

QUASILINEAR ELLIPTIC PROBLEMS WITH CRITICAL EXPONENTS AND DISCONTINUOUS NONLINEARITIES

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Abstract. Using a recent fixed point theorem in ordered Banach spaces by S. Carl and S. Heikkilä, we study the existence of weak solutions to nonlinear elliptic problems $-\operatorname{div}(x, \nabla u) = f(x, u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary condition. In particular, we prove that for some suitable function g , which may be discontinuous, and δ small enough, the p -Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = |u|^{p^*-2} u + \delta g(x, u)$$

has a positive solution which goes to 0 as $\delta \rightarrow 0^+$, where p^* is the critical exponent.

1. Introduction

Let $N \geq 2$ be an integer and $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We study the following nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}(x, \nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where f is not necessarily continuous with respect to its second variable and may have critical exponent.

This problem has been studied extensively in literature by many authors. A general method for proving the existence of solutions of (1.1), when f is a Carathéodory function and has subcritical growth, is critical point theory. When f is discontinuous and has critical exponent, the situation becomes difficult because the energy functional associated with (1.1) does not belong to C^1 class and because of the lack of compactness of embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Nevertheless, many authors dealt with this case using several methods in nonsmooth analysis. To name a few, when $a(x, \xi) = \xi$, problem (1.1) was studied in [3, 5, 10] and references therein. When $a(x, \xi) = |\xi|^{p-2} \xi$ (p -Laplace equation), it was studied in [13] for $f(x, u) = \lambda |u|^{p^*-2} u + g(x, u)$. Later, when $a(x, \xi) = |\xi|^{p(x)-2} \xi$ ($p(x)$ -Laplace equation), it was studied in [14] for $f(x, u) = \lambda |u|^{p^*(x)-2} u + g(u)$. When $a_i(x, \xi) = |\xi_i|^{p_i-2} \xi_i$ (anisotropic quasilinear elliptic equation), it was studied in [9].

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In the present paper, we exploit the fixed point theorem introduced in [6] to prove the existence of a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ of (1.1). Note that in [6], S. Carl and S. Heikkilä applied their theorem to the problem $-\Delta u = f(x, u)$ and proved the existence of one solution. However, the solution they obtained may be the trivial one if $f(x, 0) = 0$. By using a suitable set having the fixed point property, the trivial solution in our result is excluded.

As an application, in section 4 we will consider the following critical problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + \delta g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where g is not required to be Carathéodory but sup-measurable.

It is well known that if $g \equiv 0$ and Ω is a star-shaped domain then (1.2) has no nontrivial solution (see [11, 12]). The perturbation $\delta g(x, u)$ ensures the existence of a nontrivial solution to problem (1.2). The existence of a positive solution of (1.2) was studied in [2] when $p = 2$, g is the Heaviside function in u and δ is sufficiently large. This problem is also studied in [1] using a variational approach when $p = 2$ and $g(x, u) = h(x)H(u - a)u^q$, where $h(x)$ is both nonnegative and integrable on \mathbb{R}^N , H is the Heaviside function, $0 \leq q < 2^* - 1$ and $\delta > 0$. In section 4 of this paper, we will prove that when δ is sufficiently small, problem (1.2) has a positive solution which converges to 0 as $\delta \rightarrow 0^+$.

Throughout this paper, we assume that $p \in [2, \infty)$. As usual, we denote:

$$p^* = pN/(N - p) \text{ if } p < N \text{ and } p^* = \infty \text{ if } p \geq N \text{ (hence } p^* = \infty \text{ if } N = 2),$$

$$p' = p/(p - 1),$$

$$\|u\|_p = (\int_{\Omega} |u|^p)^{\frac{1}{p}}, \text{ the norm of } u \text{ in } L^p(\Omega),$$

$$\|u\|_{\infty} = \operatorname{ess\,sup}\{|u(x)| \mid x \in \Omega\}, \text{ the norm of } u \text{ in } L^{\infty}(\Omega),$$

$$\|u\| = (\int_{\Omega} |\nabla u|^p)^{\frac{1}{p}}, \text{ the norm of } u \text{ in } W_0^{1,p}(\Omega),$$

$$L_+^p(\Omega) = \{u \in L^p(\Omega) \mid u(x) \geq 0 \text{ for a.e } x \in \Omega\}.$$

It is well-known that p^* is the maximum number such that the continuous embedding $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ holds true for all $q \in [1, p^*]$. Moreover, if $q \in [1, p^*)$ then this embedding is compact.

