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CONCENTRATION COMPACTNESS PRINCIPLES FOR THE SYSTEMS OF CRITICAL ELLIPTIC EQUATIONS

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Abstract. In this paper, some important variants of the concentration compactness principle are established. By the variants, some kinds of the elliptic systems can be investigated and the existence of nontrivial solutions to the systems can be verified by the variational methods.

1. Introduction

In this paper, we are concerned with the variants of the concentration compactness principle ([17, 18]), which are important in the study of some elliptic systems.

In recent years, the elliptic problems involving the critical Sobolev or Hardy-Sobolev exponents have been studied extensively (e.g. [1], [4], [5], [7], [9], [10], [11], [15], [20], [21], [23] and the references therein), where the concentration compactness principles have played a key role. The systems of elliptic equations involving critical exponents have been also studied (e.g. [2], [3], [6], [12], [14], [19] and the references therein). However, the variants of concentration compactness principle related to critical elliptic systems can not be found and some difficulties have appeared in the investigations of these systems. In this paper, on the basis of the ideas by Lions ([17, 18]), we verify some kinds of concentration compactness principle for elliptic systems.

To continue, the following assumption is needed:

$$(\mathscr{H}_1)\ \begin{cases} N \geqslant 3,\ 1<p<N,\ \eta,\lambda,\sigma \geqslant 0,\ \eta+\lambda+\sigma>0, \\ \alpha,\beta>1,\ \alpha+\beta=p^* := \frac{Np}{N-p}. \end{cases}$$

Let $D^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x)^{1/p}$. Under the assumption $(\mathscr{H}_1)$, by the Young and Sobolev inequalities, the following best constant is well defined on $\mathscr{D} := (D^{1,p}(\mathbb{R}^N)\setminus\{0\})^2$ (e.g. [14], [16]):

$$S(\eta,\lambda,\sigma) := \inf_{(u,v)\in\mathscr{D}} \frac{\displaystyle\int_{\mathbb{R}^N}\left(|\nabla u|^p+|\nabla v|^p\right)\mathrm{d}x}{\left(\displaystyle\int_{\mathbb{R}^N}(\eta|u|^{p^*}+\lambda|v|^{p^*}+\sigma|u|^\alpha|v|^\beta)\mathrm{d}x\right)^{\frac{p}{p^*}}}. \tag{1.1}$$

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The main results of this paper are summarized in the following theorem. To the best of our knowledge, the conclusions are new when $\eta,\lambda,\sigma>0$.

THEOREM 1. *Suppose that* $(\mathscr{H}_1)$ *holds. Let* $\{(u_n,v_n)\}$ *be a bounded sequence in* $(D^{1,p}(\mathbb{R}^N))^2$ *such that* $\{(u_n,v_n)\}\rightharpoonup(u,v)$ *weakly in* $(D^{1,p}(\mathbb{R}^N))^2$, $\{|\nabla u_n|^p+|\nabla v_n|^p\}$ *converges weakly to* μ *and* $\{\eta|u_n|^{p^*}+\lambda|v_n|^{p^*}+\sigma|u_n|^\alpha|v_n|^\beta\}$ *converges tightly to* ν, *where* μ *and* ν *are nonnegative bounded measures on* $\mathbb{R}^N$. *Denote by* δ_x *the Dirac mass at* x. *Then:*

(i) *there exist an at most countable set* J *and two families* $\{x_j\}_{j\in J}\subset\mathbb{R}^N$ *and* $\{\nu_j\}_{j\in J}\subset[0,+\infty)$ *such that*

$$\nu=\eta|u|^{p^*}+\lambda|v|^{p^*}+\sigma|u|^\alpha|v|^\beta+\sum_{j\in J}\nu_j\delta_{x_j};$$

(ii) *there exists* $\{\mu_j\}_{j\in J}\subset[0,+\infty)$ *such that*

$$\mu\geqslant|\nabla u|^p+|\nabla v|^p+\sum_{j\in J}\mu_j\delta_{x_j},$$

satisfying

$$(\nu_j)^{\frac{p}{p^*}}\leqslant\mu_j/S(\eta,\lambda,\sigma),\ \ \forall j\in J.$$

