Differential
Equations
& Applications
Volume 4, Number 3 (2012), 445–457
doi:10.7153/dea-04-26

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR THE NONLINEAR KLEIN–GORDON EQUATION COUPLED WITH BORN–INFELD THEORY ON BOUNDED DOMAIN

KAIMIN TENG

(*Communicated by Pavel I. Naumkin*)

Abstract. In this paper, we prove some existence and multiple results for the following nonlinear Klein-Gordon equation coupled with Born-Infeld theory

$$\begin{cases} \Delta u = \big(m^2-(\omega+\phi)^2\big)u - f(x,u), & \text{in } \Omega,\\ \Delta\phi+\beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \Omega,\\ u=\phi=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a bounded domain with smooth boundary, and $f \in C(\overline{\Omega},\mathbb{R})$ satisfies some assumptions.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of solutions of the following Klein-Gordon equation coupled with Born-Infeld theory on bounded domain:

$$\begin{cases} \Delta u = \big(m^2-(\omega+\phi)^2\big)u - f(x,u), & \text{in } \Omega,\\ \Delta\phi+\beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & \text{in } \Omega,\\ u=\phi=0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Such class of equations deduced by coupling the Klein-Gordon equation

$$\psi_{tt} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0, \tag{1.2}$$

with the Born-Infeld theory [5],

$$\mathfrak{L}_{BI} = \frac{b^2}{4\pi}\left\{1-\sqrt{1-\frac{1}{b^2}(|E|^2-|B|^2)}\right\} \tag{1.3}$$

where $\psi = \psi(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, m is a real constant, $p \in [2,2^*)$, E is the electric field and B is the magnetic induction field. As usual, the electromagnetic field

Mathematics subject classification (2010): 26D15, 26A51, 32F99, 41A17.
Keywords and phrases: Klein-Gordon equation, Born-Infeld theory, variational methods.
This research is supported by the NSFC Grant 10961028.

Paper DEA-04-26

is described by the gauge potential (ϕ, A), $\phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$. From (ϕ, A), we obtain the electric field

$$E = -\nabla\phi - A_t$$

and the magnetic induction field

$$B = \nabla \times A.$$

Assume that ψ is a charged field and let e denote the electric charge. By gauge invariance arguments, the interaction between ψ and the electromagnetic field is usually described substituting the usual derivatives $\frac{\partial}{\partial t}$, ∇ with the gauge covariant derivatives

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + ie\phi, \quad \nabla \mapsto \nabla - ieA$$

into the Lagrangian density relative (1.2) given by

$$\mathfrak{L}_0 = \frac{1}{2}\left[|\frac{\partial \psi}{\partial t}|^2 - |\nabla\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p \tag{1.4}$$

we obtain the following Lagrangian density

$$\mathfrak{L}_0 = \frac{1}{2}\left[|\frac{\partial \psi}{\partial t} + ie\phi\psi|^2 - |\nabla\psi - ieA\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p. \tag{1.5}$$

As done in [3] and [8], let $\beta = \frac{1}{2b^2}$ and consider the second order expansion of $\mathfrak{L}_{BI}$ for $\beta \to 0^+$. Then the Lagrangian density takes the form

$$\mathfrak{L}_{BI,1} = \frac{1}{4\pi}\left[\frac{1}{2}(|E|^2 - |B|^2) + \frac{1}{4}\beta(|E|^2 - |B|^2)^2\right].$$

The nonlinear Born-Infeld-Klein-Gordon equations are the Euler-Lagrange equations of the total action

$$\mathscr{S} = \int \mathfrak{L}_{BI,1} + \mathfrak{L}_0. \tag{1.6}$$

Under the electrostatic solitary wave ansatz

$$\psi(x,t) = u(x)e^{i\omega t}, \ \ \phi = \phi(x), \ \ A = 0$$

and taking $e = 1$, where u and ϕ are real valued functions defined on Ω and ω is a positive frequency parameter, the total action in (1.6) takes the form

$$\mathscr{E}_0(u,\phi) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{8\pi}|\nabla\phi|^2 - \frac{\beta}{16\pi}|\nabla\phi|^4 + \frac{1}{2}(m^2 - (\omega+\phi)^2)u^2 - \frac{1}{p}|u|^p\right] dx \tag{1.7}$$

whose Euler-Lagrange equation is precisely (1.1) when $f(x,t) = |t|^{p-2}t$.

In recent years, the Born-Infeld nonlinear electromagnetism has regained its importance due to its relevance in the theory of superstrings and membranes [11]. Mathematically, many people considered the system coupled Klein-Gordon equation with Born-Infeld theory through using variational methods. Particularly, in Fortunato et al. [9], Mugnai [10] and D'Avenia et al. [8], they obtained the existence of infinite many radial solutions for the equation (1.1) on $\mathbb{R}^3$. Yu [17] studied the Klein-Gordon equation coupled with the original Born-Infeld theory, and obtained infinitely many solitary waves solution on bounded domain and $\mathbb{R}^3$ (radial solution), respectively. The authors in [15] studied the same problem as Yu [17] and Mugnai [10] at critical case, and obtained some existence results. Via variational methods, the existence of solitary waves solution has been studied in different systems, such as [1], [3], [2], [6], [7], [13], [16], and so on. In this paper, we are interested in the existence and multiplicity of solutions for problem (1.1) on bounded domain.

