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QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTS AND DISCONTINUOUS NONLINEARITIES

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Abstract. Using a recent fixed point theorem in ordered Banach spaces by S. Carl and S. Heikkilä, we study the existence of weak solutions to nonlinear elliptic problems $-\text{div}a(x,\nabla u) = f(x,u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition. In particular, we prove that for some suitable function g, which may be discontinuous, and δ small enough, the p-Laplace equation

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \delta g(x,u)$$

has a positive solution which goes to 0 as $\delta \to 0^+$, where p^* is the critical exponent.

1. Introduction

Let $N \geqslant 2$ be an integer and $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We study the following nonlinear elliptic problem

$$\begin{cases} -\text{div}a(x,\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is not necessarily continuous with respect to its second variable and may have critical exponent.

This problem has been studied extensively in literature by many authors. A general method for proving the existence of solutions of (1.1), when f is a Carathéodory function and has subcritical growth, is critical point theory. When f is discontinuous and has critical exponent, the situation becomes difficult because the energy functional associated with (1.1) does not belong to C^1 class and because of the lack of compactness of embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Nevertheless, many authors dealt with this case using several methods in nonsmooth analysis. To name a few, when $a(x,\xi) = \xi$, problem (1.1) was studied in [3, 5, 10] and references therein. When $a(x,\xi) = |\xi|^{p-2}\xi$ (p-Laplace equation), it was studied in [13] for $f(x,u) = \lambda|u|^{p^*-2}u + g(x,u)$. Later, when $a(x,\xi) = |\xi|^{p(x)-2}\xi$ ($p(x)$-Laplace equation), it was studied in [14] for $f(x,u) = \lambda|u|^{p^*(x)-2}u + g(u)$. When $a_i(x,\xi) = |\xi_i|^{p_i-2}\xi_i$ (anisotropic quasilinear elliptic equation), it was studied in [9].

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In the present paper, we exploit the fixed point theorem introduced in [6] to prove the existence of a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ of (1.1). Note that in [6], S. Carl and S. Heikkilä applied their theorem to the problem $-\Delta u = f(x,u)$ and proved the existence of one solution. However, the solution they obtained may be the trivial one if $f(x,0) = 0$. By using a suitable set having the fixed point property, the trivial solution in our result is excluded.

As an application, in section 4 we will consider the following critical problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \delta g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where g is not required to be Carathéodory but sup-measurable.

It is well known that if $g \equiv 0$ and Ω is a star-shaped domain then (1.2) has no nontrivial solution (see [11, 12]). The perturbation $\delta g(x,u)$ ensures the existence of a nontrivial solution to problem (1.2). The existence of a positive solution of (1.2) was studied in [2] when $p = 2$, g is the Heaviside function in u and δ is sufficiently large. This problem is also studied in [1] using a variational approach when $p = 2$ and $g(x,u) = h(x)H(u-a)u^q$, where $h(x)$ is both nonnegative and integrable on $\mathbb{R}^N$, H is the Heaviside function, $0 \leqslant q < 2^* - 1$ and $\delta > 0$. In section 4 of this paper, we will prove that when δ is sufficiently small, problem (1.2) has a positive solution which converges to 0 as $\delta \to 0^+$.

Throughout this paper, we assume that $p \in [2,\infty)$. As usual, we denote:

$$p^* = pN/(N-p) \text{ if } p < N \text{ and } p^* = \infty \text{ if } p \geqslant N \text{ (hence } p^* = \infty \text{ if } N = 2),$$
$$p' = p/(p-1),$$
$$|u|_p = \left(\textstyle\int_\Omega |u|^p\right)^{\frac{1}{p}}, \text{ the norm of } u \text{ in } L^p(\Omega),$$
$$|u|_\infty = ess\sup\{|u(x)| \mid x \in \Omega\}, \text{ the norm of } u \text{ in } L^\infty(\Omega),$$
$$\|u\| = \left(\textstyle\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}, \text{ the norm of } u \text{ in } W_0^{1,p}(\Omega),$$
$$L_+^p(\Omega) = \{u \in L^p(\Omega) \mid u(x) \geqslant 0 \text{ for a.e } x \in \Omega\}.$$

It is well-known that p^* is the maximum number such that the continuous embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ holds true for all $q \in [1,p^*]$. Moreover, if $q \in [1,p^*)$ then this embedding is compact.

