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EXISTENCE OF POSITIVE SOLUTIONS FOR A FOURTH ORDER DIFFERENTIAL INCLUSION

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Abstract. We prove an existence result for positive solutions of a fourth order differential inclusion. The proof is accomplished through the use of Green's functions and a fixed point theorem. One of the technical assumptions is explored in detail.

1. Introduction and statement of the main results

In this article, we prove the following:

THEOREM 1. *Let* $F:[0,1]\times\mathbb{R}\to\mathscr{P}([0,\infty))$ *be compact- and convex-valued, Lebesgue measurable in* t *for each* x, *upper semicontinuous in* x *for almost all* t *and for each* $r>0$ *there exists a function* $h_r\in L^1([0,1],\mathbb{R})$ *such that* $|y|\leqslant h_r(t)$ *for almost all* t, *every* $x\in\mathbb{R}$ *with* $|x|<r$ *and every* $y\in F(t,x)$. *Also, assume assumption H holds (specified later in the paper). Then, there exists a positive solution to the problem*

$$\begin{cases} u''''(t)\in F(t,u(t)),\ t\in[0,1] \\ u(0)=0,\ u'(0)=0,\ u''(1)=0,\ u'''(1)=0. \end{cases} \tag{1}$$

In this theorem, we use the following definition of a positive solution.

DEFINITION 1. A function $u:[0,1]\to\mathbb{R}$ is a positive solution to (1) if

i) $u\in AC^3([0,1],\mathbb{R})$ (by this we mean u', u'' and u''' are each absolutely continuous on $[0,1]$),
ii) $u''''(t)\in F(t,u(t))$ for almost all $t\in[0,1]$,
iii) $u(0)=0$, $u'(0)=0$, $u''(1)=0$, $u'''(1)=0$,
iv) $u(t)>0$ for all $t\in(0,1]$.

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Theorem 1 generalizes a result due to Yang [13], which proved existence of positive solutions to the problem

$$u''''(t) = g(t)f(u(t)),\, t \in [0,1]$$
$$u(0) = 0,\, u'(0) = 0,\, u''(1) = 0,\; u'''(1) = 0,$$

if $f : \mathbb{R} \to [0,\infty)$ and $g : [0,1] \to [0,\infty)$ are continuous, among other assumptions. We note that Theorem 1 greatly increases the class of problems to which Yang's theorem applies. We shall show that the basic argument of Yang still applies, with a number of modifications to account for the more general setting.

The boundary conditions are motivated in [1]. A theorem similar to Yang's result is proven in [6]. In [2], three different theorems are proven using fixed point theory for boundary value problems with a fourth-order differential inclusion with different boundary conditions than ours. Similarly in [3], two existence theorems for such problems are proven, which generalizes [11].

We shall make use of the following fixed point theorem for set-valued operators, which is a special case of Theorem 5.5 in [4].

THEOREM 2. *Let* $(X, \|\cdot\|)$ *be a Banach space over the reals, and let* $P \subseteq X$ *be a cone in* X. *Let* H_1 *and* H_2 *be real numbers such that* $H_2 > H_1 > 0$ *and let* $\Omega_i = \{u \in X \mid \|u\| < H_i\}$ *for* $i = 1,2$. *If the operator* $T : P \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to \mathscr{P}(P)$ *is compact and convex valued and is completely continuous such that either*

1) $\|w\| \leqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1, w \in T(u)$ *and* $\|w\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2, w \in T(u)$, *or*

2) $\|w\| \geqslant \|u\|$ *for* $u \in P \cap \partial\Omega_1, w \in T(u)$ *and* $\|w\| \leqslant \|u\|$ *for* $u \in P \cap \partial\Omega_2, w \in T(u)$,

then T *has a fixed point.*

2. Some lemmas

We begin with the necessary lemmas. Lemma 1 is easy to prove.