The action functional $\mathscr{E}: H_0^1(\Omega)\times\mathscr{D}\to\mathbb{R}$ is defined by

$$\mathscr{E}(u,\phi)=\int_\Omega\left[\frac{1}{2}|\nabla u|^2-\frac{1}{8\pi}|\nabla\phi|^2-\frac{\beta}{16\pi}|\nabla\phi|^4\right.$$
$$\left.+\frac{1}{2}(m^2-(\omega+\phi)^2)u^2-F(x,u)\right]dx,$$

where $F(x,t)=\int_0^t f(x,s)ds$. From the structure of $\mathscr{E}$, we see that $\mathscr{E}$ is strongly indefinite, namely it is unbounded both from below and from above on infinite dimensional subspaces. To avoid this indefiniteness, we will use the reduction method, by which we are led to study an one variable functional that does not present such a strongly indefinite nature. Thus, we can use the critical point theorems to problem (1.1).

Assume that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

(f_0) $f\in C(\bar{\Omega}\times\mathbb{R},\mathbb{R})$, and $f(x,0)=0$;

(f_1) there are constants $a_1,a_2>0$ such that

$$|f(x,t)|\leqslant a_1+a_2|t|^s,$$

where $1<s<\frac{n+2}{n-2}$ $(n\geqslant 3)$;

(f_2) $\lim\limits_{t\to 0}\frac{f(x,t)}{t}=0$;

(f_3) there exist $\mu>2$ and $R>0$ such that $tf(x,t)\geqslant\mu F(x,t)>0$ for $|t|\geqslant R$ and $x\in\Omega$;

(f_4) $f(x,-u)=-f(x,u)$, for all $u\in\mathbb{R}$ and $x\in\Omega$.

The main results we provide in this paper are the following theorems.

THEOREM 1. *Assume that*

$$m^2>\frac{\mu}{\mu-2}\omega^2-\lambda_1 \ \textit{and} \ f \ \textit{satisfies} \ (f_0)\text{-}(f_3).$$

Here λ_1 *is the first eigenvalue of eigenvalue problem* $(-\Delta,H_0^1(\Omega))$. *Then problem* (1.1) *has at least one nontrivial solution.*

REMARK 1. By the assumption (f_0), we can obtain that $(u,\phi)=(0,0)$ is a solution of problem (1.1).

In addition, if f also satisfies the symmetric assumption , we may apply the symmetric Mountain-pass theorem to obtain infinite many solutions for problem (1.1).

THEOREM 2. *Assume that*

$$m^2 > \frac{\mu}{\mu-2}\omega^2 - \lambda_1 \ \textit{and} \ f \ \textit{satisfies} \ (f_0)\text{-}(f_4).$$

Here λ_1 is the first eigenvalue of eigenvalue problem $(-\Delta, H_0^1(\Omega))$. Then problem (1.1) has infinitely many solutions.

2. Preliminaries Lemmas

In this section, we present the variational framework for problem (1.1) and also give some preliminaries which are useful later.

Throughout this paper, we denote $\|\cdot\|_s$ the L^s space for $1 \leqslant s \leqslant +\infty$. Let $H_0^1(\Omega)$ denote the usual Sobolev space with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}},$$

and $\mathscr{D}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathscr{D}} = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla u|^4 dx\right)^{\frac{1}{4}}.$$

It is obvious that $\mathscr{D}(\Omega)$ is a reflexive and separable Banach space, continuously embedded in $H_0^1(\Omega)$. Moreover, from Sobolev's imbedding theorem (see [16]), $\mathscr{D}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$.

A fundamental tool in our analysis will be the following Proposition.

PROPOSITION 1. *For every $u \in H_0^1(\Omega)$, there exists a unique $\phi_u \in \mathscr{D}(\Omega)$ such that ϕ_u satisfies the second equation in problem (1.1) and*

$$-\omega \leqslant \phi_u \leqslant 0. \tag{2.1}$$

Moreover, the map $\Phi : H_0^1(\Omega) \to \mathscr{D}(\Omega)$, $u \mapsto \phi_u$ is a continuous map.