REMARK 1. Suppose that $\{u_n\}\subset L^1(\mathbb{R}^N)$. Then $\{u_n\}$ is called a tight sequence, if for any $\varepsilon>0$, there exists $R>0$, such that

$$\int_{\mathbb{R}^N\setminus B_R(0)}|u_n(x)|\mathrm{d}x<\varepsilon,\quad\forall n\in\mathbb{N}.$$

The convergence of $\{u_n\}\subset L^1(\mathbb{R}^N)$ is called converging tightly, if $\{u_n\}$ is a tight sequence. On the other hand, since μ is a nonnegative bounded measure on $\mathbb{R}^N$, from Theorem 1 (ii) it follows that

$$\sum_{j\in J}(\nu_j)^{\frac{p}{p^*}}\leqslant(S(\eta,\lambda,\sigma))^{-1}\sum_{j\in J}\mu_j\leqslant(S(\eta,\lambda,\sigma))^{-1}\int_{\mathbb{R}^N}d\mu<\infty.$$

This paper is organized as follows: Theorem 1 is proved in Section 2, some variants and applications of Theorem 1 are given in Section 3.

2. Proof of Theorem 1

The proof of Theorem 1 follows the idea similar to that of [17] and some preliminary results are needed.

LEMMA 1. (see [17], Lemma 1.2) *Let* μ,ν *be two nonnegative bounded measures on* $\mathbb{R}^N$ *satisfying for some constant* $C_0\geqslant0$,

$$\Big(\int_{\mathbb{R}^N}|\varphi|^{p^*}\mathrm{d}\nu\Big)^{\frac{1}{p^*}}\leqslant C_0\Big(\int_{\mathbb{R}^N}|\varphi|^p\mathrm{d}\mu\Big)^{\frac{1}{p}},\quad\forall\varphi\in C_0^\infty(\mathbb{R}^N).\tag{2.1}$$

Then there exist an at most countable set J and two families

$$\{x_j\}_{j\in J} \subset \mathbb{R}^N \quad \textit{and} \quad \{\nu_j\}_{j\in J} \subset [0,+\infty)$$

such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j} \quad \textit{and} \quad \mu \geqslant C_0^{-p} \sum_{j\in J} (\nu_j)^{\frac{p}{p^*}} \delta_{x_j}. \tag{2.2}$$

Thus, in particular,

$$\sum_{j\in J} (\nu_j)^{\frac{p}{p^*}} < \infty. \tag{2.3}$$

If in addition: $\nu(\mathbb{R}^N)^{\frac{1}{p^*}} \geqslant C_0 \mu(\mathbb{R}^N)^{\frac{1}{p}}$, J reduces to a single point and $\nu = \gamma\delta_{x_0} = \gamma^{-\frac{p}{p^*}} C_0^p \mu$ for some $x_0 \in \mathbb{R}^N$ and $\gamma \geqslant 0$.

LEMMA 2. *Assume that $u,v \in D^{1,p}(\mathbb{R}^N)$, J is an at most countable set, $\{x_j\}_{j\in J}$ is a set of distinct points in $\mathbb{R}^N$ and $\{\nu_j\}_{j\in J}$ is a set of nonnegative real numbers such that $\sum_{j\in J}(\nu_j)^{\frac{p}{p^*}} < \infty$. Then the measure $\nu = \eta|u|^{p^*} + \lambda|v|^{p^*} + \sigma|u|^{\alpha}|v|^{\beta} + \sum_{j\in J} \nu_j \delta_{x_j}$ is the tight limit of a sequence $\{\eta|u_n|^{p^*} + \lambda|v_n|^{p^*} + \sigma|u_n|^{\alpha}|v_n|^{\beta}\}$, where $\{(u_n,v_n)\}$ converges weakly in $(D^{1,p}(\mathbb{R}^N))^2$ to (u,v).*

Proof. Take $\varphi^{(1)}, \varphi^{(2)} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (\eta|\varphi^{(1)}|^{p^*} + \lambda|\varphi^{(2)}|^{p^*} + \sigma|\varphi^{(1)}|^{\alpha}|\varphi^{(2)}|^{\beta}) \mathrm{d}x = 1.$$