For $p_1, p_2 \in (1, p^*]$ if $p^* < \infty$ or $p_1, p_2 \in (1, \infty)$ if $p^* = \infty$, denote by K_1 and K_2 the inverses of Sobolev coefficients of continuous embedding $W_0^{1,p}(\Omega) \subset L^{p_1}(\Omega)$ and $W_0^{1,p}(\Omega) \subset L^{p_2}(\Omega)$, respectively, that means

$$K_1 = \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{p_1}}{\|u\|}, \quad K_2 = \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{p_2}}{\|u\|}.$$

Suppose that $C_i, i \in \{0, 1, 2, 3\}$ are positive constants. We impose the following hypotheses on a and f :

- (A1) $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function;
- (A2) $|a(x, \xi)| \leq k_0(x) + C_0|\xi|^{p-1}$, for a.e $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $k_0 \in L^{p'}(\Omega)$;
- (A3) $(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq C_3|\xi - \xi'|^p$ for a.e $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$;
- (F1) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is sup-measurable, i.e $x \mapsto f(x, u(x))$ is measurable in Ω whenever $u : \Omega \rightarrow \mathbb{R}$ is measurable;
- (F2) $|f(x, s)| \leq k_1(x) + C_1|s|^{p_1-1}$, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$, where $k_1 \in L^{p'_1}(\Omega)$;
- (Q1) $q : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function;
- (Q2) $|q(x, s)| \leq k_2(x) + C_2|s|^{p_2-1}$, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}$, where $k_2 \in L^{p'_2}(\Omega)$;
- (Q3) $(q(x, s) - q(x, s'))(s - s') \geq 0$ for a.e $x \in \Omega$ and for all $s, s' \in \mathbb{R}$;
- (FQ) $s \mapsto f(x, s) + q(x, s)$ is increasing for a.e $x \in \Omega$.

Our main result is the following theorem:

THEOREM 1.1. *Assume that the conditions (A1)-(A3), (F1)-(F2), (Q1)-(Q3) and (FQ) are satisfied, then problem (1.1) possesses a weak solution u in the following cases:*

- (i) $1 < p_1, p_2 < p$,
- (ii) $p_1 = p_2 = p$ and $C_3 - C_1K_1^{p_1} - C_2K_2^{p_2} > 0$,
- (iii) $p_1 = p$, $1 < p_2 < p$ and $C_3 - C_1K_1^{p_1} > 0$,
- (iv) $p_2 = p$, $1 < p_1 < p$ and $C_3 - C_2K_2^{p_2} > 0$,
- (v) $p < p_1, p_2 \leq p^*$ and $|k_0|_{p'} + |k_1|_{p'_1} + |k_2|_{p'_2} < \varepsilon$, where ε is sufficiently small. Moreover, $\|u\| \rightarrow 0$ as $\varepsilon \rightarrow 0$, keeping other constants C_i fixed.

Furthermore, if \underline{u} is a subsolution of (1.1) then the above solution u can be chosen in $\{w \in W_0^{1,p}(\Omega) \mid \underline{u} \leq w\}$ in cases (i)-(iv). If (1.1) has a subsolution \underline{u} such that $\|\underline{u}^+\|$ is sufficiently small then the above solution u can also be chosen in $\{w \in W_0^{1,p}(\Omega) \mid \underline{u} \leq w\}$ in case (v).

As an example of functions satisfying all assumptions of Theorem 1.1, we may take $a(x, \xi) = |\xi|^{p-2}\xi$, $f(x, s) = |s|^{p^*-2}s + \delta g(x, s)$ and $q(x, s) = 0$, where

$$g(x, s) = \begin{cases} 0 & \text{if } s < 0, \\ 1/2 & \text{if } s = 0, \\ 1 & \text{if } s > 0, \end{cases}$$

and δ is sufficiently small. In section 4, we will prove that the problem associated with this example actually has a positive weak solution.

The next theorem is well-known in sub-super solution theory. For example, see [7, 8] and references therein. In this paper, we will introduce a new and simple proof using the fixed point theorem mentioned in [6].

THEOREM 1.2. *Assume that conditions (A1)-(A3), (F1)-(F2), (Q1)-(Q3) and (FQ) are satisfied and problem (1.1) has a subsolution \underline{u} and a supersolution \bar{u} satisfying $\underline{u} \leq \bar{u}$. Then (1.1) has a weak solution in $[\underline{u}, \bar{u}]$.*

2. Preliminaries

In this section, we recall some basic concepts and tools on ordered Banach spaces which will be used in the sequel. See [6] for more details on these definitions and notations.

