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EXISTENCE RESULTS FOR SECOND ORDER THREE–POINT BOUNDARY VALUE PROBLEMS

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Abstract. The paper is devoted to the study of second order differential equations and systems with nonlinear three point boundary conditions. The existence of solutions is proved using fixed point theorems: Banach's and Boyd-Wong's contraction principles, Perov's and Schauder's fixed point theorems.

1. Introduction

Multi-point boundary value problems that arise from different areas of applied mathematics and physics have received a lot of attention in the literature in the last decades (see for example [11], [13], [20], [31], [33], [36], [37] and references therein). For example, a number of problems in the theory of elastic stability can be treated as a multi-point problem [44] and also the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be handled as a multi-point boundary value problem [39]. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [28], [29]. Then, Gupta [22] has studied three point boudary-value problems for nonlinear ordinary differential equations. Since then, applying various methods of nonlinear analysis, many authors studied more general nonlinear multi-point boundary value problems (we refer the reader to [15], [16], [23], [24], [25], [35], [36], [38]). For additional backgrounds and results, we refer the reader to the monograph by Agarwal, O'Regan and Wong [1], as well as to the contributions from [3], [14], [18], [19] and [34].

In paper [21], Guo discussed the existence and uniqueness of solutions of a two-point boundary value problem for second order nonlinear impulsive integro-differential equations of mixed type on an infinite interval in a Banach space E. In paper [32], Liu also studied the existence of at least one solution of a two-point boundary value

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problem for second-order nonlinear ordinary differential equations in a Banach space. Being directly inspired by [21], [32], in paper [12], by using the Sadovskii fixed point theorem, the authors study the existence of at least one solution for the second-order three-point boundary value problem

$$\begin{cases} u''(t)+f(t,u(t),u'(t))=\theta, & 0<t<1, \\ u(0)=\theta,\, u(1)=\alpha u(\eta), \end{cases}$$

in a Banach space E, where θ is the zero element of E, $I=[0,1], 0<\alpha<1, 0<\eta<\frac{1}{\alpha}$, $f\in C[I\times E\times E,E]$ and they also obtain the existence of at least one positive solution. Next, by using Krasnoselskii's fixed point theorem of cone expansion–compression type and under suitable conditions, in paper [43], Sun presents the existence of single and multiple positive solutions to the nonlinear second-order m-point boundary value problem

$$\begin{cases} u''(t)+\lambda a(t)f(u(t))=0, & 0<t<1, \\ u(0)=0,\, u(1)=\sum\limits_{i=1}^{m-2} a_i u(\xi_i), \end{cases}$$

where λ is a positive parameter, $a_i \geqslant 0$ for $i=1,2,\ldots,m-3$ and $a_{m-2}>0$, ξ_i satisfy

$$0<\xi_1<\xi_2<\ldots<\xi_{m-2}<1 \quad \text{and} \quad \sum_{i=1}^{m-2} a_i u(\xi_i)<1.$$

Here, the author manages to derive an explicit interval of λ such that for any λ in this interval, the existence of at least one positive solution to the boundary value problem is guaranteed, and the existence of at least two solutions for λ in an appropriate interval is also discussed.

On the other hand, motivated by the works in [27], [45], [46], the purpose of paper [17] is to show the existence of multiple positive solutions to the multipoint boundary-value problem for the one-dimensional p-Laplacian

$$\begin{cases} (\phi_p(u'))'+q(t)f(t,u)=0, & 0<t<1, \\ u(0)=\sum\limits_{i=1}^{m-2} a_i u(\xi_i), & u(1)=\sum\limits_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where $\phi_p(s)=|s|^{p-2}s$. By using a fixed point theorem in a cone, the authors present sufficient conditions for the existence of positive solutions.

As we have mentioned, three-point boundary-value problems for differential equations were presented and studied by many authors (see [16], [22], [26], [36] and the references cited there). However, to the author's knowledge, three-point boundary value problems for differential systems have not received as much attention as necessary in the literature.