LEMMA 1. *If a function* $u \in AC^3([0,1],\mathbb{R})$ *satisfies the boundary conditions in (1) and* $u''''(t) \geqslant 0$ *for almost all* $t \in [0,1]$, *then*

$$u'''(t) \leqslant 0,\; u''(t) \geqslant 0,\; u'(t) \geqslant 0 \;\text{ and }\; u(t) \geqslant 0 \text{ for all } t \in [0,1].$$

Next, we define $a : [0,1] \to \mathbb{R}$ by

$$a(t) = \frac{3}{2}t^2 - \frac{1}{2}t^3. \tag{2}$$

LEMMA 2. *If* $u \in AC^3([0,1],\mathbb{R})$ *satisfies the hypotheses of Lemma 1, then*

$$a(t)u(1) \leqslant u(t) \leqslant tu(1) \text{ for all } t \in [0,1].$$

Proof. This is the same as Lemma 2.2 in [13], except there he assumed $u \in C^4([0,1])$ and $u''''(t) \geqslant 0$ for all $0 \leqslant t \leqslant 1$. Our change basically requires no modifications in Yang's proof. □

LEMMA 3. *Let* $F : [0,1] \times \mathbb{R} \to \mathscr{P}([0,\infty))$. *Assume* u *is a nonnegative solution of* (1). *Then, for all* $t \in [0,1]$, u *satisfies*

$$u'''(t) \leqslant 0,\ u''(t) \geqslant 0,\ u'(t) \geqslant 0 \ \text{ and } u(t) \geqslant 0,$$
$$a(t)u(1) \leqslant u(t) \leqslant tu(1).$$

Proof. Note that $u''''(t) \in F(t,u(t)) \in \mathscr{P}([0,\infty))$ a.e on $[0,1]$, so we can apply Lemmas 1 and 2. □

Denote by X the Banach space $C([0,1],\mathbb{R})$ with the max norm. Define a cone P on X by

$$P = \left\{ v \in X \mid v(1) \geqslant 0,\ a(t)v(1) \leqslant v(t) \leqslant tv(1) \ \text{ for all } t \in [0,1] \right\}.$$

From Lemma 3, we have:

LEMMA 4. *Let* $F : [0,1] \times \mathbb{R} \to \mathscr{P}([0,\infty))$. *Assume* u *is a nonnegative solution of* (1). *Then,* $u \in P$.

LEMMA 5. *If* $u \in P$, *then* $u(1) = \|u\|$.

Proof. Follows quickly from the definition of P. □

Now let G denote the Green's function for problem (1) (see [13]). For each $u \in X$, let $S(u) = \left\{ v \in L^1([0,T],\mathbb{R}) \mid v(t) \in F(t,u(t)) \text{ a.e. on } [0,1] \right\}$. We note that $S(u)$ is well-known to be nonempty for each $u \in X$ under the conditions of Theorem 1 - see for example p. 227-228 of [9]. We define an operator $T : P \to \mathscr{P}(X)$ by

$$T(u) = \left\{ w \in X \mid w(t) = \int_0^1 G(t,s)v(s)ds \text{ where } v \in S(u) \right\}. \tag{3}$$

We need only verify that T has a fixed point as in [3], [5], [12] and others. From Theorem 3.2 in [8] we have that T is convex and compact valued and completely continuous.

LEMMA 6. $T(P) \subseteq P$.

Proof. Let $u \in P$ and $w \in Tu$. From (3) it follows that w satisfies the boundary conditions in (1). Also, from (3) we have $w''''(t) = v(t) \in F(t,u(t))$ a.e. on $[0,1]$ and hence $w'''' \geqslant 0$ a.e. on $[0,1]$. Lemma 6 then follows from Lemma 2. □

We define the following constants:

$$A = \int_0^1 G(1,s)a(s)ds \quad \text{and} \quad B = \int_0^1 G(1,s)ds.$$

In [13] (see also [5]), the following constants were defined:

$$\limsup_{x\to 0^+}\frac{f(x)}{x}=F_0 \quad \text{and} \quad \liminf_{x\to\infty}\frac{f(x)}{x}=f_\infty$$

and it was assumed that

$$BF_0<1<Af_\infty.$$

From this, we deduce that he intended for F_0 to be finite, and from his proof it is clear that f_∞ is also assumed to be finite. In [7], this is extended to continuous f's which are time-dependent as follows:

$$\limsup_{x\to 0^+}\max_{t\in[0,1]}\frac{f(t,x)}{x}=F_0 \quad \text{and} \quad \liminf_{x\to\infty}\min_{t\in[0,1]}\frac{f(t,x)}{x}=f_\infty.$$