The subset X_+ of a normed space X is called an *order cone* iff the following are true:

- X_+ is closed, convex, nonempty and $X_+ \neq \{0\}$;
- If $u \in X_+$ and $\alpha \geq 0$, then $\alpha u \in X_+$;
- If $u \in X_+$ and $-u \in X_+$, then $u = 0$.

A Banach space (normed space) $(X, \|\cdot\|)$ endowed with a partial ordering \leq induced by an order cone X_+ by $x \leq y$ iff $y - x \in X_+$ is called an *ordered Banach space* (*ordered normed space*).

Let X be an ordered normed space with the ordering \leq and the order cone X_+ , P be a subset of X , a be an element of P and $\{a_n\}$ be a sequence in P . We say that:

- a is a *sup-center* (respectively, *inf-center*) of P if $\sup\{a, y\}$ (respectively, $\inf\{a, y\}$) exists and belongs to P for all $y \in P$;
- P is (*weakly*) *sequentially order compact* if increasing and decreasing sequences of P have (weak) limits in P ;
- P has the *fixed point property* if each increasing mapping $G : P \rightarrow P$ has a fixed point;
- X is a *Banach semilattice* if X is an ordered Banach space satisfying $\|x^+\| \leq \|x\|$ and $\|x^-\| \leq \|x\|$ for all $x \in X$ where $x^+ = \sup\{0, x\}$ and $x^- = \inf\{0, x\}$;
- $\{a_n\}$ is *increasing* (resp., *decreasing*) if $a_n \leq a_m$ (resp., $a_n \geq a_m$) whenever $n \leq m$;
- $\{a_n\}$ is *bounded* if there exists a constant $C > 0$ such that $\|a_n\| \leq C$ for all $n \in \mathbb{N}$;
- X_+ is a (*weakly*) *fully regular order cone* if each bounded and increasing sequence of X_+ is (weakly) convergent.

We also denote

$$B'_X(b, R) = \{x \in X \mid \|x - b\| \leq R\} \quad \text{and} \quad [b_1, b_2]_X = \{x \in X \mid b_1 \leq x \leq b_2\},$$

where $b, b_1, b_2 \in X$.

The following fixed point theorem is proved in [6]:

LEMMA 2.1. *Let P be a weakly sequentially order compact subset of an ordered normed space X having a sup-center or an inf-center. Then P has the fixed point property.*

We now introduce some concrete types of sets having the fixed point property beside closed balls as in [6]. Lemma 2.2 below slightly generalizes [6, Corollary 2].

LEMMA 2.2. *Let X be a Banach semilattice which is reflexive or has a weakly fully regular order cone X_+ and Y be a closed subspace of X . Moreover, assume that for every $x \in Y$, we have $x^+ \in Y$. Then any nonempty set P of the following types possesses the fixed point property:*

- (i) $P = B'_Y(a, R)$, where $a \in Y$ and $R > 0$,
- (ii) $P = B'_Y(a, R) \cap \{x \in X \mid b \leq x\}$, where $a \in Y, b \in X$ and $R > 0$,
- (iii) $P = B'_Y(a, R) \cap [b_1, b_2]_X$, where $a \in Y, b_1, b_2 \in X, a \leq b_2$ and $R > 0$.

Proof. Note that if X is reflexive then its order cone X_+ is weakly fully regular. Moreover, every set P of types (i), (ii) or (iii) is a weakly closed subset of X because it is convex and closed. Also, it is clear that P is bounded.

We show that P is weakly sequentially order compact. In order to do this, let $\{x_n\} \subset P$ be an increasing sequence. Put $y_n = x_n - x_0, n = 1, 2, \dots$, then $\{y_n\} \subset X_+$ is an bounded and increasing sequence, which weakly converges to some y because X_+ is weakly fully regular.

Therefore, $\{x_n\}$ is also weakly convergent and its weak limit is $y + x_0$, which belongs to P since P is a weakly closed subset of X . The case that $\{x_n\}$ is decreasing can be done similarly.

We prove that a is a sup-center of P . Let $y \in P$ then $\sup\{a, y\} = (y - a)^+ + a$.