For $p_1, p_2 \in (1,p^*]$ if $p^* < \infty$ or $p_1, p_2 \in (1,\infty)$ if $p^* = \infty$, denote by K_1 and K_2 the inverses of Sobolev coefficients of continuous embedding $W_0^{1,p}(\Omega) \subset L^{p_1}(\Omega)$ and $W_0^{1,p}(\Omega) \subset L^{p_2}(\Omega)$, respectively, that means

$$K_1 = \sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{|u|_{p_1}}{\|u\|}, \quad K_2 = \sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{|u|_{p_2}}{\|u\|}.$$

Suppose that $C_i, i \in \{0,1,2,3\}$ are positive constants. We impose the following hypotheses on a and f:

(A1) $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function;
(A2) $|a(x,\xi)|\leqslant k_0(x)+C_0|\xi|^{p-1}$, for a.e $x\in\Omega$ and for all $\xi\in\mathbb{R}^N$, where $k_0\in L^{p'}(\Omega)$;
(A3) $(a(x,\xi)-a(x,\xi'))\cdot(\xi-\xi')\geqslant C_3|\xi-\xi'|^p$ for a.e $x\in\Omega$ and for all $\xi,\xi'\in\mathbb{R}^N$;
(F1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is sup-measurable, i.e $x\mapsto f(x,u(x))$ is measurable in Ω whenever $u:\Omega\to\mathbb{R}$ is measurable;
(F2) $|f(x,s)|\leqslant k_1(x)+C_1|s|^{p_1-1}$, for a.e $x\in\Omega$ and for all $s\in\mathbb{R}$, where $k_1\in L^{p_1'}(\Omega)$;
(Q1) $q:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function;
(Q2) $|q(x,s)|\leqslant k_2(x)+C_2|s|^{p_2-1}$, for a.e $x\in\Omega$ and for all $s\in\mathbb{R}$, where $k_2\in L^{p_2'}(\Omega)$;
(Q3) $(q(x,s)-q(x,s'))(s-s')\geqslant 0$ for a.e $x\in\Omega$ and for all $s,s'\in\mathbb{R}$;
(FQ) $s\mapsto f(x,s)+q(x,s)$ is increasing for a.e $x\in\Omega$.

Our main result is the following theorem:

THEOREM 1.1. *Assume that the conditions* (A1)-(A3), (F1)-(F2), (Q1)-(Q3) *and* (FQ) *are satisfied, then problem* (1.1) *possesses a weak solution u in the following cases:*

(i) $1<p_1,p_2<p$,
(ii) $p_1=p_2=p$ *and* $C_3-C_1K_1^{p_1}-C_2K_2^{p_2}>0$,
(iii) $p_1=p$, $1<p_2<p$ *and* $C_3-C_1K_1^{p_1}>0$,
(iv) $p_2=p$, $1<p_1<p$ *and* $C_3-C_2K_2^{p_2}>0$,
(v) $p<p_1,p_2\leqslant p^*$ *and* $|k_0|_{p'}+|k_1|_{p_1'}+|k_2|_{p_2'}<\varepsilon$, *where ε is sufficiently small. Moreover,* $\|u\|\to 0$ *as* $\varepsilon\to 0$, *keeping other constants C_i fixed.*

Furthermore, if $\underline{u}$ is a subsolution of (1.1) *then the above solution u can be chosen in* $\{w\in W_0^{1,p}(\Omega)\mid \underline{u}\leqslant w\}$ *in cases* (i)-(iv). *If* (1.1) *has a subsolution $\underline{u}$ such that $\|\underline{u}^+\|$ is sufficiently small then the above solution u can also be chosen in* $\{w\in W_0^{1,p}(\Omega)\mid \underline{u}\leqslant w\}$ *in case* (v).

As an example of functions satisfying all assumptions of Theorem 1.1, we may take $a(x,\xi)=|\xi|^{p-2}\xi$, $f(x,s)=|s|^{p^*-2}s+\delta g(x,s)$ and $q(x,s)=0$, where

$$g(x,s)=\begin{cases}0 & \text{if } s<0,\\ 1/2 & \text{if } s=0,\\ 1 & \text{if } s>0,\end{cases}$$

and δ is sufficiently small. In section 4, we will prove that the problem associated with this example actually has a positive weak solution.

The next theorem is well-known in sub-super solution theory. For example, see [7, 8] and references therein. In this paper, we will introduce a new and simple proof using the fixed point theorem mentioned in [6].

THEOREM 1.2. *Assume that conditions* (A1)-(A3), (F1)-(F2), (Q1)-(Q3) *and* (FQ) *are satisfied and problem* (1.1) *has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ satisfying $\underline{u}\leqslant\overline{u}$. Then* (1.1) *has a weak solution in* $[\underline{u},\overline{u}]$.