Otherwise, set

$$\int_{\mathbb{R}^N} (\eta|\varphi^{(1)}|^{p^*} + \lambda|\varphi^{(2)}|^{p^*} + \sigma|\varphi^{(1)}|^{\alpha}|\varphi^{(2)}|^{\beta}) \mathrm{d}x =: \mathscr{C} > 0.$$

Replacing $\varphi^{(i)}$ by $\bar{\varphi}^{(i)} := \varphi^{(i)}/\mathscr{C}^{1/p^*}$, $i = 1,2$, we have $\bar{\varphi}^{(1)}, \bar{\varphi}^{(2)} \in C_0^\infty(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} (\eta|\bar{\varphi}^{(1)}|^{p^*} + \lambda|\bar{\varphi}^{(2)}|^{p^*} + \sigma|\bar{\varphi}^{(1)}|^{\alpha}|\bar{\varphi}^{(2)}|^{\beta}) \mathrm{d}x = 1.$$

Then for all $x_0 \in \mathbb{R}^N, n \in \mathbb{N}$, $\varphi_n^{(i)}(x - x_0) := n^{\frac{p-N}{p}} \varphi^{(i)}(\frac{x-x_0}{n})$ satisfies

$$\int_{\mathbb{R}^N} |\nabla\varphi_n^{(i)}|^p \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla\varphi^{(i)}|^p \mathrm{d}x, \quad i = 1,2,$$

$$\int_{\mathbb{R}^N} \left(\eta|\varphi_n^{(1)}|^{p^*} + \lambda|\varphi_n^{(2)}|^{p^*} + \sigma|\varphi_n^{(1)}|^{\alpha}|\varphi_n^{(2)}|^{\beta}\right) \mathrm{d}x = 1,$$

$$\varphi_n^{(i)} \underset{n}{\rightharpoonup} 0 \ \text{ weakly in } \ D^{1,p}(\mathbb{R}^N), \quad i = 1,2.$$

Furthermore, for any $x \neq x_0, i = 1,2$, $|\varphi_n^{(i)}| \to 0$ as $n \to \infty$. Then

$$\eta|\varphi_n^{(1)}|^{p^*} + \lambda|\varphi_n^{(2)}|^{p^*} + \sigma|\varphi_n^{(1)}|^{\alpha}|\varphi_n^{(2)}|^{\beta} \underset{n}{\longrightarrow} \delta_{x_0}.$$

For any finite $J' \subset J$, define

$$\psi_n^{(i)}(x-x_j) := \sum_{j\in J'} (\nu_j)^{\frac{1}{p^*}} \varphi_n^{(i)}(x-x_j),$$

where $\operatorname{Supp}\varphi_n^{(i)}(x-x_j)$ are disjoint for $j \in J'$. Then we have

$$\int_{\mathbb{R}^N} |\nabla \psi_n^{(i)}|^p \mathrm{d}x = \sum_{j\in J'} (\nu_j)^{\frac{p}{p^*}} \int_{\mathbb{R}^N} |\nabla \varphi^{(i)}|^p \mathrm{d}x \leqslant \sum_{j\in J} (\nu_j)^{\frac{p}{p^*}} \int_{\mathbb{R}^N} |\nabla \varphi^{(i)}|^p \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \Big(\eta|\psi_n^{(1)}|^{p^*} + \lambda|\psi_n^{(2)}|^{p^*} + \sigma|\psi_n^{(1)}|^{\alpha}|\psi_n^{(2)}|^{\beta}\Big)\mathrm{d}x = \sum_{j\in J'} \nu_j,$$

$$\eta|\psi_n^{(1)}|^{p^*} + \lambda|\psi_n^{(2)}|^{p^*} + \sigma|\psi_n^{(1)}|^{\alpha}|\psi_n^{(2)}|^{\beta} \underset{n}{\longrightarrow} \sum_{j\in J'} \nu_j \delta_{x_j},$$

$$\psi_n^{(i)} \underset{n}{\rightharpoonup} 0 \text{ weakly in } D^{1,p}(\mathbb{R}^N), \quad i = 1,2.$$