In case (i), from the Banach semilattice property, we get

$$\|\sup\{a, y\} - a\| = \|(y - a)^+\| \leq \|y - a\| \leq R.$$

Moreover, $\sup\{a, y\} \in Y$ since $a, y \in Y$. Therefore, $\sup\{a, y\} \in B'_Y(a, R) = P$.

In case (ii), $y \in P \subset B'_Y(a, R)$, hence $\sup\{a, y\} \in B'_Y(a, R)$ from the above proof. Moreover, $b \leq y$ implies $b \leq \sup\{a, y\}$. Therefore,

$$\sup\{a, y\} \in B'_Y(a, R) \cap \{x \in X \mid b \leq x\} = P.$$

In case (iii), $y \in P \subset B'_Y(a, R)$, hence $\sup\{a, y\} \in B'_Y(a, R)$ from the above proof. Moreover, $b_1 \leq y \leq b_2$, and $a \leq b_2$, so $b_1 \leq \sup\{a, y\} \leq b_2$. Therefore,

$$\sup\{a, y\} \in B'_Y(a, R) \cap [b_1, b_2]_X = P. \quad \square$$

The following remark is useful in applications:

REMARK 2.1. For $p > 1$, Sobolev spaces $W^{1,p}(\Omega)$ are Banach semilattices with the usual ordering and reflexive. Hence for $X = W^{1,p}(\Omega)$ and $Y = W_0^{1,p}(\Omega)$, all sets of types (i), (ii) or (iii) mentioned in Lemma 2.2 have the fixed point property.

3. Proof of Theorem 1.1 and Theorem 1.2

For $u \in W^{1,p}(\Omega)$ and $\varphi \in W_0^{1,p}(\Omega)$, we define

$$\langle Au, \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi,$$

$$\langle Fu, \varphi \rangle = \int_{\Omega} f(x, u) \varphi,$$

$$\langle Qu, \varphi \rangle = \int_{\Omega} q(x, u) \varphi.$$

We call

- $u \in W_0^{1,p}(\Omega)$ a weak solution of (1.1) if $\langle Au, \varphi \rangle = \langle Fu, \varphi \rangle$ for all $\varphi \in W_0^{1,p}(\Omega)$,

- $u \in W^{1,p}(\Omega)$ a subsolution of (1.1) if $u|_{\partial\Omega} \leq 0$ and $\langle Au, \varphi \rangle \leq \langle Fu, \varphi \rangle$ for all $\varphi \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$,

- $u \in W^{1,p}(\Omega)$ a supersolution of (1.1) if $u|_{\partial\Omega} \geq 0$ and $\langle Au, \varphi \rangle \geq \langle Fu, \varphi \rangle$ for all $\varphi \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$.

Due to (A1)-(A3) and (Q1)-(Q3), the operator $A + Q : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is bijective and if $(A + Q)u \leq (A + Q)v$ then $u \leq v$. The reader may find detailed proofs for these facts in the appendix.

Now, consider the operator $G : W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)$ defined by

$$G = (A + Q)^{-1} \circ (F + Q).$$

Then $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1) if and only if u is a fixed point of G .

Moreover, from (FQ), the operator $F + Q$ is increasing. Therefore G is increasing, too.

From now on, $B'(u, R)$ denotes the closed ball in $W_0^{1,p}(\Omega)$ of center $u \in W^{1,p}(\Omega)$ and radius $R > 0$.

Proof of Theorem 1.1. From the above arguments, all we have to do is to construct P which has the fixed point property such that if $v \in P$ then $G(v) \in P$.

We look for P of type $P = B'(0, R)$. This type of set has the fixed point property by Lemma 2.2 and Remark 2.1.

Set $u = G(v)$, then $(A + Q)u = (F + Q)v$.

Choose $\xi' = 0$ in (A3), $s' = 0$ in (Q3) and using (A2), (Q2), we have

$$\begin{aligned} \langle (A + Q)u, u \rangle &= \int_{\Omega} a(x, \nabla u) \cdot \nabla u + \int_{\Omega} q(x, u)u \\ &\geq \int_{\Omega} (C_3 |\nabla u|^p - |k_0(x)| \cdot |\nabla u|) - \int_{\Omega} |k_2(x)| \cdot |u| \\ &\geq C_3 \|u\|^p - (|k_0|_{p'} + K_2 |k_2|_{p_2'}) \|u\|. \end{aligned} \tag{3.1}$$