Proof. We consider the minimizing argument on the functional defined as

$$E_u(\phi) = \int_\Omega \left[\frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4 + (\omega + \frac{1}{2}\phi)u^2\phi\right] dx.$$

Obviously, the functional E_u is well defined on $\mathscr{D}(\Omega)$ and is convex, coercive, and weakly lower semi-continuous. Indeed, The coercivity of E_u on $\mathscr{D}(\Omega)$ is the following fact that

$$E_u(\phi) = \int_\Omega \left[\frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4 + \frac{1}{2}(\omega+\phi)^2u^2 - \frac{1}{2}\omega^2u^2\right]dx$$
$$\geqslant \int_\Omega \left[\frac{1}{8\pi}|\nabla\phi|^2 + \frac{\beta}{16\pi}|\nabla\phi|^4\right]dx - \frac{\omega}{2}\int_\Omega u^2dx.$$

The convexity and weakly lower semi-continuity of E_u on $\mathscr{D}(\Omega)$ is obviously true. Hence, there is a minimizer ϕ_u of the functional E_u on $\mathscr{D}(\Omega)$ and it is correspond to the weak solution of the second equation in (1.1). The uniqueness is follows form the fact the operator $\mathscr{A} : \mathscr{D}(\Omega) \to (\mathscr{D}(\Omega))'$ defined by

$$\langle \mathscr{A}u, v\rangle = \langle(-\Delta u - \beta\Delta_4 u + 4\pi u^2, v\rangle = \int_\Omega \left[\nabla u\cdot\nabla v + \beta|\nabla u|^2\nabla u\cdot\nabla v + 4\pi u^2 v\right]dx$$

is strictly monotone (see appendix B, Theorem 5 in [4]).

Multiplying the second equation of problem (1.1) by $\Phi^+ = \max\{\Phi(u), 0\}$ and $(\omega+\Phi(u))^-$, respectively, through simple calculation, we could get the conclusion (2.1).

Finally, we assume that $u_n \to u$ in $H_0^1(\Omega)$, our purpose is to prove that $\Phi(u_n) \to \Phi(u)$ in $\mathscr{D}(\Omega)$. Since $\Phi(u_n)$ and $\Phi(u)$ satisfy the second equation of (1.1), that is,

$$\int_\Omega \left[\nabla\Phi(u_n)\cdot\nabla\varphi + \beta|\nabla\Phi(u_n)|^2\nabla\Phi(u_n)\cdot\nabla\varphi\right]dx = \int_\Omega (\omega + \frac{1}{2}\Phi(u_n))\Phi(u_n)u_n^2\varphi dx$$

and

$$\int_\Omega \left[\nabla\Phi(u)\cdot\nabla\varphi + \beta|\nabla\Phi(u)|^2\nabla\Phi(u)\cdot\nabla\varphi\right]dx = \int_\Omega (\omega + \frac{1}{2}\Phi(u))\Phi(u)u^2\varphi dx$$

for any $\varphi \in \mathscr{D}(\Omega)$, let us take the difference between them, take $\varphi = \Phi(u_n) - \Phi$ and apply the inequality

$$[(|x|^{p-2}x - |y|^{p-2}y)(x-y)] \geqslant c_p|x-y|^p, \text{ for } x, y \in \mathbb{R}^N,\ p \geqslant 2, \tag{2.2}$$

we get

$$C\big(\|\nabla(\Phi(u_n)-\Phi(u))\|_2^2 + \|\nabla(\Phi(u_n)-\Phi(u))\|_4^4\big)$$
$$\leqslant 4\pi\int_\Omega \big[\omega|u_n^2 - u^2||\Phi(u_n)-\Phi| + |\Phi(u_n)||\Phi(u_n)-\Phi(u)|u_n^2$$
$$+ |\Phi||\Phi(u_n)-\Phi(u)|u^2\big]dx.$$

By the Hölder's inequality and (2.1), we have

$$\|\nabla(\Phi(u_n)-\Phi(u))\|_2^2 + \|\nabla(\Phi(u_n)-\Phi(u))\|_4^4$$
$$\leqslant C\Big(\|u_n - u\|_2^2 + \|u_n\|_{\frac{12}{5}}^2\|\Phi(u_n)-\Phi(u)\|_6 + \|u\|_{\frac{12}{5}}^2\|\Phi(u_n)-\Phi(u)\|_6\Big)$$

$$\leqslant C_1\Big(\|u_n-u\|^2+\|u_n\|^2\|\nabla(\Phi(u_n)-\Phi(u))\|_2+\|u\|^2\|\nabla(\Phi(u_n)-\Phi(u))\|_2\Big).$$

From the assumption $u_n \to u$ in $H_0^1(\Omega)$, we complete the conclusion.

REMARK 2. By the expression of the functional E_u, we see that $\Phi(u)=\Phi(-u)$, for every $u\in H_0^1(\Omega)$.

REMARK 3. From the proof of Proposition (1), we see that if $u\in L^p(\Omega)$ with $2\leqslant p<2^*$, the uniqueness of Φ is true, too. Indeed, we get he following conclusion: for every $u\in L^p(\Omega)$, there exists a unique $\Phi(u)\in\mathscr{D}(\Omega)$ such that $E_u(\Phi(u)=\inf\limits_{\phi\in\mathscr{D}(\Omega)}E_u(\phi)$.