Increasing J' to J and by a diagonal procedure we obtain a sequence $\bar{\psi}_n$ such that

$$\int_{\mathbb{R}^N} |\nabla \bar{\psi}_n^{(i)}(x)|^p \mathrm{d}x \leqslant \sum_{j\in J} (\nu_j)^{\frac{p}{p^*}} \int_{\mathbb{R}^N} |\nabla \varphi^{(i)}(x)|^p \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} (\eta|\bar{\psi}_n^{(1)}|^{p^*} + \lambda|\bar{\psi}_n^{(2)}|^{p^*} + \sigma|\bar{\psi}_n^{(1)}|^{\alpha}|\bar{\psi}_n^{(2)}|^{\beta})\mathrm{d}x \underset{n}{\longrightarrow} \sum_{j\in J} \nu_j,$$

$$\eta|\bar{\psi}_n^{(1)}|^{p^*} + \lambda|\bar{\psi}_n^{(2)}|^{p^*} + \sigma|\bar{\psi}_n^{(1)}|^{\alpha}|\bar{\psi}_n^{(2)}|^{\beta} \underset{n}{\longrightarrow} \sum_{j\in J} \nu_j \delta_{x_j} \text{ tightly},$$

$$\bar{\psi}_n^{(i)} \underset{n}{\rightharpoonup} 0 \text{ weakly in } D^{1,p}(\mathbb{R}^N), \quad i = 1,2.$$

Finally we set $u_n = u + \bar{\psi}_n^{(1)}$, $v_n = v + \bar{\psi}_n^{(2)}$. Then one can checks that $\{(u_n, v_n)\}$ has the required properties. Furthermore,

$$|\nabla u_n|^p + |\nabla v_n|^p \longrightarrow |\nabla u|^p + |\nabla v|^p + \Big(\int_{\mathbb{R}^N} (|\nabla \varphi^{(1)}|^p + |\nabla \varphi^{(2)}|^p)\mathrm{d}x\Big) \sum_{j\in J} (\nu_j)^{\frac{p}{p^*}} \delta_{x_j}.$$

The proof is thus complete. □

PROOF OF THEOREM 1. For convenience, we denote positive constants as C.

Case (i): $u = v = 0$.

For all $\varphi \in C_0^\infty(\mathbb{R}^N)$, from (1.1) it follows that

$$\Big(\int_{\mathbb{R}^N} |\varphi|^{p^*}(\eta|u_n|^{p^*} + \lambda|v_n|^{p^*} + \sigma|u_n|^{\alpha}|v_n|^{\beta})\mathrm{d}x\Big)^{\frac{1}{p^*}} \leqslant (S(\eta,\lambda,\sigma))^{-\frac{1}{p}} \Big(\int_{\mathbb{R}^N} (|\nabla(\varphi u_n)|^p + |\nabla(\varphi v_n)|^p)\mathrm{d}x\Big)^{\frac{1}{p}}. \quad (2.4)$$

Note that

$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^N}\eta|\varphi|^{p^*}(|u_n|^{p^*}+\lambda|v_n|^{p^*}+\sigma|u_n|^{\alpha}|v_n|^{\beta})\mathrm{d}x\Big)^{\frac{1}{p^*}}=\Big(\int_{\mathbb{R}^N}|\varphi|^{p^*}\mathrm{d}\nu\Big)^{\frac{1}{p^*}}$$

and

$$\Big|\Big(\int_{\mathbb{R}^N}(|\nabla(\varphi u_n)|^p+|\nabla(\varphi v_n)|^p)\mathrm{d}x\Big)^{\frac{1}{p}}-\Big(\int_{\mathbb{R}^N}|\varphi|^p(|\nabla u_n|^p+|\nabla v_n|^p)\mathrm{d}x\Big)^{\frac{1}{p}}\Big|$$
$$\leqslant C\Big(\int_{\mathbb{R}^N}|\nabla\varphi|^p(|u_n|^p+|v_n|^p)\mathrm{d}x\Big)^{\frac{1}{p}}. \quad (2.5)$$

Since $\varphi\in C_0^\infty(\mathbb{R}^N)$, by the Rellich theorem ([22]), we deduce that the right hand side of (2.5) goes to 0 as $n\to\infty$. Then from (2.4) and (2.5) it follows that

$$\Big(\int_{\mathbb{R}^N}|\varphi|^{p^*}\mathrm{d}\nu\Big)^{\frac{1}{p^*}}\leqslant(S(\eta,\lambda,\sigma))^{-\frac{1}{p}}\Big(\int_{\mathbb{R}^N}|\varphi|^p\mathrm{d}\mu\Big)^{\frac{1}{p}},\quad\forall\varphi\in C_0^\infty(\mathbb{R}^N), \quad (2.6)$$

which together with Lemma 1 implies the conclusions of Theorem 1.