From (3.1) and growth conditions (F2), (Q2) on f and g , we have

$$\begin{aligned} C_3 \|u\|^p - (|k_0|_{p'} + K_2 |k_2|_{p_2'}) \|u\| &\leq \langle (A + Q)u, u \rangle = \langle (F + Q)v, u \rangle \\ &\leq |k_1|_{p_1'} |u|_{p_1} + C_1 |v|_{p_1'}^{p_1-1} |u|_{p_1} + |k_2|_{p_2'} |u|_{p_2} + C_2 |v|_{p_2'}^{p_2-1} |u|_{p_2} \\ &\leq (K_1 |k_1|_{p_1'} + K_2 |k_2|_{p_2'} + C_1 K_1 |v|_{p_1'}^{p_1-1} + C_2 K_2 |v|_{p_2'}^{p_2-1}) \|u\| \\ &\leq (K_1 |k_1|_{p_1'} + K_2 |k_2|_{p_2'} + C_1 K_1^{p_1} \|v\|^{p_1-1} + C_2 K_2^{p_2} \|v\|^{p_2-1}) \|u\|. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} C_3 \|u\|^{p-1} &\leq |k_0|_{p'} + K_1 |k_1|_{p_1'} + 2K_2 |k_2|_{p_2'} + C_1 K_1^{p_1} \|v\|^{p_1-1} + C_2 K_2^{p_2} \|v\|^{p_2-1} \\ &= M + C_1 K_1^{p_1} \|v\|^{p_1-1} + C_2 K_2^{p_2} \|v\|^{p_2-1}, \end{aligned}$$

where $M = |k_0|_{p'} + K_1|k_1|_{p'_1} + 2K_2|k_2|_{p'_2}$.

We must find $R > 0$ such that if $\|v\| \leq R$ then $\|u\| \leq R$. This property will be satisfied if $M + C_1K_1^{p_1}R^{p_1-1} + C_2K_2^{p_2}R^{p_2-1} \leq C_3R^{p-1}$ or $p(R) \geq 0$, where

$$p(t) = C_3t^{p-1} - (C_1K_1^{p_1}t^{p_1-1} + C_2K_2^{p_2}t^{p_2-1} + M).$$

- (i) If $1 < p_1, p_2 < p$ then $p(R) \geq 0$ for R large enough.
- (ii) If $p_1 = p_2 = p$ and $C_3 - C_1K_1^{p_1} - C_2K_2^{p_2} > 0$ then $\lim_{t \rightarrow +\infty} p(t) = +\infty$. Thus, $p(R) \geq 0$ for R large enough.
- (iii) If $p_1 = p, 1 < p_2 < p$ and $C_3 - C_1K_1^{p_1} > 0$ then $\lim_{t \rightarrow +\infty} p(t) = +\infty$. Thus, $p(R) \geq 0$ for R large enough.
- (iv) This case is similar to (iii).
- (v) If $p_1, p_2 > p$ then

$$\bar{p}(t) = C_3t^{p-1} - (C_1K_1^{p_1}t^{p_1-1} + C_2K_2^{p_2}t^{p_2-1}) > 0$$

for all $t \in (0, t_0]$ where $t_0 > 0$ is sufficiently small. Note that $\lim_{t \rightarrow 0^+} \bar{p}(t) = 0$.

If we fix $t \in (0, t_0]$ and choose k_0, k_1, k_2 such that

$$|k_0|_{p'} + |k_1|_{p'_1} + |k_2|_{p'_2} < (\max\{1, K_1, 2K_2\})^{-1} \bar{p}(t),$$

then $M < \bar{p}(t)$ and $p(t) = \bar{p}(t) - M > 0$. It implies that for all ε small enough, we can choose $R_\varepsilon > 0$ such that if $|k_0|_{p'} + |k_1|_{p'_1} + |k_2|_{p'_2} < \varepsilon$, then $p(R_\varepsilon) > 0$. Moreover, $\lim_{\varepsilon \rightarrow 0} R_\varepsilon = 0$.

The above arguments imply the existence of $R > 0$ such that if $v \in B'(0, R)$ then $G(v) = u \in B'(0, R)$.

Next, suppose that \underline{u} is a subsolution of (1.1). We consider

$$P = B'(0, R) \cap \{w \in W^{1,p}(\Omega) \mid \underline{u} \leq w\}$$

with R obtained above. Note that

$$\underline{u}^+ \in W_0^{1,p}(\Omega) \cap \{w \in W^{1,p}(\Omega) \mid \underline{u} \leq w\}.$$

In the cases of (i)-(iv), we may choose R large enough if necessary. In case (v), we may choose \underline{u} such that $\|\underline{u}^+\|$ is small enough. Therefore, P is nonempty and P has the fixed point property by Lemma 2.2.