2. Preliminaries

In this section, we recall some basic concepts and tools on ordered Banach spaces which will be used in the sequel. See [6] for more details on these definitions and notations.

The subset X_+ of a normed space X is called an *order cone* iff the following are true:
- X_+ is closed, convex, nonempty and $X_+ \neq \{0\}$;
- If $u \in X_+$ and $\alpha \geqslant 0$, then $\alpha u \in X_+$;
- If $u \in X_+$ and $-u \in X_+$, then $u = 0$.

A Banach space (normed space) $(X, \|\cdot\|)$ endowed with a partial ordering $\leqslant$ induced by an order cone X_+ by $x \leqslant y$ iff $y - x \in X_+$ is called an *ordered Banach space* (*ordered normed space*).

Let X be an ordered normed space with the ordering $\leqslant$ and the order cone X_+, P be a subset of X, a be an element of P and $\{a_n\}$ be a sequence in P. We say that:

- a is a *sup-center* (respectively, *inf-center*) of P if $\sup\{a,y\}$ (respectively, $\inf\{a,y\}$) exists and belongs to P for all $y \in P$;
- P is *(weakly) sequentially order compact* if increasing and decreasing sequences of P have (weak) limits in P;
- P has *the fixed point property* if each increasing mapping $G : P \to P$ has a fixed point;
- X is a *Banach semilattice* if X is an ordered Banach space satisfying $\|x^+\| \leqslant \|x\|$ and $\|x^-\| \leqslant \|x\|$ for all $x \in X$ where $x^+ = \sup\{0,x\}$ and $x^- = \inf\{0,x\}$;
- $\{a_n\}$ is *increasing* (resp., *decreasing*) if $a_n \leqslant a_m$ (resp., $a_n \geqslant a_m$) whenever $n \leqslant m$;
- $\{a_n\}$ is *bounded* if there exists a constant $C > 0$ such that $\|a_n\| \leqslant C$ for all $n \in \mathbb{N}$;
- X_+ is a *(weakly) fully regular order cone* if each bounded and increasing sequence of X_+ is (weakly) convergent.

We also denote

$$B'_X(b,R) = \{x \in X \mid \|x - b\| \leqslant R\} \quad \text{and} \quad [b_1, b_2]_X = \{x \in X \mid b_1 \leqslant x \leqslant b_2\},$$

where $b, b_1, b_2 \in X$.

The following fixed point theorem is proved in [6]:

LEMMA 2.1. *Let P be a weakly sequentially order compact subset of an ordered normed space X having a sup-center or an inf-center. Then P has the fixed point property.*

We now introduce some concrete types of sets having the fixed point property beside closed balls as in [6]. Lemma 2.2 bellow slightly generalizes [6, Corollary 2].

LEMMA 2.2. *Let X be a Banach semilattice which is reflexive or has a weakly fully regular order cone X_+ and Y be a closed subspace of X. Moreover, assume that for every $x \in Y$, we have $x^+ \in Y$. Then any nonempty set P of the following types possesses the fixed point property:*

(i) $P = B'_Y(a,R)$, *where* $a \in Y$ *and* $R > 0$,

(ii) $P = B'_Y(a,R) \cap \{x \in X \mid b \leqslant x\}$, *where* $a \in Y, b \in X$ *and* $R > 0$,

(iii) $P = B'_Y(a,R) \cap [b_1,b_2]_X$, *where* $a \in Y, b_1, b_2 \in X, a \leqslant b_2$ *and* $R > 0$.

Proof. Note that if X is reflexive then its order cone X_+ is weakly fully regular. Moreover, every set P of types (i), (ii) or (iii) is a weakly closed subset of X because it is convex and closed. Also, it is clear that P is bounded.

We show that P is weakly sequentially order compact. In order to do this, let $\{x_n\} \subset P$ be an increasing sequence. Put $y_n = x_n - x_0$, $n = 1,2,\ldots$, then $\{y_n\} \subset X_+$ is an bounded and increasing sequence, which weakly converges to some y because X_+ is weakly fully regular.

Therefore, $\{x_n\}$ is also weakly convergent and its weak limit is $y + x_0$, which belongs to P since P is a weakly closed subset of X. The case that $\{x_n\}$ is decreasing can be done similarly.

We prove that a is a sup-center of P. Let $y \in P$ then $\sup\{a,y\} = (y-a)^+ + a$.