REMARK 4. The proof of Proposition 1 implies that $\forall s,t\in\mathbb{R}$, $\forall u\in L^2(\Omega)$, we have

$$\int_\Omega\Big|\frac{\nabla\Phi(su)-\nabla\Phi(tu)}{s-t}\Big|^2+\omega u^2(t+s)\frac{\Phi(su)-\Phi(tu)}{s-t}$$
$$+u^2\frac{\Phi(su)-\Phi(tu)}{s-t}\Big(s^2\frac{\Phi(su)-\Phi(tu)}{s-t}+(s+t)\Phi(tu)\Big)\leqslant 0.$$

Indeed, $\forall s,t\in\mathbb{R}$, $\forall u\in L^2(\Omega)$, $\Phi(su)$ and $\Phi(tu)$ satisfy

$$\int_\Omega(1+\beta|\nabla\Phi(su)|^2)\nabla\Phi(su))\cdot(\nabla\Phi(su)-\Phi(tu))dx$$
$$=4\pi\int_\Omega s^2u^2(\omega+\Phi(su))(\Phi(su)-\Phi(tu))dx$$

and

$$\int_\Omega(1+\beta|\nabla\Phi(tu)|^2)\nabla\Phi(tu))\cdot(\nabla\Phi(tu)-\Phi(su))dx$$
$$=4\pi\int_\Omega t^2u^2(\omega+\Phi(tu))(\Phi(tu)-\Phi(su))dx.$$

Let us add them together, apply (2.2), we get

$$\int_\Omega\Big|\nabla\Phi(su)-\nabla\Phi(tu)\Big|^2$$
$$\leqslant 4\pi\int_\Omega\Big[u^2\omega(s^2-t^2)(\Phi(su)-\Phi(tu))$$
$$+u^2s^2(\Phi(su)-\Phi(tu))^2+u^2(s^2-t^2)\Phi(tu)(\Phi(su)-\Phi(tu))\Big]dx.$$

Hence, the above inequality is divided by $(s-t)^2$, we obtain the conclusion.

LEMMA 1. *If* $\{\phi_n\} \subset \mathscr{D}(\Omega)$ *is bounded and* $\phi_n \rightharpoonup \phi$ *in* $\mathscr{D}(\Omega)$*, then* ϕ_n *uniformly converges to* ϕ *in* $\bar{\Omega}$.

Proof. Let $\{\phi_{nk}\}$ be an any subsequence of $\{\phi_n\}$, then $\{\nabla\phi_{n_k}\}$ is uniformly bounded in $L^2(\Omega)$ and $L^4(\Omega)$. By Morrey's inequality, we get

$$\|\phi_{n_k}\|_{C^{0,\frac{1}{4}}(\bar{\Omega})} \leqslant C\|\phi_{n_k}\|_{\mathscr{D}(\Omega)}$$

which implies that $\{\phi_{n_k}\}$ is equicontinuous on $\bar{\Omega}$. By the fact that $\mathscr{D}(\Omega)$ is continuously embedded in $L^\infty(\Omega)$, we see that $\{\phi_{n_k}\}$ is uniformly bounded in $\bar{\Omega}$. Apply the Arzelá-Ascoli theorem, we can extract a subsequence, which is denoted by $\{\phi_{n_{k_l}}\}$, such that $\phi_{n_{k_l}} \to \phi$ uniformly in $\bar{\Omega}$. Because the subsequence $\{\phi_{n_k}\}$ is arbitrary, we have $\phi_n \to \phi$ uniformly in $\bar{\Omega}$.

REMARK 5. By Lemma 1, we can also prove that E_u on $\mathscr{D}(\Omega)$ is weakly sequentially lower semi-continuous. Indeed, assume that $\{\phi_n\} \subset \mathscr{D}(\Omega)$ and $\phi_n \rightharpoonup \phi$ weakly in $\mathscr{D}(\Omega)$. By Lemma 1, we know that $\phi_n \to \phi$ uniformly in $\bar{\Omega}$. Hence, $\phi_n \to \phi$ in $L^1(\Omega)$. Since $\phi_n \rightharpoonup \phi$ weakly in $\mathscr{D}(\Omega)$, then $\phi_n \rightharpoonup \phi$ weakly in $\mathscr{D}^{1,2}(\Omega)$. Hence, $\nabla\phi_n \rightharpoonup \nabla\phi$ weakly in $L^2(\Omega)$ and then we get $\nabla\phi_n \rightharpoonup \nabla\phi$ weakly in $L^1(\Omega)$. Apply Theorem 1.6 in [14], we get $E_u(\phi) \leqslant \liminf_{n\to\infty} E_u(\phi_n)$.

By the Lemma 1, we can improve the result of Proposition 1.