Assume that $v \in P$, we prove that $G(v) \in P$.

Since $v \in P \subset B'(0, R)$, we have $G(v) \in B'(0, R)$ from the above proof.

On the other hand, because \underline{u} is a subsolution of (1.1) and $\underline{u} \leq v$, we have

$$(A + Q)(\underline{u}) \leq (F + Q)(\underline{u}) \leq (F + Q)(v).$$

Thus, $\underline{u} \leq (A + Q)^{-1} \circ (F + Q)v = G(v)$. Therefore, $G(v) \in P$.

Theorem 1.1 has been proved completely. \square

Proof of Theorem 1.2. As before, we look for P of type $P = B'(u_0, R) \cap [\underline{u}, \bar{u}]$ such that if $v \in P$ then $G(v) \in P$. This type of set has the fixed point property by Lemma 2.2 and Remark 2.1.

Set $u = G(v)$, then $(A + Q)u = (F + Q)v$. From (3.2) and $v \in [\underline{u}, \bar{u}]$ we have

$$\begin{aligned} C_3 \|u\|^{p-1} &\leq M + C_1 K_1 |v|_{p_1}^{p_1-1} + C_2 K_2 |v|_{p_2}^{p_2-1} \\ &\leq M + C_1 K_1 (|\underline{u}|_{p_1} + |\bar{u}|_{p_1})^{p_1-1} + C_2 K_2 (|\underline{u}|_{p_2} + |\bar{u}|_{p_2})^{p_2-1}, \end{aligned}$$

which means that $\|u\|$ is bounded by a positive number R_0 independent of $v \in [\underline{u}, \bar{u}]$. Choose $u_0 \in W_0^{1,p}(\Omega) \cap [\underline{u}, \bar{u}]$, $R = R_0 + \|u_0\|$ and set $P = B'(u_0, R) \cap [\underline{u}, \bar{u}]$.

If $v \in P$ then $v \in [\underline{u}, \bar{u}]$. Thus, $\|u\| \leq R_0$ by the above proof. Therefore,

$$\|u - u_0\| \leq \|u\| + \|u_0\| \leq R_0 + \|u_0\| = R, \text{ i.e. } u \in B'(u_0, R).$$

On the other hand, from definition of \underline{u} and monotonicity of $F + Q$, we have

$$(A + Q)\underline{u} \leq (F + Q)\underline{u} \leq (F + Q)v = (A + Q)u.$$

Thus, $\underline{u} \leq u$.

Similarly, $(A + Q)u = (F + Q)v \leq (F + Q)\bar{u} \leq (A + Q)\bar{u}$. Thus, $u \leq \bar{u}$.

Consequently, $G(v) = u \in P$, as desired. \square

4. Quasilinear elliptic problems with critical exponents and discontinuous nonlinearities

In this section, we study problem (1.2). Suppose that $N \geq 3$ and $2 \leq p < N$, hence $p^* < +\infty$. Let $C > 0$ be a positive constant, we impose the following conditions on g :

- (G1) $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is sup-measurable;
- (G2) $\liminf_{s \rightarrow 0^+} (g(x, s)/s^{p-1}) = +\infty$, uniformly in x ;
- (G3) $|g(x, s)| \leq Cs^{p^*-1}$, for a.e $x \in \Omega$ and for all $s > 1$;
- (G4) $\bar{g} \in L^{p^*/(p^*-1)}(\Omega)$, where $\bar{g}(x) = \sup_{0 \leq s \leq 1} |g(x, s)|$ for a.e $x \in \Omega$;
- (H1) $h : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Carathéodory function;
- (H2) $|h(x, s)| \leq Cs^{p^*-1}$, for a.e $x \in \Omega$ and for all $s \in \mathbb{R}_+$;
- (H3) $(h(x, s) - h(x, s'))(s - s') \geq 0$ for a.e $x \in \Omega$ and for all $s, s' \in \mathbb{R}_+$;
- (GH) $s \mapsto g(x, s) + h(x, s)$ is increasing for a.e $x \in \Omega$ and $s \in \mathbb{R}_+$.