Case (ii): $u\neq0$ or $v\neq0$.

Set $\bar{u}_n=u_n-u$, $\bar{v}_n=v_n-v$. From the Brezis–Lieb lemma ([7]) it follows that

$$\int_{\mathbb{R}^N}|\varphi|^{p^*}|u_n|^{p^*}\mathrm{d}x-\int_{\mathbb{R}^N}|\varphi|^{p^*}|\bar{u}_n|^{p^*}\mathrm{d}x\longrightarrow\int_{\mathbb{R}^N}|\varphi|^{p^*}|u|^{p^*}\mathrm{d}x,$$
$$\int_{\mathbb{R}^N}|\varphi|^{p^*}|v_n|^{p^*}\mathrm{d}x-\int_{\mathbb{R}^N}|\varphi|^{p^*}|\bar{v}_n|^{p^*}\mathrm{d}x\longrightarrow\int_{\mathbb{R}^N}|\varphi|^{p^*}|v|^{p^*}\mathrm{d}x.$$

Arguing as in [12] we have

$$\int_{\mathbb{R}^N}|\varphi|^{p^*}|u_n|^{\alpha}|v_n|^{\beta}\mathrm{d}x-\int_{\mathbb{R}^N}|\varphi|^{p^*}|\bar{u}_n|^{\alpha}|\bar{v}_n|^{\beta}\mathrm{d}x\longrightarrow\int_{\mathbb{R}^N}|\varphi|^{p^*}|u|^{\alpha}|v|^{\beta}\mathrm{d}x.$$

Since $\{(\bar{u}_n,\bar{v}_n)\}$ is bounded in $(D^{1,p}(\mathbb{R}^N))^2$, and $\{|\bar{u}_n|^{p^*}\}$, $\{|\bar{v}_n|^{p^*}\}$ and $\{|\bar{u}_n|^{\alpha}|\bar{v}_n|^{\beta}\}$ are tight, by Lemma 2 we have

$$\nu=\eta|u|^{p^*}+\lambda|v|^{p^*}+\sigma|u|^{\alpha}|v|^{\beta}+\sum_{j\in J}\nu_j\delta_j.$$

Passing to the limit as $n\to\infty$ in (2.4) and applying the Rellich theorem, we deduce for all $\varphi\in C_0^\infty(\mathbb{R}^N)$ that

$$\Big(\int_{\mathbb{R}^N}|\varphi|^{p^*}\mathrm{d}\nu\Big)^{\frac{1}{p^*}}(S(\eta,\lambda,\sigma))^{\frac{1}{p}}$$
$$\leqslant\Big(\int_{\mathbb{R}^N}|\varphi|^p\mathrm{d}\mu\Big)^{\frac{1}{p}}+C\Big(\int_{\mathbb{R}^N}|\nabla\varphi|^p(|u_n|^p+|v_n|^p)\mathrm{d}x\Big)^{\frac{1}{p}}. \quad (2.7)$$

Choose $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that $0\leqslant\varphi\leqslant1$, $\varphi(0)=1$ and $\mathrm{Supp}\,\varphi=B(0,1)$. For all $\varepsilon>0$ and $j\in J$, applying the Hölder inequality and (2.7) with $\varphi(\frac{x-x_j}{\varepsilon})$, we have

$$(\nu_j)^{\frac{1}{p^*}}(S(\eta,\lambda,\sigma))^{\frac{1}{p}}-(\mu(B(x_j,\varepsilon)))^{\frac{1}{p}}$$

$$\leqslant \frac{C}{\varepsilon}\Big(\int_{B(x_j,\varepsilon)} |\nabla\varphi(\frac{x-x_j}{\varepsilon})|^p(|u_n|^p+|v_n|^p)\mathrm{d}x\Big)^{\frac{1}{p}}$$