PROPOSITION 2. $\Phi : L^2(\Omega) \to \mathscr{D}(\Omega)$ *is continuous. Moreover, if* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$*, then* $\Phi(u_n) \to \Phi(u)$ *in* $\mathscr{D}(\Omega)$.

Proof. Assume that $u_n \to u$ strongly in $L^2(\Omega)$. By Proposition 1, we know that $\{\Phi(u_{n_k})\}$ is uniformly bounded in $\mathscr{D}(\Omega)$. Hence, it is also uniformly bounded in $L^\infty(\Omega)$. Then there exists a subsequence $\{u_{n_{k_l}}\}$ and $g \in \mathscr{D}(\Omega)$, such that $\Phi(u_{n_{k_l}}) \rightharpoonup g$ in $\mathscr{D}(\Omega)$. By Lemma 1, we have $\Phi(u_{n_{k_l}}) \to g$ uniformly in $\bar{\Omega}$. For the rest of the proof, we only need to show that $g = \Phi(u)$.

Since $\Phi(u_{n_{k_l}})$ is the minimizer of $E_{u_{n_{k_l}}}$ on $\mathscr{D}(\Omega)$ and $u_{n_{k_l}} \to u$ in $L^2(\Omega)$, we have

$$E_{u_{n_{k_l}}}(\Phi(u_{n_{k_l}})) \leqslant E_{u_{n_{k_l}}}(\Phi(u)) \to E_u(\Phi(u)).$$

Therefore, by $\{\Phi(u_{n_{k_l}})\}$ is uniformly bounded in $L^\infty(\Omega)$, converges to g uniformly in $\bar{\Omega}$, $u_{n_{k_l}} \to u$ in $L^2(\Omega)$, and dominated convergence theorem, we get

$$\left|\int_\Omega \left[\Phi(u_{n_{k_l}})u_{n_{k_l}}^2 - gu^2\right] dx\right| = \left|\int_\Omega (\Phi(u_{n_{k_l}}) - g)u^2 dx + \int_\Omega \Phi(u_{n_{k_l}})(u_{n_{k_l}}^2 - u^2) dx\right| \to 0.$$

Hence,

$$\int_\Omega \Phi(u_{n_{k_l}})u_{n_{k_l}}^2\,dx \to \int_\Omega gu^2dx.$$

Similarly, we can prove that

$$\int_\Omega \Phi(u_{n_{k_l}})^2u_{n_{k_l}}^2\,dx \to \int_\Omega g^2u^2dx.$$

Hence, by the weak lower semicontinuity of $E_{u_{n_{k_l}}}$, we get

$$E_u(g) \leqslant \liminf_{l\to\infty} E_{u_{n_{k_l}}}(\Phi(u_{n_{k_l}})) \leqslant \lim_{l\to\infty} E_{u_{n_{k_l}}}(\Phi(u)) = E_u(\Phi(u)).$$

By the uniqueness result in Proposition 1, $g = \Phi(u)$.

The second conclusion is obviously.

REMARK 6. From the proof of Proposition 2, we see that $\Phi : L^p(\Omega) \to \mathscr{D}(\Omega)$ is continuous if $2 \leqslant p < 2^*$.

Define $J : H_0^1(\Omega) \to \mathbb{R}$ as

$$\begin{aligned} J(u) =& \mathscr{E}(u,\Phi(u)) \\ =& \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}(m^2-(\omega+\Phi(u))^2)u^2 - \frac{1}{8\pi}|\nabla\Phi|^2 \\ & - \frac{\beta}{16\pi}|\nabla\Phi|^4 - F(x,u)\Big]dx. \end{aligned}$$

Fixed $u \in H_0^1(\Omega)$, since $\Phi(u)$ is the solution of the second equation in (1.1), we have

$$\Big\langle \frac{1}{4\pi}\Delta\Phi(u) + \frac{\beta}{4\pi}\Delta_4\Phi(u) - \Phi(u)u^2, \Phi(u)\Big\rangle = \langle \omega u^2, \Phi(u)\rangle,$$

i.e.

$$-\int_\Omega \Big[\frac{1}{4\pi}|\nabla\Phi(u)|^2 + \frac{\beta}{4\pi}|\nabla\Phi(u)|^4 - \Phi^2(u)u^2\Big]dx = \omega\int_\Omega u^2\Phi(u)dx. \tag{2.3}$$

Using (2.3), we get the other form of the functional J,

$$\begin{aligned} J(u) = \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}(m^2-\omega^2)u^2 + \frac{1}{2}\Phi^2(u)u^2 + \frac{1}{8\pi}|\nabla\Phi|^2 \\ + \frac{3\beta}{16\pi}|\nabla\Phi|^4 - F(x,u)\Big]dx. \end{aligned}$$

PROPOSITION 3. *$J \in C^1(H_0^1(\Omega),\mathbb{R})$ in the sense of Fréchet and*

$$\langle J'(u),v\rangle = \int_\Omega \Big[\nabla u\cdot\nabla v + (m^2-(\omega+\Phi(u))^2)uv) - f(x,u)v\Big]dx$$

for all $u,v \in H_0^1(\Omega)$.

