $n = 0$ . On the other hand,

$$\int_{t_0}^\infty \int_v^\infty r^{-1/3}(u) \left[ \int_u^\infty q(s) ds \right]^{1/3} du dv = \frac{\beta^{1/3}}{4^{1/3}} \int_{t_0}^\infty \frac{1}{v} dv = \infty,$$

i.e., (2.14) holds, so every nonoscillatory solution  $x(t)$  of (E<sub>3</sub>) tends to zero as  $t \rightarrow \infty$ .

### 3. Extension

All our results here hold immediately true also for the following third order advanced differential equations

$$\left[ r(t) |x''(t)|^{\gamma-1} x''(t) \right]' + q(t) |x(\tau(t))|^{\gamma-1} x(\tau(t)) = 0, \tag{\tilde{E}}$$

for which the hypothesis (H<sub>1</sub>) is assumed to hold and instead of (H<sub>2</sub>) we assume only that  $\gamma > 0$ .

We illustrate all our results in the following example.

EXAMPLE 4. Consider the third order differential equation

$$\left(t^a |x''(t)|^{\gamma-1} x''(t)\right)' + \frac{\beta}{t^b} |x(t^c)|^{\gamma-1} x(t^c) = 0, \quad t \geq 1, \quad (E_4)$$

where  $b > 0$ ,  $\beta > 0$ ,  $\gamma > a > 0$ , and  $c \geq 1$ .

Here

$$Q(t) = \beta \frac{s^{\gamma c - \gamma - b + 1}}{\gamma c - \gamma - b + 1} \Big|_t^\infty$$

and we have the following criteria:

PROPOSITION 1. If  $\gamma c - \gamma - b + 1 \geq 0$ , then  $(E_4)$  has property (A) by Corollary 1.

PROPOSITION 2. If  $\gamma c - \gamma - b + 1 < 0$  and  $\gamma c - \gamma - b + 1 - b/\gamma + 1/\gamma - a/\gamma + 2 \geq 0$ , then  $(E_4)$  has property (A) by Corollary 2.

PROPOSITION 3. If  $\gamma c - \gamma - b + 1 < 0$ ,  $\gamma c - \gamma - b + 1 - b/\gamma + 1/\gamma - a/\gamma + 2 < 0$ , and  $c - b/\gamma + 1/\gamma - a/\gamma + 1 > 0$ , then  $(E_4)$  has property (A) by Theorem 1.

PROPOSITION 4. If  $\gamma c - \gamma - b + 1 < 0$ ,  $\gamma c - \gamma - b + 1 - b/\gamma + 1/\gamma - a/\gamma + 2 < 0$ ,  $c - b/\gamma + 1/\gamma - a/\gamma + 1 = 0$ , and

$$\frac{\gamma \beta^{1/\gamma}}{(\gamma - a)(b + \gamma - \gamma c - 1)^{1+1/\gamma}} > \frac{1}{(\gamma + 1)^{1+1/\gamma}},$$

then  $(E_4)$  has property (A) by Theorem 1.

REMARK 2. Note that Proposition 4 of Example 4 includes also the results of Example 2.

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