$$\leqslant \frac{C}{\varepsilon}\Big(\int_{B(x_j,\varepsilon)} (|u_n|^p+|v_n|^p)^{\frac{p^*}{p}}\mathrm{d}x\Big)^{\frac{1}{p^*}}\Big(\int_{B(x_j,\varepsilon)} |\nabla\varphi(\frac{x-x_j}{\varepsilon})|^N\mathrm{d}x\Big)^{\frac{1}{N}}$$

$$\leqslant C\Big(\Big(\int_{B(x_j,\varepsilon)} |u_n|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}+\Big(\int_{B(x_j,\varepsilon)} |v_n|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}\mathrm{d}x\Big)^{\frac{1}{p}},$$

which implies that $\mu(\{x_j\})\geqslant 0$ and

$$\mu \geqslant (\nu_j)^{\frac{p}{p^*}} S(\eta,\lambda,\sigma)\delta_{x_j}, \quad \forall j \in J,$$

$$\mu \geqslant \sum_{j\in J}(\nu_j)^{\frac{p}{p^*}} S(\eta,\lambda,\sigma)\delta_{x_j} =: \mu_1.$$

By the weak convergence we have $\mu \geqslant |\nabla u|^p+|\nabla v|^p$. From the fact that $|\nabla u|^p+|\nabla v|^p$ and μ_1 are orthogonal, we conclude Theorem 1.

3. Some variants and applications

We now study some variants of Theorem 1. The following assumption is needed:

$$(\mathscr{H}_2)\ \begin{cases} N\geqslant 3,\ 1<p<N,\ \eta,\lambda,\sigma\geqslant 0,\ \eta+\lambda+\sigma>0,\ 0\leqslant t<p, \ \ \tau<\bar{\tau}, \\ 1<\alpha,\beta<p^*(t)-1, \ \ \alpha+\beta=p^*(t). \end{cases}$$

Consider the following Hardy and Hardy-Sobolev inequalities ([8], [13]):

$$\int_{\mathbb{R}^N}\frac{|u|^p}{|x-\xi|^p}\mathrm{d}x \leqslant \frac{1}{\bar{\tau}}\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x, \ \ \forall u\in C_0^\infty(\mathbb{R}^N), \ \ \xi\in\mathbb{R}^N, \tag{3.1}$$

$$\Big(\int_{\mathbb{R}^N}\frac{|u|^{p^*(t)}}{|x-\xi|^t}\mathrm{d}x\Big)^{\frac{p}{p^*(t)}} \leqslant C(t)\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x,\ \forall u\in C_0^\infty(\mathbb{R}^N),\ \xi\in\mathbb{R}^N, \tag{3.2}$$

where $C(t)$ is a positive constant depending on t, $\bar{\tau}=((N-p)/p)^p$ is the best Hardy constant and $p^*(t)=p(N-t)/N-p)$ is the critical Hardy-Sobolev exponent.

Under the assumption $(\mathscr{H}_2)$, by (3.1), (3.2) and the Young inequality, the following best constant is well defined on $\mathscr{D}=(D^{1,p}(\mathbb{R}^N)\setminus\{0\})^2$ (e.g. [14], [16]):

$$S_{\eta,\lambda,\sigma}(\tau,t) := \inf_{(u,v)\in\mathscr{D}} \frac{\displaystyle\int_{\mathbb{R}^N}\Big(|\nabla u|^p+|\nabla v|^p-\tau\frac{|u|^p+|v|^p}{|x|^p}\Big)\mathrm{d}x}{\displaystyle\Big(\int_{\mathbb{R}^N}\frac{\eta|u|^{p^*(t)}+\lambda|v|^{p^*(t)}+\sigma|u|^\alpha|v|^\beta}{|x|^t}\mathrm{d}x\Big)^{\frac{p}{p^*(t)}}}.$$

Note that $S_{\eta,\lambda,\sigma}(0,0)=S(\eta,\lambda,\sigma)$.

By the argument of [17, 18] and the proof of Theorem 1, we obtain the following Theorems 2 and 3, the variants of Theorem 1. The proofs are omitted.