Motivated by paper [10], in Section 2, we study the three-point boundary value problem for second order differential equations:

$$\begin{cases} u''=f(t,u,u'), & 0<t<t_0, \\ u(0)=0,\, u(t_0)=g(u(\eta)), \end{cases} \tag{1.1}$$

where $0 < \eta < t_0 < 1$, f, g are continuous functions and u is sought in $C^1[0,t_0]$. Our tools here are Banach's and Schauder's fixed point theorems. Even in the particular case $g(s) = s$, our results compared to those already published in [12], [17], [25], [35] or [43], bring to the reader some novelty elements by treating differential systems of this type using the technique based on convergent to zero matrices and vector-valued norms. Therefore, in Section 3, we discuss differential systems of the type

$$\begin{cases} u''(t) = f(t,u(t),v(t)), \\ v''(t) = g(t,u(t),v(t)), \\ u(0) = 0, \ u(t_0) = \phi(u(\eta),v(\eta)), \\ v(0) = 0, \ v(t_0) = \psi(u(\eta),v(\eta)), \end{cases} \quad (0 < t < t_0)$$

by using Perov's and Schauder's fixed point theorems (see for example [2], [41]) and the technique based on convergent to zero matrices and vector-valued norms. Section 4 is devoted to the problem

$$\begin{cases} u'' = f(t,u,u'), \quad 0 < t < 1, \\ u(0) = 0, \ u(t_0) = g(u(\eta)). \end{cases} \tag{1.2}$$

Compared to problem (1.1), even if the three-boundary condition is the same, equation (1.2) is considered on the larger interval $[0,1]$. Finally, in Section 5, a similar strategy is applied to a system of two second order differential equations.

To conclude this introduction, we recall some notions that are used in what follows. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as} \quad k \to \infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [41], [42]):

(a) $I - M$ is nonsingular and $(I-M)^{-1} = I + M + M^2 + \ldots$ (where I stands for the unit matrix of the same order as M);

(b) the eigenvalues of M are located inside the unit disc of the complex plane;

(c) $I - M$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.

Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbb{R}^n$ such that

(i) $d(u,v) \geqslant 0$ for all $u,v \in X$ and if $d(u,v) = 0$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u,v \in X$;

(iii) $d(u,v) \leqslant d(u,w) + d(w,v)$ for all $u,v,w \in X$.

Here, if $x,y \in \mathbb{R}^n$, $x = (x_1,x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n)$, by $x \leqslant y$ we mean $x_i \leqslant y_i$ for $i = 1,2,\ldots,n$. We call the pair (X,d) a *generalized metric space*. For such a space convergence and completeness are similar to those in usual metric spaces.

An operator $T : X \to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that

$$d(T(u),T(v)) \leqslant Md(u,v) \quad \text{for all } u,v \in X.$$

Also recall Banach's, Perov's and Schauder's fixed point theorems:

THEOREM 1.1. (Banach Contraction Principle) *If $T : X \to X$ is contractive on a complete metric space X then T has a unique fixed point in X.*

The analogue of Banach's Contraction Principle for generalized metric spaces is the following theorem of Perov (see [2], [41]):

THEOREM 1.2. (Perov) *Let (X,d) be a complete generalized metric space and $T : X \to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d(T^k(u_0),u^*) \leqslant M^k(I-M)^{-1}d(u_0,T(u_0)) \text{ for all } k \in \mathbf{N}.$$

THEOREM 1.3. (Schauder) *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

2. Existence results for equations

Consider problem (1.1). Here are some hypotheses:

(H1) there exist $a,b,c > 0$ such that

$$\begin{cases} |f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant a\,|u-\overline{u}| + b\,|v-\overline{v}|\,, \\ |g(u) - g(\overline{u})| \leqslant c\,|u-\overline{u}|\,, \end{cases} \tag{2.1}$$

for $t \in [0,t_0]$ and $u,v,\overline{u},\overline{v} \in \mathbb{R}$.

(H2) there exist $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2 > 0$ such that

$$|f(t,u,v)| \leqslant \alpha_1\,|u| + \alpha_2\,|v| + \alpha_3 \ \text{ and } \ |g(u)| \leqslant \beta_1\,|u| + \beta_2, \tag{2.2}$$

for $t \in [0,t_0]$ and $u,v \in \mathbb{R}$.

EXAMPLE 2.1. (a) For instance, function

$$f_1(t,u,v) = \alpha \sin 3u + \beta v$$

satisfies (2.1) for $a = 3\alpha$ and $b = \beta$.