Proof. It is suffice to prove that

$$\lim_{\|v\|\to 0}\frac{J(u+v)-J(u)-DJ(u)v}{\|v\|}=0,$$

where

$$DJ(u)v=\int_\Omega[\nabla\cdot\nabla v+(m^2-(\omega+\Phi(u))^2)uv-f(x,u)v]dx.$$

We split $J(u+v)-J(u)-DJ(u)v$ into three parts, that is $J(u+v)-J(u)-DJ(u)v=I_1+I_2+I_3$, where:

$$I_1=\int_\Omega\left[\frac{1}{2}|\nabla(u+v)|^2-\frac{1}{2}|\nabla u|^2-\nabla u\cdot\nabla v\right]dx,$$
$$I_2=-\int_\Omega\left[F(x,u+v)-F(x,u)-f(x,u)v\right]dx,$$

and

$$I_3=E_u(\Phi(u))-E_{u+v}(\Phi(u+v))+\int_\Omega\left[\frac{1}{2}(m^2-\omega^2)v^2+(2\omega+\Phi(u))\Phi(u)uv\right]dx.$$

By the assumptions (f_0), (f_1), combing with the proof of Proposition B.10 in [12] and Proposition 3.1 in [17], we can complete the proof.

For problem (1.1), similarly as Fortunato et al. in [3](see Proposition 3.5), we can prove a relationship between the critical points of the functional $\mathscr{E}$ and J:

LEMMA 2. *The following statements are equivalent:*

(i) $(u,\phi)\in H_0^1(\Omega)\times\mathscr{D}(\Omega)$ *is a critical point of* $\mathscr{E}$ *(i.e.,* (u,ϕ) *is a solution of (1.1));*

(ii) u *is a critical point of* J *and* $\phi=\Phi(u)$.

Proof. $(ii)\Rightarrow(i)$ Obviously.

$(i)\Rightarrow(ii)$ Let $\mathscr{E}'_u(u,\phi)$ and $\mathscr{E}'_\phi(u,\phi)$ denote the partial derivatives of $\mathscr{E}$ at $(u,\phi)\in H_0^1(\Omega)\times\mathscr{D}(\Omega)$. Then for every $v\in H_0^1(\Omega)$ and $\psi\in\mathscr{D}(\Omega)$, we have

$$\mathscr{E}'_u(u,\phi)[v]=\int_\Omega\left[\nabla u\cdot\nabla v+[m^2-(\omega+\phi)^2]uv-f(x,u)v\right]dx, \tag{2.4}$$

$$\mathscr{E}'_\phi(u,\phi)[\psi]=\int_\Omega\left[\frac{1}{4\pi}(\nabla\phi\cdot\nabla\psi+\beta|\nabla\phi|^2\nabla\phi\cdot\nabla\psi)+(\omega+\phi)]u^2\psi\right]dx. \tag{2.5}$$

By standard computations, using the hypotheses (f_0) and (f_1), we can prove that $\mathscr{E}'_u(u,\phi)$ and $\mathscr{E}'_\phi(u,\phi)$ are continuous. Therefore, $\mathscr{E}\in C^1(H_0^1(\Omega)\times\mathscr{D}(\Omega))$. From (2.4) and (2.5), it is easy to obtain that its critical points are solutions of problem (1.1). By Proposition 1, we know that $\phi=\Phi(u)$.

3. Proof of main results

LEMMA 3. *Suppose f satisfies (f_0)-(f_3), the functional J satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that

$$J(u_n) \text{ bounded and } J'(u_n) \to 0. \tag{3.1}$$

Hence, by hypothesis (f_3) and (3.1), we have

$$\begin{aligned}
M + \|u_n\| &\geqslant \mu J(u_n) - \langle J'(u_n), u_n\rangle \\
&= \Big(\frac{\mu}{2} - 1\Big)\int_\Omega \big(|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2\big)dx + \int_\Omega \big(f(x,u_n)u_n \\
&\quad - \mu F(x,u_n)\big)dx + \int_\Omega \Big(\frac{\mu}{8\pi}|\nabla\Phi(u_n)|^2 + \frac{3\beta\mu}{16\pi}|\nabla\Phi(u_n)|^4 \\
&\quad + \frac{\mu}{2}\Phi(u_n)^2u_n^2 + \Phi(u_n)^2u_n^2 + 2\omega\Phi(u_n)^2u_n^2\Big)dx \\
&\geqslant \Big(\frac{\mu}{2} - 1\Big)\int_\Omega \big(|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2\big)dx + \int_\Omega \big(2\omega\Phi(u_n)u_n^2 dx \\
&\quad + \Phi(u_n)^2u_n^2\big)dx - C \\
&\geqslant \Big(\frac{\mu}{2} - 1\Big)\int_\Omega \big(|\nabla u_n|^2 + (m^2 - \omega^2)u_n^2\big)dx - \omega^2\int_\Omega u_n^2 dx - C \\
&= \Big(\frac{\mu}{2} - 1\Big)\int_\Omega |\nabla u_n|^2 dx + [(\frac{\mu}{2} - 1)m^2 - \frac{\mu}{2}\omega^2]\int_\Omega u_n^2 dx - C,
\end{aligned}$$