THEOREM 4.1. *Assume that the condition (G1)-(G4), (H1)-(H3) and (GH) are satisfied. Then there exists a positive number δ_0 such that (1.2) has a positive solution u_δ for all $\delta \in (0, \delta_0)$. Moreover, $\|u_\delta\| \rightarrow 0$ as $\delta \rightarrow 0^+$.*

Proof. We will apply Theorem 1.1 with $a(x, \xi) = |\xi|^{p-2}\xi$,

$$f(x, s) = \begin{cases} |s|^{p^*-2}s + \delta g(x, s) & \text{if } s \geq 0, \\ \delta g(x, 0) & \text{if } s < 0, \end{cases}$$

and

$$q(x, s) = \begin{cases} \delta h(x, s) & \text{if } s \geq 0, \\ \delta h(x, 0) & \text{if } s < 0. \end{cases}$$

For $\delta > 0$ small, by (G3) and (G4) we have $|f(x, s)| \leq C'|s|^{p^*-1} + \delta \bar{g}(x)$ for some positive constant C' . Therefore, (A1)-(A3), (F1)-(F2), (Q1)-(Q3) and (FQ) are satisfied with $k_0 = k_2 = 0$, $k_1 = \delta \bar{g}$.

From (v) in Theorem 1.1, there exists $\delta_0 > 0$ such that problem (1.1) with a, f, q as above has a weak solution for all $\delta \in (0, \delta_0)$ and this solution converges to 0 as $\delta \rightarrow 0$. In order to ensure that this solution is positive, and therefore, also a solution of (1.2), we have to show that (1.2) has a sequence of positive subsolutions converging to 0 in $W_0^{1,p}(\Omega)$.

Let λ_1 be the first eigenvalue and $\varphi_1 > 0$ be a corresponding eigenfunction of the eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$ in Ω with zero Dirichlet boundary condition.

Setting $u_\varepsilon = \varepsilon \varphi_1$. We look for $\varepsilon > 0$ such that $-\Delta_p u_\varepsilon \leq u_\varepsilon^{p^*-1} + \delta g(x, u_\varepsilon)$, or equivalently, $\lambda_1(\varepsilon \varphi_1)^{p-1} \leq u_\varepsilon^{p^*-1} + \delta g(x, \varepsilon \varphi_1)$.

By (G2), there exists $s_0 > 0$ such that $g(x, s) \geq (\lambda_1/\delta)s^{p-1}$ for all $s \in (0, s_0)$ and a.e $x \in \Omega$. Thus, u_ε is a subsolution of problem (1.2) if $\varepsilon \in (0, s_0/|\varphi_1|_\infty)$.

Now the existence of a positive solution u_δ to problem (1.2) and the convergence of $\{u_\delta\}$ follow immediately from Theorem 1.1. \square

REMARK 4.1. Two trivial examples of g satisfying Theorem 4.1 are $g(x, s) = |s|^{q-2}s$ with $1 < q < p$ and

$$g(x, s) = \begin{cases} 0 & \text{if } s < a, \\ 1/2 & \text{if } s = a, \\ 1 & \text{if } s > a, \end{cases}$$

where $a \leq 0$.

We choose $h \equiv 0$ in both examples above.

5. Appendix

For the reader's convenience, in this appendix we prove the fundamental facts in theory of monotone operators saying that the operator $A + Q$ is bijective and $(A + Q)^{-1}$ is increasing. These facts are used in the proof of Theorem 1.1.

LEMMA 5.1. *Assume that conditions (A1)-(A3) and (Q1)-(Q3) are satisfied, then operator $A + Q : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is bijective and if $(A + Q)u \leq (A + Q)v$ then $u \leq v$.*

Proof. Our proof has 4 steps.

Step 1. ($A + Q$ is monotone) Let $u, v \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} & \langle (A + Q)u - (A + Q)v, u - v \rangle \\ &= \int_{\Omega} [a(x, \nabla u) - a(x, \nabla v)](\nabla u - \nabla v) dx + \int_{\Omega} [q(x, u) - q(x, v)](u - v) dx \\ &\geq \int_{\Omega} C_3 |\nabla u - \nabla v|^p dx \\ &= C_3 \|u - v\|^p, \end{aligned}$$

by (A3) and (Q3). Therefore, $A + Q$ is monotone.

Step 2. ($A + Q$ is hemicontinuous) Let $u, v, w \in W_0^{1,p}(\Omega)$.

• We prove that: $\lim_{t \rightarrow 0} \langle A(u + tv), w \rangle = \langle Au, w \rangle$.

Setting $h_t(x) = a(x, \nabla u(x) + t \nabla v(x)) \nabla w(x)$, $t \in [-1, 1]$.