In case (i), from the Banach semilattice property, we get

$$\|\sup\{a,y\} - a\| = \|(y-a)^+\| \leqslant \|y-a\| \leqslant R.$$

Moreover, $\sup\{a,y\} \in Y$ since $a, y \in Y$. Therefore, $\sup\{a,y\} \in B'_Y(a,R) = P$.

In case (ii), $y \in P \subset B'_Y(a,R)$, hence $\sup\{a,y\} \in B'_Y(a,R)$ from the above proof. Moreover, $b \leqslant y$ implies $b \leqslant \sup\{a,y\}$. Therefore,

$$\sup\{a,y\} \in B'_Y(a,R) \cap \{x \in X \mid b \leqslant x\} = P.$$

In case (iii), $y \in P \subset B'_Y(a,R)$, hence $\sup\{a,y\} \in B'_Y(a,R)$ from the above proof. Moreover, $b_1 \leqslant y \leqslant b_2$, and $a \leqslant b_2$, so $b_1 \leqslant \sup\{a,y\} \leqslant b_2$. Therefore,

$$\sup\{a,y\} \in B'_Y(a,R) \cap [b_1,b_2]_X = P. \qquad \square$$

The following remark is useful in applications:

REMARK 2.1. For $p > 1$, Sobolev spaces $W^{1,p}(\Omega)$ are Banach semilattices with the usual ordering and reflexive. Hence for $X = W^{1,p}(\Omega)$ and $Y = W_0^{1,p}(\Omega)$, all sets of types (i), (ii) or (iii) mentioned in Lemma 2.2 have the fixed point property.

3. Proof of Theorem 1.1 and Theorem 1.2

For $u \in W^{1,p}(\Omega)$ and $\varphi \in W_0^{1,p}(\Omega)$, we define
$\langle Au, \varphi \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla \varphi$,
$\langle Fu, \varphi \rangle = \int_\Omega f(x,u)\varphi$,
$\langle Qu, \varphi \rangle = \int_\Omega q(x,u)\varphi$.

We call

- $u \in W_0^{1,p}(\Omega)$ a weak solution of (1.1) if $\langle Au, \varphi \rangle = \langle Fu, \varphi \rangle$ for all $\varphi \in W_0^{1,p}(\Omega)$,

- $u \in W^{1,p}(\Omega)$ a subsolution of (1.1) if $u|_{\partial\Omega} \leqslant 0$ and $\langle Au, \varphi \rangle \leqslant \langle Fu, \varphi \rangle$ for all $\varphi \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$,

- $u \in W^{1,p}(\Omega)$ a supersolution of (1.1) if $u|_{\partial\Omega} \geqslant 0$ and $\langle Au, \varphi \rangle \geqslant \langle Fu, \varphi \rangle$ for all $\varphi \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$.

Due to (A1)-(A3) and (Q1)-(Q3), the operator $A+Q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bijective and if $(A+Q)u \leqslant (A+Q)v$ then $u \leqslant v$. The reader may find detailed proofs for these facts in the appendix.

Now, consider the operator $G : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$G = (A+Q)^{-1} \circ (F+Q).$$

Then $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1) if and only if u is a fixed point of G.

Moreover, from (FQ), the operator $F+Q$ is increasing. Therefore G is increasing, too.

From now on, $B'(u,R)$ denotes the closed ball in $W_0^{1,p}(\Omega)$ of center $u \in W^{1,p}(\Omega)$ and radius $R > 0$.

Proof of Theorem 1.1. From the above arguments, all we have to do is to construct P which has the fixed point property such that if $v \in P$ then $G(v) \in P$.

We look for P of type $P = B'(0,R)$. This type of set has the fixed point property by Lemma 2.2 and Remark 2.1.

Set $u = G(v)$, then $(A+Q)u = (F+Q)v$.

Choose $\xi' = 0$ in (A3), $s' = 0$ in (Q3) and using (A2), (Q2), we have

$$\begin{aligned} \langle (A+Q)u, u \rangle &= \int_\Omega a(x, \nabla u) \cdot \nabla u + \int_\Omega q(x,u)u \\ &\geqslant \int_\Omega (C_3 |\nabla u|^p - |k_0(x)|.|\nabla u|) - \int_\Omega |k_2(x)|.|u| \\ &\geqslant C_3 \|u\|^p - (|k_0|_{p'} + K_2 |k_2|_{p_2'}) \|u\|. \end{aligned} \tag{3.1}$$