where $M > 0$ and $C > 0$ are constants. We denote

$$g_n = \Big(\frac{\mu}{2} - 1\Big)\int_\Omega |\nabla u_n|^2 dx + \Big[\Big(\frac{\mu}{2} - 1\Big)m^2 - \frac{\mu}{2}\omega^2\Big]\int_\Omega u_n^2 dx.$$

If

$$\Big(\frac{\mu}{2} - 1\Big)m^2 - \frac{\mu}{2}\omega^2 \geqslant 0, \quad \text{i.e. } m^2 \geqslant \frac{\mu}{\mu - 2}\omega^2,$$

it is easy to get that

$$g_n \geqslant \Big(\frac{\mu}{2} - 1\Big)\int_\Omega |\nabla u_n|^2 dx.$$

Thus, the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. If

$$\Big(\frac{\mu}{2} - 1\Big)m^2 - \frac{\mu}{2}\omega^2 < 0, \quad \text{i.e. } m^2 < \frac{\mu}{\mu - 2}\omega^2,$$

by the characterization of the first eigenvalue λ_1 of eigenvalue problem $\big(-\Delta, H_0^1(\Omega)\big)$, we have

$$g_n \geqslant \left\{\Big(\frac{\mu}{2} - 1\Big) + \frac{[(\frac{\mu}{2} - 1)m^2 - \frac{\mu}{2}\omega^2]}{\lambda_1}\right\}\int_\Omega |\nabla u_n|^2 dx.$$

From the hypothesis $\frac{\mu}{\mu-2}\omega^2-\lambda_1<m^2$, we also obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Next, we shall show that u_n is strongly convergence in $H_0^1(\Omega)$. In fact, By the boundedness of u_n in $H_0^1(\Omega)$, going if necessary to a subsequence, we may assume that there exists $u\in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H_0^1(\Omega), \\ u_n \to u, & \text{in } L^p(\Omega) \text{ for } 2\leqslant p<2^*. \end{cases} \tag{3.2}$$

Since

$$\begin{aligned}\langle J'(u_n),u_n-u\rangle &= \int_\Omega \left[\nabla u_n\cdot\nabla(u_n-u)+(m^2-(\omega+\Phi(u_n))^2)u_n(u_n-u)\right]dx \\ &\qquad -\int_\Omega f(x,u_n)(u_n-u)dx \\ &= \int_\Omega |\nabla(u_n-u)|^2dx-\int_\Omega \nabla u\cdot\nabla(u_n-u)dx-\int_\Omega f(x,u_n)(u_n-u)dx \\ &\qquad +\int_\Omega (m^2-(\omega+\Phi(u_n))^2)u_n(u_n-u)]dx \\ &= \int_\Omega |\nabla(u_n-u)|^2dx+\mathbb{I}+\mathbb{II}+\mathbb{III}.\end{aligned}$$

By the weak continuity of u_n in $H_0^1(\Omega)$, it is easy to obtain that $\mathbb{I}\to 0$. By Proposition 1, Hölder's inequality and (3.2), we can get that $\mathbb{II}\to 0$. By hypothesis (f_1), Hölder's inequality and (3.2), we also can get that $\mathbb{III}\to 0$. Therefore, by (3.1), is is not difficult to conclude that $u_n\to u$ in $H_0^1(\Omega)$.

Using $E_u(\Phi(u))\leqslant 0$, it is easy to verify that

$$\begin{aligned}\frac{1}{2}\int_\Omega(|\nabla u|^2+m^2u^2)dx-\int_\Omega F(x,u(x))dx &\geqslant J(u) \\ &\geqslant \frac{1}{2}\int_\Omega \left(|\nabla u|^2+(m^2-\omega^2)u^2\right)dx \\ &\qquad -\int_\Omega F(x,u(x))dx.\end{aligned}$$

Indeed,

$$\begin{aligned}J(u) &= \int_\Omega\left[\frac{1}{2}|\nabla u|^2+\frac{1}{2}(m^2-(\omega+\Phi(u))^2)u^2-\frac{1}{8\pi}|\nabla\Phi|^2\right. \\ &\qquad \left.-\frac{\beta}{16\pi}|\nabla\Phi|^4-F(x,u)\right]dx \\ &\leqslant \int_\Omega\left[\frac{1}{2}|\nabla u|^2+\frac{1}{2}(m^2-(\omega+\Phi(u))^2)u^2-F(x,u)\right]dx \\ &\leqslant \frac{1}{2}\int_\Omega(|\nabla u|^2+m^2u^2)dx-\int_\Omega F(x,u(x))dx,\end{aligned}$$

hence, the left inequality holds. On the other hand, the right inequality can be obtained by the fact that $E_u(\Phi(u)) \leqslant 0$. From hypotheses (f_1)-(f_3), we can get the following Lemma.