(b) An example of function f satisfying (2.2) is

$$f_2(t,u,v) = \gamma u \cos v + \frac{1}{1+u^2}v + 1$$

with $\alpha_1 = \gamma, \alpha_2 = \alpha_3 = 1$.

Notice that any function f satisfying (2.1) also satisfies (2.2), but not conversely. Similar remarks hold for function g. Thus, condition (H1) implies condition (H2) and so, the existence results based on (H2) are more general than those obtained using (H1).

2.1. Application of Banach's Contraction Principle

We begin this section by pointing out that problem (1.1) can be written equivalently as

$$u(t) = \int_0^{t_0} G(t,s) f\left(s, u(s), u'(s)\right) ds + \frac{t}{t_0} g(u(\eta)), \tag{2.3}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} -\frac{t(t_0-s)}{t_0}, & 0 \leqslant t \leqslant s \leqslant t_0, \\ -\frac{s(t_0-t)}{t_0}, & 0 \leqslant s \leqslant t \leqslant t_0. \end{cases} \tag{2.4}$$

We observe that u is a solution of (1.1) if and only if u is a fixed point of the operator $T : C^1[0,t_0] \to C^1[0,t_0]$, defined by

$$(Tu)(t) = \int_0^{t_0} G(t,s) f\left(s, u(s), u'(s)\right) ds + \frac{t}{t_0} g(u(\eta)), \tag{2.5}$$

where $C^1[0,t_0]$ denotes the space of all continuously differentiable functions defined on $[0,t_0]$, equipped with the norm

$$\|u\|_{C^1[0,t_0]} = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty\right\}.$$

Here, $\|w\|_\infty = \max_{0 \leqslant t \leqslant t_0} |w(t)|$. The space $C^1[0,t_0]$ is a Banach space with respect to the norm $\|u\|_{C^1[0,t_0]}$ (see, e.g., [40], pp. 148-149).

THEOREM 2.1. *If* f, g *satisfy* (H1) *with*

$$\frac{a+b}{2} t_0 + \frac{c}{t_0} < 1, \tag{2.6}$$

then problem (1.1) *has a unique solution. Moreover, this solution can be obtained as limit of the sequence of succesive approximations.*

Proof. Using (H1) we have:

$$\begin{aligned} |T(u)(t) - T(\overline{u})(t)| = | & \int_0^{t_0} G(t,s) f(s, u(s), u'(s)) ds + \frac{t}{t_0} g(u(\eta)) \\ & - \int_0^{t_0} G(t,s) f(s, \overline{u}(s), \overline{u}'(s)) ds - \frac{t}{t_0} g(\overline{u}(\eta))| \\ \leqslant & \int_0^{t_0} |G(t,s)| \cdot |f(s, u(s), u'(s)) - f(s, \overline{u}(s), \overline{u}'(s))| ds \\ & + \frac{t}{t_0} |g(u(\eta) - g(\overline{u}(\eta))| \\ \leqslant & \int_0^{t_0} |G(t,s)| \cdot (a|u(s) - \overline{u}(s)| + b\left|u'(s) - \overline{u}'(s)\right|) ds \end{aligned}$$

$$+\frac{t}{t_0}\cdot c\,|u(\eta)-\overline{u}(\eta)|\,.$$

Moreover,

$$\max_{t\in[0,t_0]}\int_0^{t_0}|G(t,s)|\,ds=\max_{t\in[0,t_0]}\frac{t\,(t_0-t)}{2}=\frac{t_0^2}{8}. \tag{2.7}$$

Taking the maximum, since $t\leqslant t_0,$ it follows that

$$\begin{aligned}\|T(u)-T(\overline{u})\|_\infty &\leqslant a\frac{t_0^2}{8}\|u-\overline{u}\|_\infty+b\frac{t_0^2}{8}\left\|u'-\overline{u}'\right\|_\infty+c\,\|u-\overline{u}\|_\infty\\ &\leqslant\left(\frac{a+b}{8}t_0^2+c\right)\|u-\overline{u}\|_{C^1[0,t_0]}\,.\end{aligned}$$