From (3.1) and growth conditions (F2), (Q2) on f and g, we have

$$\begin{aligned} C_3 \|u\|^p &- (|k_0|_{p'} + K_2 |k_2|_{p_2'}) \|u\| \\ &\leqslant \langle (A+Q)u, u \rangle = \langle (F+Q)v, u \rangle \\ &\leqslant |k_1|_{p_1'} |u|_{p_1} + C_1 |v|_{p_1}^{p_1 - 1} |u|_{p_1} + |k_2|_{p_2'} |u|_{p_2} + C_2 |v|_{p_2}^{p_2 - 1} |u|_{p_2} \\ &\leqslant (K_1 |k_1|_{p_1'} + K_2 |k_2|_{p_2'} + C_1 K_1 |v|_{p_1}^{p_1 - 1} + C_2 K_2 |v|_{p_2}^{p_2 - 1}) \|u\| \\ &\leqslant (K_1 |k_1|_{p_1'} + K_2 |k_2|_{p_2'} + C_1 K_1^{p_1} \|v\|^{p_1 - 1} + C_2 K_2^{p_2} \|v\|^{p_2 - 1}) \|u\|. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} C_3 \|u\|^{p-1} &\leqslant |k_0|_{p'} + K_1 |k_1|_{p_1'} + 2K_2 |k_2|_{p_2'} + C_1 K_1^{p_1} \|v\|^{p_1 - 1} + C_2 K_2^{p_2} \|v\|^{p_2 - 1} \\ &= M + C_1 K_1^{p_1} \|v\|^{p_1 - 1} + C_2 K_2^{p_2} \|v\|^{p_2 - 1}, \end{aligned}$$

where $M = |k_0|_{p'} + K_1|k_1|_{p_1'} + 2K_2|k_2|_{p_2'}$.

We must find $R > 0$ such that if $\|v\| \leqslant R$ then $\|u\| \leqslant R$. This property will be satisfied if $M + C_1K_1^{p_1}R^{p_1-1} + C_2K_2^{p_2}R^{p_2-1} \leqslant C_3R^{p-1}$ or $p(R) \geqslant 0$, where

$$p(t) = C_3t^{p-1} - (C_1K_1^{p_1}t^{p_1-1} + C_2K_2^{p_2}t^{p_2-1} + M).$$

(i) If $1 < p_1, p_2 < p$ then $p(R) \geqslant 0$ for R large enough.

(ii) If $p_1 = p_2 = p$ and $C_3 - C_1K_1^{p_1} - C_2K_2^{p_2} > 0$ then $\lim_{t\to+\infty} p(t) = +\infty$. Thus, $p(R) \geqslant 0$ for R large enough.

(iii) If $p_1 = p, 1 < p_2 < p$ and $C_3 - C_1K_1^{p_1} > 0$ then $\lim_{t\to+\infty} p(t) = +\infty$. Thus, $p(R) \geqslant 0$ for R large enough.

(iv) This case is similar to (iii).

(v) If $p_1, p_2 > p$ then

$$\overline{p}(t) = C_3t^{p-1} - (C_1K_1^{p_1}t^{p_1-1} + C_2K_2^{p_2}t^{p_2-1}) > 0$$

for all $t \in (0, t_0]$ where $t_0 > 0$ is sufficiently small. Note that $\lim_{t\to 0^+} \overline{p}(t) = 0$.

If we fix $t \in (0, t_0]$ and choose k_0, k_1, k_2 such that

$$|k_0|_{p'} + |k_1|_{p_1'} + |k_2|_{p_2'} < (\max\{1, K_1, 2K_2\})^{-1}\overline{p}(t),$$

then $M < \overline{p}(t)$ and $p(t) = \overline{p}(t) - M > 0$. It implies that for all ε small enough, we can choose $R_\varepsilon > 0$ such that if $|k_0|_{p'} + |k_1|_{p_1'} + |k_2|_{p_2'} < \varepsilon$, then $p(R_\varepsilon) > 0$. Moreover, $\lim_{\varepsilon\to 0} R_\varepsilon = 0$.

The above arguments imply the existence of $R > 0$ such that if $v \in B'(0,R)$ then $G(v) = u \in B'(0,R)$.

Next, suppose that $\underline{u}$ is a subsolution of (1.1). We consider

$$P = B'(0,R) \cap \{w \in W^{1,p}(\Omega) \mid \underline{u} \leqslant w\}$$

with R obtained above. Note that

$$\underline{u}^+ \in W_0^{1,p}(\Omega) \cap \{w \in W^{1,p}(\Omega) \mid \underline{u} \leqslant w\}.$$

In the cases of (i)-(iv), we may choose R large enough if necessary. In case (v), we may choose $\underline{u}$ such that $\|\underline{u}^+\|$ is small enough. Therefore, P is nonempty and P has the fixed point property by Lemma 2.2.