We note that [7] allows $f_\infty=\infty$. For our purposes, the important thing about the first of these is that it implies the condition:

$$\begin{cases}\text{for all } \varepsilon>0 \text{ there exists } H>0 \text{ such that for all } t\in[0,1] \text{ and all } x\in(0,H),\\ f(t,x)\leqslant(F_0+\varepsilon)x.\end{cases} \tag{4}$$

(A similar comment holds for f_∞.) How should this be extended to f's which are only Lebesgue measurable in t? It first appears that the natural choice would be

$$\limsup_{x\to 0^+}\operatorname{ess\,sup}_{t\in[0,1]}\frac{f(t,x)}{x}=F_0 \quad \text{and} \quad \liminf_{x\to\infty}\operatorname{ess\,inf}_{t\in[0,1]}\frac{f(t,x)}{x}=f_\infty.$$

However, if we rewrite this choice of F_0 in the form of (4), we get:

$$\begin{cases}\text{for all } \varepsilon>0 \text{ there exists } H>0 \text{ such that for all } x\in(0,H),\\ \operatorname{ess\,sup}_{t\in[0,1]}f(t,x)\leqslant(F_0+\varepsilon)x.\end{cases}$$

The problem with this is in defining $\operatorname{ess\,sup}_{t\in[0,1]}f(t,x)$: the choice of the set of full measure on which the supremum is taken is dependent on x. Hence this is not quite the same idea as (4). What we really need is the following:

$$\begin{cases}\text{for all } \varepsilon>0 \text{ there exists } H>0 \text{ such that there is a set of full measure } S\\ \text{such that for all } t\in S \text{ and all } x\in(0,H),\ f(t,x)\leqslant(F_0+\varepsilon)x.\end{cases}$$

We denote this condition as

$$\limsup_{x\to 0^+}\frac{f(t,x)}{x}=F_0 \ \text{ uniformly for almost all } t\in[0,1]. \tag{5}$$

The paper [12] also considers the case in which f is measurable in t. In [10], a variation of Li's assumption was made for the continuous f case. In place of the F_0 condition, it was assumed that there exists a function $c\in C([0,1],(0,\infty))$ such that

$$\lim_{|x|\to 0}\frac{f(t,x)}{x}=c(t) \text{ for all } t\in[0,1].$$

In this spirit, and noting that the proof really only requires an inequality, we generalize (5) as:

$$\begin{cases} \text{there exists } c \in L^1([0,1],[0,\infty)) \text{ such that} \\ \limsup_{x\to 0^+} \frac{f(t,x)}{x} \leqslant c(t) \text{ uniformly for almost all } t \in [0,1]. \end{cases}$$

Finally, extending to set-valued functions, we obtain the following assumption H:

$$\begin{cases} \text{there exists } c \in L^1\left([0,1],[0,\infty)\right) \text{ such that} \\ \limsup_{x\to 0^+} \sup\left\{\frac{y}{x} : y \in F(t,x)\right\} \leqslant c(t) \text{ uniformly for almost all } t \in [0,1], \end{cases} \tag{6}$$

$$\begin{cases} \text{there exists a measurable } d : [0,1] \to [0,\infty] \text{ such that} \\ \liminf_{x\to\infty} \inf\left\{\frac{y}{x} : y \in F(t,x)\right\} \geqslant d(t) \text{ uniformly for almost all } t \in [0,1] \end{cases} \tag{7}$$

and

$$\int_0^1 G(1,s)c(s)ds < 1 < \int_0^1 G(1,s)a(s)d(s)ds.$$

For clarity, we emphasize that (6) should be interpreted as there exists a function $c \in L^1([0,1],[0,\infty))$ such that for all $\varepsilon > 0$ there exists $H > 0$ such that there is a set of full measure S such that for all $t \in S$, all $x \in (0,H)$ and all $y \in F(t,x)$, $y \leqslant [c(t)+\varepsilon]x$, and analogously for (7). Note that it can be shown that the corresponding assumption in [13] is a special case of this.

3. The proof of Theorem 1

Now we have the necessary pieces to prove Theorem 1.