THEOREM 2. *Suppose* $(\mathscr{H}_2)$ *holds and* $t=0$. *Let* $\{(u_n,v_n)\}$ *be a bounded sequence in* $(D^{1,p}(\mathbb{R}^N))^2$ *such that:*

$$\{(u_n,v_n)\} \rightharpoonup (u,v) \ \text{weakly in } (D^{1,p}(\mathbb{R}^N))^2,$$

$$\{|\nabla u_n|^p + |\nabla v_n|^p\} \ \text{converges weakly to } \mu,$$

$$\left\{\frac{|u_n|^p+|v_n|^p}{|x|^p}\right\} \ \text{converge tightly to } \omega,$$

$$\left\{\eta|u_n|^{p^*}+\lambda|v_n|^{p^*}+\sigma|u_n|^\alpha|v_n|^\beta\right\} \text{converge tightly to } \nu,$$

where μ,ω *and* ν *are bounded nonnegative measures on* $\mathbb{R}^N$. *Then:*
(i) *there exist* $\omega_0,\nu_0\in[0,\infty)$, *an at most countable set* J *and two families* $\{x_j\}_{j\in J}\subset \mathbb{R}^N\setminus\{0\}$ *and* $\{\nu_j\}_{j\in J}\subset[0,+\infty)$ *such that*

$$\omega=\frac{|u|^p+|v|^p}{|x|^p}+\omega_0, \quad \nu=\eta|u|^{p^*}+\lambda|v|^{p^*}+\sigma|u|^\alpha|v|^\beta+\nu_0+\sum_{j\in J}\nu_j\delta_{x_j};$$

(ii) *there exist* $\mu_0\in[0,\infty)$, $\{\mu_j\}_{j\in J}\subset[0,+\infty)$ *such that*

$$\mu\geqslant|\nabla u|^p+|\nabla v|^p+\mu_0+\sum_{j\in J}\mu_j\delta_{x_j},$$

$$(\nu_0)^{\frac{p}{p^*}}\leqslant(\mu_0-\tau\omega_0)/S_{\eta,\lambda,\sigma}(\tau,0),$$

$$(\nu_j)^{\frac{p}{p^*}}\leqslant\mu_j/S_{\eta,\lambda,\sigma}(0,0)=\mu_j/S(\eta,\lambda,\sigma), \ \ \forall j\in J,$$

and therefore

$$\sum_{j\in J}(\nu_j)^{\frac{p}{p^*}}<\infty.$$

THEOREM 3. *Suppose* $(\mathscr{H}_2)$ *holds and* $t>0$. *Let* $\{(u_n,v_n)\}$ *be a bounded sequence in* $(D^{1,p}(\mathbb{R}^N))^2$ *such that:*

$$\{(u_n,v_n)\} \rightharpoonup (u,v) \ \text{weakly in } (D^{1,p}(\mathbb{R}^N))^2,$$

$$\{|\nabla u_n|^p + |\nabla v_n|^p\} \ \text{converges weakly to } \mu,$$

$$\left\{\frac{|u_n|^p+|v_n|^p}{|x|^p}\right\} \ \text{converge tightly to } \omega,$$

$$\left\{\frac{\eta|u_n|^{p^*(t)}+\lambda|v_n|^{p^*(t)}+\sigma|u_n|^\alpha|v_n|^\beta}{|x|^t}\right\} \text{converge tightly to } \nu,$$

where μ,ω *and* ν *are bounded nonnegative measures on* $\mathbb{R}^N$. *Then:*
(i) *there exist* $\omega_0,\nu_0\in[0,+\infty)$ *such that*

$$\omega=\frac{|u|^p+|v|^p}{|x|^p}+\omega_0, \quad \nu=\frac{\eta|u|^{p^*(t)}+\lambda|v|^{p^*(t)}+\sigma|u|^\alpha|v|^\beta}{|x|^t}+\nu_0\delta_0;$$

(ii) *there exists* $\mu_0 \in [0,+\infty)$ *such that*

$$\mu \geqslant |\nabla u|^p + |\nabla v|^p + \mu_0, \quad (\nu_0)^{\frac{p}{p^*(t)}} \leqslant (\mu_0 - \tau\omega_0)/S_{\eta,\lambda,\sigma}(\tau,t).$$