By Young's inequality, we have

$$|h_t(x)| \leq \frac{1}{p'} |a(x, \nabla u(x) + t \nabla v(x))|^{p'} + \frac{1}{p} |\nabla w(x)|^p.$$

On the other hand, by (A2)

$$\begin{aligned} |a(x, \nabla u(x) + t \nabla v(x))|^{p'} &\leq (|k_0(x)| + C_0 |\nabla u(x) + t \nabla v(x)|^{p-1})^{p'} \\ &\leq 2^{p'} |k_0(x)|^{p'} + 2^{p'} C_0^{p'} |\nabla u(x) + t \nabla v(x)|^p \\ &\leq 2^{p'} |k_0(x)|^{p'} + 2^{p+p'} C_0^{p'} |\nabla u(x)|^p + 2^{p+p'} C_0^{p'} |\nabla v(x)|^p. \end{aligned}$$

Therefore,

$$|h_t(x)| \leq h(x),$$

with

$$h(x) = \frac{2^{p'}}{p'} |k_0(x)|^{p'} + \frac{2^{p+p'} C_0^{p'}}{p'} |\nabla u(x)|^p + \frac{2^{p+p'} C_0^{p'}}{p'} |\nabla v(x)|^p + \frac{1}{p} |\nabla w(x)|^p.$$

From the hypothesis, we have $h \in L^1(\Omega)$.

We have $|h_t(x)| \leq h(x)$, $\forall t \in [-1, 1]$ and $\lim_{t \rightarrow 0} h_t(x) = a(x, \nabla u(x)) \nabla w(x)$ for a.e $x \in \Omega$. By Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{t \rightarrow 0} \langle A(u + tv), w \rangle = \langle Au, w \rangle.$$

• Similarly, we have: $\lim_{t \rightarrow 0} \langle Q(u + tv), w \rangle = \langle Qu, w \rangle$.

• From the above proof,

$\lim_{t \rightarrow 0} \langle (A + Q)(u + tv), w \rangle = \langle (A + Q)u, w \rangle$, i.e $A + Q$ is hemicontinuous.

Step 3. ($A + Q$ is coercive) From (3.1), for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ we have

$$\frac{\langle (A + Q)u, u \rangle}{\|u\|} \geq C_3 \|u\|^{p-1} - (|k_0|_{p'} + K_2 |k_2|_{p_2'}),$$

i.e $A + Q$ is coercive.

Applying Browder’s theorem in [4], from steps 1, 2 and 3 we conclude that $A + Q$ is surjective.

Step 4. (Assuming that $(A + Q)u \leq (A + Q)v$, we prove that $u \leq v$) By the hypothesis, we have

$$\langle (A + Q)v - (A + Q)u, \varphi \rangle \geq 0, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega).$$

Choose $\varphi = (u - v)_+$ as a test function and put $\Omega_+ = \{x \in \Omega \mid v(x) \leq u(x)\}$, we have

$$\begin{aligned} 0 &\leq \int_{\Omega} [a(x, \nabla v) - a(x, \nabla u)] \nabla (u - v)_+ dx + \int_{\Omega} [q(x, v) - q(x, u)] (u - v)_+ dx \\ &= \int_{\Omega_+} [a(x, \nabla v) - a(x, \nabla u)] \nabla (u - v) dx + \int_{\Omega_+} [q(x, v) - q(x, u)] (u - v) dx. \end{aligned}$$

Thus,

$$\int_{\Omega_+} [a(x, \nabla v) - a(x, \nabla u)] (\nabla v - \nabla u) dx + \int_{\Omega_+} [q(x, v) - q(x, u)] (v - u) dx \leq 0.$$

On the other hand, by (Q3) we have:

$$\int_{\Omega_+} [q(x, v) - q(x, u)] (v - u) dx \geq 0,$$

and by (A3)

$$\begin{aligned} \int_{\Omega_+} [a(x, \nabla v) - a(x, \nabla u)] (\nabla v - \nabla u) dx &\geq C_3 \int_{\Omega_+} |\nabla v - \nabla u|^p dx \\ &= C_3 \int_{\Omega} |\nabla (u - v)_+|^p dx \\ &= C_3 \|(u - v)_+\|^p. \end{aligned}$$

Thus, $(u - v)_+ = 0$, i.e $u \leq v$.

Next, suppose that $(A + Q)u = (A + Q)v$, the above proof implies that $u \leq v$ and $v \leq u$. So $A + Q$ is injective. \square

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