Assume that $v \in P$, we prove that $G(v) \in P$.

Since $v \in P \subset B'(0,R)$, we have $G(v) \in B'(0,R)$ from the above proof.

On the other hand, because $\underline{u}$ is a subsolution of (1.1) and $\underline{u} \leqslant v$, we have

$$(A+Q)(\underline{u}) \leqslant (F+Q)(\underline{u}) \leqslant (F+Q)(v).$$

Thus, $\underline{u} \leqslant (A+Q)^{-1} \circ (F+Q)v = G(v)$. Therefore, $G(v) \in P$.

Theorem 1.1 has been proved completely. $\square$

Proof of Theorem 1.2. As before, we look for P of type $P = B'(u_0,R)\cap[\underline{u},\overline{u}]$ such that if $v\in P$ then $G(v)\in P$. This type of set has the fixed point property by Lemma 2.2 and Remark 2.1.

Set $u = G(v)$, then $(A+Q)u = (F+Q)v$. From (3.2) and $v\in[\underline{u},\overline{u}]$ we have

$$\begin{aligned} C_3\|u\|^{p-1} &\leqslant M + C_1K_1|v|_{p_1}^{p_1-1} + C_2K_2|v|_{p_2}^{p_2-1} \\ &\leqslant M + C_1K_1(|\underline{u}|_{p_1} + |\overline{u}|_{p_1})^{p_1-1} + C_2K_2(|\underline{u}|_{p_2} + |\overline{u}|_{p_2})^{p_2-1}, \end{aligned}$$

which means that $\|u\|$ is bounded by a positive number R_0 independent of $v\in[\underline{u},\overline{u}]$. Choose $u_0 \in W_0^{1,p}(\Omega)\cap[\underline{u},\overline{u}]$, $R = R_0 + \|u_0\|$ and set $P = B'(u_0,R)\cap[\underline{u},\overline{u}]$.

If $v\in P$ then $v\in[\underline{u},\overline{u}]$. Thus, $\|u\|\leqslant R_0$ by the above proof. Therefore,

$$\|u-u_0\| \leqslant \|u\| + \|u_0\| \leqslant R_0 + \|u_0\| = R, \text{ i.e } u\in B'(u_0,R).$$

On the other hand, from definition of $\underline{u}$ and monotonicity of $F+Q$, we have

$$(A+Q)\underline{u} \leqslant (F+Q)\underline{u} \leqslant (F+Q)v = (A+Q)u.$$

Thus, $\underline{u}\leqslant u$.

Similarly, $(A+Q)u = (F+Q)v \leqslant (F+Q)\overline{u} \leqslant (A+Q)\overline{u}$. Thus, $u\leqslant\overline{u}$.

Consequently, $G(v) = u \in P$, as desired. □

4. Quasilinear elliptic problems with critical exponents and discontinuous nonlinearities

In this section, we study problem (1.2). Suppose that $N\geqslant 3$ and $2\leqslant p < N$, hence $p^* < +\infty$. Let $C>0$ be a positive constant, we impose the following conditions on g:

(G1) $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is sup-measurable;

(G2) $\liminf_{s\to 0^+}(g(x,s)/s^{p-1}) = +\infty$, uniformly in x;

(G3) $|g(x,s)|\leqslant Cs^{p^*-1}$, for a.e $x\in\Omega$ and for all $s>1$;

(G4) $\overline{g}\in L^{p^*/(p^*-1)}(\Omega)$, where $\overline{g}(x) = \sup_{0\leqslant s\leqslant 1}|g(x,s)|$ for a.e $x\in\Omega$;

(H1) $h:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is a Carathéodory function;

(H2) $|h(x,s)|\leqslant Cs^{p^*-1}$, for a.e $x\in\Omega$ and for all $s\in\mathbb{R}_+$;

(H3) $(h(x,s)-h(x,s'))(s-s')\geqslant 0$ for a.e $x\in\Omega$ and for all $s,s'\in\mathbb{R}_+$;

(GH) $s\mapsto g(x,s)+h(x,s)$ is increasing for a.e $x\in\Omega$ and $s\in\mathbb{R}_+$.