Theorems 1-3 are crucial for studying elliptic systems. For example, consider the following problem:

$$\begin{cases} Lu = |u|^{2^*-2}u + \dfrac{\sigma\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u + a_1u + a_2v, \\ Lv = |v|^{2^*-2}v + \dfrac{\sigma\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + a_2u + a_3v, \\ u,v \in H_0^1(\Omega), \end{cases} \tag{3.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain such that:

$$0 \in \Omega,\ L := -\Big(\Delta \cdot + \tau\frac{\cdot}{|x|^2}\Big),\ \sigma \geqslant 0,\ \alpha,\beta > 1,\ \tau < \Big(\frac{N-2}{2}\Big)^2,$$
$$\alpha+\beta = 2^*,\ 2^* := \frac{2N}{N-2} \text{ is the critical Sobolev exponent,}$$

the space $H_0^1(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to $(\int_\Omega |\nabla \cdot|^2 \,\mathrm{d}x)^{1/2}$,

$$a_i > 0,\ i = 1,2,3,\ \ a_1a_3 - a_2^2 > 0,\ \ 0 < \lambda_1 \leqslant \lambda_2 < \Lambda_1(\tau),$$

where λ_1 and λ_2 are the eigenvalues of the matrix

$$A := \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix},$$

and $\Lambda_1(\tau)$ is the first eigenvalue of the operator L on $H_0^1(\Omega)$. The energy functional of (3.3) is defined on $H_0^1(\Omega)\times H_0^1(\Omega)$ by

$$\begin{aligned} J(u,v) := \frac{1}{2}\int_\Omega\Big(|\nabla u|^2 + |\nabla v|^2 - \tau\frac{u^2+v^2}{|x|^2}\Big)\mathrm{d}x - \frac{\sigma}{2^*}\int_\Omega |u|^{\alpha}|v|^{\beta}\mathrm{d}x \\ - \frac{1}{2^*}\int_\Omega\big(|u|^{2^*} + |v|^{2^*}\big)\mathrm{d}x - \frac{1}{2}\int_\Omega\big(a_1u^2 + 2a_2uv + a_3v^2\big)\mathrm{d}x. \end{aligned}$$

Applying Theorem 2 with $p = 2$, we can verify the following local $(PS)_c$ condition.

LEMMA 3. *$J(u,v)$ satisfies the $(PS)_c$ condition for all* $c < \frac{1}{N}(S_{1,1,\sigma}(\tau,0))^{\frac{N}{2}}$.

Furthermore, by the variational arguments we can investigate the nontrivial solutions to (3.3) (e.g. [14]).

Theorem 3 is also useful in studying the elliptic systems related to (3.1) and (3.2). For example, consider the following problem:

$$\begin{cases} Lu = \dfrac{\sigma\alpha}{2^*(r)} \dfrac{|u|^{\alpha-2}|v|^{\beta}u}{|x|^r} + \eta \dfrac{|u|^{2^*(s)-2}u}{|x-\xi_1|^s} + a_1u + a_2v, \\ Lv = \dfrac{\sigma\beta}{2^*(r)} \dfrac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^r} + \lambda \dfrac{|v|^{2^*(t)-2}v}{|x-\xi_2|^t} + a_2u + a_3v, \\ (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega), \end{cases} \tag{3.4}$$

where $\Omega \subset \mathbb{R}^N\,(N \geqslant 3)$ is a bounded domain with the smooth boundary $\partial\Omega$ such that the points $0, \xi_1, \xi_2 \in \Omega$, $L = -(\Delta \cdot + \tau \frac{\cdot}{|x|^2})$, $\eta, \lambda, \sigma \geqslant 0$, $a_1, a_2, a_3 \in \mathbb{R}$, $0 < r,s,t < 2$, $1 < \alpha, \beta < 2^*(r) - 1$, $\alpha + \beta = 2^*(r)$, $\tau < \bar{\tau}$, $2^*(r), 2^*(s)$ and $2^*(t)$ are the critical Hardy–Sobolev exponents. Applying Theorem 3 with $p = 2$, we can verify that the energy functional corresponding to (3.4) satisfies some kinds of local $(PS)_c$ conditions and therefore the existence of nontrivial solutions to (3.4) can be investigated.

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