LEMMA 4. *The functional J satisfies the Mountain Pass geometry structure, that is:*

(i) *there exist $\alpha > 0$ and $\rho > 0$ such that $J(u) \geqslant \alpha$, for $\|u\| = \rho$;*
(ii) *there exists $\tilde{u} \in H_0^1(\Omega)$ such that $J(\tilde{u}) < 0$ and $\|\tilde{u}\| > \rho$.*

Proof. By the hypotheses (f_1) and (f_3), and by the standard arguments, the conclusion (ii) holds. We only need to show the conclusion (i) is true. In fact, if $m^2 \geqslant \omega^2$, by the hypotheses (f_1) and (f_2), and by the standard arguments, we can prove that (i) holds. If $m^2 < \omega^2$, we have

$$\begin{aligned} J(u) &\geqslant \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + (m^2-\omega^2)u^2\right)dx - \int_\Omega F(x,u(x))dx \\ &\geqslant \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2}\frac{\omega^2-m^2}{\lambda_1}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u(x))dx \\ &= \frac{\lambda_1-\omega^2+m^2}{2\lambda_1}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u(x))dx. \end{aligned}$$

By $m^2 > \frac{\mu}{\mu-2}\omega^2 - \lambda_1$, we get $m^2 > \omega^2 - \lambda_1$, that is, $\lambda_1 - \omega^2 + m^2 > 0$. Therefore, by the similar argument as the case of $m^2 \geqslant \omega^2$, the conclusion (i) can be completed.

PROOF OF THEOREM 1 Combining with Lemma 3 and Lemma 4, and we can apply Theorem 2.2 in [12], the conclusion is completed.

PROOF OF THEOREM 2 From Remark (2) and Remark (1), we know that the functional J is even function and $J(0) = 0$. By modifying the proof of (ii) in Lemma 4 and combining with Lemma 4, using Theorem 9.2 in [12], we can complete the proof.

REFERENCES

[1] A. AZZOLLINI, L. PISANI, A. POMPONIO, *Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system*, to appear on Proc. Roy. Soc. Edinburgh Sect. A.

[2] A. AZZOLLINI, A. POMPONIO, *Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations*, Topol. Methods Nonlinear Anal., **35** (2010), 33–42.

[3] V. BENCI, D. FORTUNATO, *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Rev. Math. Phys., **4** (2002), 409–420.

[4] V. BENCI, P. D'AVENIA, D. FORTUNATO, L. PISANI, *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Rational Mech. Anal., **154** (2000), 297–324.

[5] M. BORN, L. INFELD, *Foundations of the new field theory*, Proc. R. Soc. Lond. A, **144** (1934), 425–451.

[6] T. D'APRILE, D. MUGNAI, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. R. Soc. Edinb. Sect. A, **134** (2004), 1–14.

[7] T. D'APRILE, D. MUGNAI, *Non-Existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud., **4** (2004), 307–322.

[8] P. D' AVENIA, L. PISANI, *Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations*, Electron. J. Diff. Eqns, **26** (2002), 1–13.

[9] D. FORTUNATO, L. ORSINA, L. PISANI, *Born-Infeld type equations for electrostatic fields*, J. Math. Phys., **43** (2002), 5698–5706.

[10] D. MUGNAI, *Coupled Klein-Gordon and Born-Infeld type equations: looking for solitary waves*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci, **460** (2004), 1519–1528.

[11] J. POLCHINSKI, *TASI lectures on D-branes*, arXiv:hep-th/9611050; R. ARGURIO, *Brane physics in M-theory*, hep-th/9807171; K.G. SAVVIDY, *Born-Infeld action in string theory*, hep-th/9906075.

[12] P. H. RABINOWITZ, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, American Mathematical Society, CBMS Regional Conference Series in Mathematics, 1986.

[13] W. A. STRAUSS, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., **55** (1977), 149–162.

[14] M. STRUWE, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, 3rd edition, Springer-Verlag, 2008.

[15] K. M. TENG, K. J. ZHANG, *Existence of solitary wave solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical Sobolev exponent*, Nonlinear Analysis, **74** (2011), 4241–4251.

[16] M. WILLEM, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, **24**, Birkhäuser, Boston, 1996.

[17] Y. YU, *Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory*, Ann. I. Poincaré, **27** (2010), 351–376.

(Received October 15, 2011)

Kaimin Teng
Department of Mathematics
Taiyuan University of Technology
Taiyuan, Shanxi 030024
P. R. China
e-mail: tengkaimin@yahoo.com.cn