PROOF OF THEOREM 1. We shall show that Theorem 2 applies. Choose $\varepsilon > 0$. From assumption H, we know that there exists a $H_1 > 0$ such that

$$\frac{y}{x} \leqslant c(t) + \varepsilon \tag{8}$$

for almost all $t \in [0,1]$, all $x \in (0,H_1]$ and all $y \in F(t,x)$. Choose $u \in P$ with $\|u\| = H_1$ and let $w \in T(u)$. Then, there exists a $v \in S(u)$ (see (3)) such that

$$w(t) = \int_0^1 G(t,s)v(s)ds.$$

Note that $v(s) \in F(s,u(s))$ a.e. on $[0,1]$, and hence $v(s) \leqslant [c(t)+\varepsilon]u(s)$ a.e. on $[0,1]$ by (8). We then have

$$w(1) = \int_0^1 G(1,s)v(s)ds \leqslant \int_0^1 G(1,s)\left[c(s)+\varepsilon\right]u(s)ds$$

$$\leqslant \|u\| \left[\int_0^1 G(1,s)c(s)ds + \varepsilon \int_0^1 G(1,s)ds \right]$$
$$< \|u\| \left[1 + \varepsilon \int_0^1 G(1,s)ds \right].$$

Since ε is arbitrary, we have $w(1) \leqslant \|u\|$ and, applying Lemma 5, we conclude $\|w\| \leqslant \|u\|$. We have thus verified the first half of requirement 1 in Theorem 2.

To verify the second half, we proceed as follows.

Case 1: $\int_0^1 G(1,s)a(s)d(s)ds < \infty$.

Since $\int_0^1 G(1,s)a(s)d(s)ds > 1$, it is possible to choose $\delta > 0$ and $c \in (0,1)$ such that

$$\int_c^1 G(1,s)d(s)a(s)ds - \delta \int_c^1 G(1,s)a(s)ds > 1. \tag{9}$$

We know from Assumption H that there exists a $H > 0$ such that

$$\frac{y}{x} \geqslant d(t) - \delta \tag{10}$$

for almost all $t \in [0,1]$, all $x \in [H,\infty)$ and all $y \in F(t,x)$. Choose $K \in L^1([0,1])$ defined by

$$K(t) = d(t) - \delta.$$

Case 2: $\int_0^1 G(1,s)a(s)d(s)ds = \infty$.

Choose $k \in L^1([0,1])$, $\delta > 0$ and $c \in (0,1)$ such that $k(t) \leqslant d(t)$ a.e. on $[0,1]$ and

$$\int_c^1 G(1,s)k(s)a(s)ds - \delta \int_c^1 G(1,s)a(s)ds > 1. \tag{11}$$

From the definition of f_∞, we know that there exists a $H > 0$ such that

$$\frac{y}{x} \geqslant k(t) - \delta \tag{12}$$

for almost all $t \in [0,1]$, all $x \in [H,\infty)$ and all $y \in F(t,x)$. Let

$$K(t) = k(t) - \delta.$$

Now, in the following we take K and c as in Case 1 or Case 2, let

$$H_2 = \max\left\{Hc^{-2}, 2H_1\right\}.$$

We then have $H_2 > H_1$, and for $u \in P$ with $\|u\| = H_2$ and $t \in [c,1]$, we have

$$u(t) \geqslant a(t)u(1) = a(t)H_2 \tag{13}$$

and hence

$$u(t) \geqslant H_2 t^2 \geqslant H_2 c^2 \geqslant H \quad \text{for all } t \in [c,1],$$

using the fact that a, defined in (2), satisfies $a(t) \geqslant t^2$ for all $t \in [0,1]$. Let $w \in T(u)$. Then, there exists a $v \in S(u)$ such that

$$w(t) = \int_0^1 G(t,s)v(s)ds.$$

Note that $v(s) \in F(s,u(s))$ a.e. on $[0,1]$, and hence $v(s) \geqslant K(s)u(s)$ a.e. on $[0,1]$ by (10) or (12). Then,

$$\begin{aligned} w(1) = \int_0^1 G(1,s)v(s)ds &\geqslant \int_c^1 G(1,s)v(s)ds \\ &\geqslant \int_c^1 G(1,s)K(s)u(s)ds \\ &\geqslant \int_c^1 G(1,s)K(s)H_2a(s)ds \text{ (by (13))} \\ &= \|u\| \int_c^1 G(1,s)K(s)a(s)ds \\ &> \|u\| \text{ (by (9) or (11)).} \end{aligned}$$

Applying Lemma 5, we conclude $\|w\| \geqslant \|u\|$, verifying the second half of 1 in Theorem 2. Applying Theorem 2, we reach our desired conclusion. □

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