THEOREM 4.1. *Assume that the condition* (G1)-(G4), (H1)-(H3) *and* (GH) *are satisfied. Then there exists a positive number δ_0 such that* (1.2) *has a positive solution u_δ for all $\delta\in(0,\delta_0)$. Moreover, $\|u_\delta\|\to 0$ as $\delta\to 0^+$.*

Proof. We will apply Theorem 1.1 with $a(x,\xi)=|\xi|^{p-2}\xi$,

$$f(x,s)=\begin{cases}|s|^{p^*-2}s+\delta g(x,s) & \text{if } s\geqslant 0,\\ \delta g(x,0) & \text{if } s<0,\end{cases}$$

and

$$q(x,s)=\begin{cases}\delta h(x,s) & \text{if } s\geqslant 0,\\ \delta h(x,0) & \text{if } s<0.\end{cases}$$

For $\delta>0$ small, by (G3) and (G4) we have $|f(x,s)|\leqslant C'|s|^{p^*-1}+\delta\overline{g}(x)$ for some positive constant C'. Therefore, (A1)-(A3), (F1)-(F2), (Q1)-(Q3) and (FQ) are satisfied with $k_0=k_2=0$, $k_1=\delta\overline{g}$.

From (v) in Theorem 1.1, there exists $\delta_0>0$ such that problem (1.1) with a,f,q as above has a weak solution for all $\delta\in(0,\delta_0)$ and this solution converges to 0 as $\delta\to 0$. In order to ensure that this solution is positive, and therefore, also a solution of (1.2), we have to show that (1.2) has a sequence of positive subsolutions converging to 0 in $W_0^{1,p}(\Omega)$.

Let λ_1 be the first eigenvalue and $\varphi_1>0$ be a corresponding eigenfunction of the eigenvalue problem $-\Delta_p u=\lambda|u|^{p-2}u$ in Ω with zero Dirichlet boundary condition.

Setting $u_\varepsilon=\varepsilon\varphi_1$. We look for $\varepsilon>0$ such that $-\Delta_p u_\varepsilon\leqslant u_\varepsilon^{p^*-1}+\delta g(x,u_\varepsilon)$, or equivalently, $\lambda_1(\varepsilon\varphi_1)^{p-1}\leqslant u_\varepsilon^{p^*-1}+\delta g(x,\varepsilon\varphi_1)$.

By (G2), there exists $s_0>0$ such that $g(x,s)\geqslant(\lambda_1/\delta)s^{p-1}$ for all $s\in(0,s_0)$ and a.e $x\in\Omega$. Thus, u_ε is a subsolution of problem (1.2) if $\varepsilon\in(0,s_0/|\varphi_1|_\infty)$.

Now the existence of a positive solution u_δ to problem (1.2) and the convergence of $\{u_\delta\}$ follow immediately from Theorem 1.1. □

REMARK 4.1. Two trivial examples of g satisfying Theorem 4.1 are $g(x,s)=|s|^{q-2}s$ with $1<q<p$ and

$$g(x,s)=\begin{cases}0 & \text{if } s<a,\\ 1/2 & \text{if } s=a,\\ 1 & \text{if } s>a,\end{cases}$$

where $a\leqslant 0$.

We choose $h\equiv 0$ in both examples above.

5. Appendix

For the reader's convenience, in this appendix we prove the fundamental facts in theory of monotone operators saying that the operator $A+Q$ is bijective and $(A+Q)^{-1}$ is increasing. These facts are used in the proof of Theorem 1.1.

LEMMA 5.1. *Assume that conditions* (A1)-(A3) *and* (Q1)-(Q3) *are satisfied, then operator* $A+Q:W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega)$ *is bijective and if* $(A+Q)u\leqslant(A+Q)v$ *then* $u\leqslant v$.

Proof. Our proof has 4 steps.

Step 1. ($A+Q$ is monotone) Let $u,v \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\langle (A+Q)u-(A+Q)v, u-v\rangle \\
&\qquad = \int_\Omega [a(x,\nabla u)-a(x,\nabla v)](\nabla u-\nabla v)dx + \int_\Omega [q(x,u)-q(x,v)](u-v)dx \\
&\qquad \geqslant \int_\Omega C_3|\nabla u-\nabla v|^p dx \\
&\qquad = C_3\|u-v\|^p,
\end{aligned}$$

by (A3) and (Q3). Therefore, $A+Q$ is monotone.

Step 2. ($A+Q$ is hemicontinuous) Let $u,v,w \in W_0^{1,p}(\Omega)$.

• We prove that: $\lim_{t\to 0}\langle A(u+tv),w\rangle = \langle Au,w\rangle$.

Setting $h_t(x) = a(x,\nabla u(x)+t\nabla v(x))\nabla w(x),\ t\in[-1,1]$.

By Young's inequality, we have

$$|h_t(x)| \leqslant \frac{1}{p'}|a(x,\nabla u(x)+t\nabla v(x))|^{p'} + \frac{1}{p}|\nabla w(x)|^p.$$

On the other hand, by (A2)

$$\begin{aligned}
|a(x,\nabla u(x)+t\nabla v(x))|^{p'} &\leqslant (|k_0(x)|+C_0|\nabla u(x)+t\nabla v(x)|^{p-1})^{p'} \\
&\leqslant 2^{p'}|k_0(x)|^{p'} + 2^{p'}C_0^{p'}|\nabla u(x)+t\nabla v(x)|^p \\
&\leqslant 2^{p'}|k_0(x)|^{p'} + 2^{p+p'}C_0^{p'}|\nabla u(x)|^p + 2^{p+p'}C_0^{p'}|\nabla v(x)|^p.
\end{aligned}$$

Therefore,

$$|h_t(x)| \leqslant h(x),$$

with

$$h(x) = \frac{2^{p'}}{p'}|k_0(x)|^{p'} + \frac{2^{p+p'}C_0^{p'}}{p'}|\nabla u(x)|^p + \frac{2^{p+p'}C_0^{p'}}{p'}|\nabla v(x)|^p + \frac{1}{p}|\nabla w(x)|^p.$$

From the hypothesis, we have $h\in L^1(\Omega)$.

We have $|h_t(x)|\leqslant h(x),\ \forall t\in[-1,1]$ and $\lim_{t\to 0} h_t(x) = a(x,\nabla u(x))\nabla w(x)$ for a.e $x\in\Omega$. By Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{t\to 0}\langle A(u+tv),w\rangle = \langle Au,w\rangle.$$

• Similarly, we have: $\lim_{t\to 0}\langle Q(u+tv),w\rangle = \langle Qu,w\rangle$.

• From the above proof,

$\lim_{t\to 0}\langle (A+Q)(u+tv),w\rangle = \langle (A+Q)u,w\rangle$, i.e $A+Q$ is hemicontinuous.

Step 3. ($A+Q$ is coercive) From (3.1), for $u\in W_0^{1,p}(\Omega)\setminus\{0\}$ we have

$$\frac{\langle (A+Q)u,u\rangle}{\|u\|} \geqslant C_3\|u\|^{p-1} - \big(|k_0|_{p'} + K_2|k_2|_{p_2'}\big),$$

i.e $A+Q$ is coercive.

Applying Browder's theorem in [4], from steps 1, 2 and 3 we conclude that $A+Q$ is surjective.

Step 4. (Assuming that $(A+Q)u \leqslant (A+Q)v$, we prove that $u \leqslant v$) By the hypothesis, we have

$$\langle (A+Q)v-(A+Q)u,\varphi\rangle \geqslant 0,\ \forall \varphi \in W_0^{1,p}(\Omega)\cap L_+^p(\Omega).$$

Choose $\varphi=(u-v)_+$ as a test function and put $\Omega_+=\{x\in\Omega \mid v(x)\leqslant u(x)\}$, we have

$$\begin{aligned}0 &\leqslant \int_\Omega [a(x,\nabla v)-a(x,\nabla u)]\nabla(u-v)_+dx+\int_\Omega [q(x,v)-q(x,u)](u-v)_+dx\\ &= \int_{\Omega_+}[a(x,\nabla v)-a(x,\nabla u)]\nabla(u-v)dx+\int_{\Omega_+}[q(x,v)-q(x,u)](u-v)dx.\end{aligned}$$

Thus,

$$\int_{\Omega_+}[a(x,\nabla v)-a(x,\nabla u)](\nabla v-\nabla u)dx+\int_{\Omega_+}[q(x,v)-q(x,u)](v-u)dx\leqslant 0.$$

On the other hand, by (Q3) we have:

$$\int_{\Omega_+}[q(x,v)-q(x,u)](v-u)dx\geqslant 0,$$

and by (A3)

$$\begin{aligned}\int_{\Omega_+}[a(x,\nabla v)-a(x,\nabla u)](\nabla v-\nabla u)dx &\geqslant C_3\int_{\Omega_+}|\nabla v-\nabla u|^p dx\\ &= C_3\int_\Omega |\nabla(u-v)_+|^p dx\\ &= C_3\|(u-v)_+\|^p.\end{aligned}$$

Thus, $(u-v)_+=0$, i.e $u\leqslant v$.

Next, suppose that $(A+Q)u=(A+Q)v$, the above proof implies that $u\leqslant v$ and $v\leqslant u$. So $A+Q$ is injective. □

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