Furthermore,

$$(Tu)^{'}(t)=\int_0^{t_0}G_t(t,s)f\left(s,u(s),u'(s)\right)ds+\frac{1}{t_0}g(u(\eta)),$$

where

$$G_t(t,s)=\frac{\partial G}{\partial t}(t,s)=\begin{cases}\frac{s}{t_0}-1, & 0\leqslant t\leqslant s\leqslant t_0,\\ \frac{s}{t_0}\quad\ \ , & 0\leqslant s\leqslant t\leqslant t_0.\end{cases}$$

Then

$$\begin{aligned}\left|(Tu)^{'}(t)-(T\overline{u})^{'}(t)\right| &=\Big|\int_0^{t_0}G_t(t,s)f\left(s,u(s),u'(s)\right)ds+\frac{1}{t_0}g(u(\eta))\\ &\qquad-\int_0^{t_0}G_t(t,s)f\left(s,\overline{u}(s),\overline{u}'(s)\right)ds-\frac{1}{t_0}g(\overline{u}(\eta))\Big|\\ &\leqslant\int_0^{t_0}|G_t(t,s)|\cdot\left|f\left(s,u(s),u'(s)\right)-f\left(s,\overline{u}(s),\overline{u}'(s)\right)\right|ds\\ &\qquad+\frac{1}{t_0}|g(u(\eta)-g(\overline{u}(\eta))|\\ &\leqslant\int_0^{t_0}|G_t(t,s)|\cdot(a\,|u(s)-\overline{u}(s)|+b\left|u'(s)-\overline{u}'(s)\right|)ds\\ &\qquad+\frac{1}{t_0}\cdot c\,|u(\eta)-\overline{u}(\eta)|\,.\end{aligned}$$

Since

$$\int_0^{t_0}|G_t(t,s)|\,ds=\int_0^{t}\frac{s}{t_0}ds+\int_0^{t_0}\left(\frac{s}{t_0}-1\right)ds\leqslant\frac{t_0}{2}\quad\text{for } t\in[0,t_0],$$

it follows that

$$\left|(Tu)^{'}(t)-(T\overline{u})^{'}(t)\right|\leqslant a\frac{t_0}{2}\|u-\overline{u}\|_\infty+b\frac{t_0}{2}\left\|u'-\overline{u}'\right\|_\infty+\frac{c}{t_0}\|u-\overline{u}\|_\infty\,.$$

Hence

$$\left\|(Tu)^{'}-(T\overline{u})^{'}\right\|_\infty\leqslant a\frac{t_0}{2}\|u-\overline{u}\|_\infty+b\frac{t_0}{2}\left\|u'-\overline{u}'\right\|_\infty+\frac{c}{t_0}\|u-\overline{u}\|_\infty$$

$$\leqslant \left(\frac{a+b}{2}t_0+\frac{c}{t_0}\right)\|u-\overline{u}\|_{C^1[0,t_0]}.$$

Therefore

$$\|Tu-T\overline{u}\|_{C^1[0,t_0]}\leqslant \max\left\{\frac{a+b}{8}t_0^2+c,\frac{a+b}{2}t_0+\frac{c}{t_0}\right\}\|u-\overline{u}\|_{C^1[0,t_0]}.$$

Since $t_0\leqslant 1,$ we have

$$\frac{a+b}{8}t_0^2+c\leqslant \frac{a+b}{8}t_0^2+\frac{c}{t_0}\leqslant \frac{a+b}{2}t_0+\frac{c}{t_0},$$

and hence

$$\max\left\{\frac{a+b}{8}t_0^2+c,\frac{a+b}{2}t_0+\frac{c}{t_0}\right\}=\frac{a+b}{2}t_0+\frac{c}{t_0}.$$

Thus we obtain that

$$\|Tu-T\overline{u}\|_{C^1[0,t_0]}\leqslant \left(\frac{a+b}{2}t_0+\frac{c}{t_0}\right)\|u-\overline{u}\|_{C^1[0,t_0]}.$$

Since

$$\frac{a+b}{2}t_0+\frac{c}{t_0}<1,$$

then T is a contraction and Banach's Contraction Principle can be applied.

2.2. Application of Schauder's fixed point theorem

Under the weaker hypothesis (H2), we have the following existence result as a consequence of Schauder's fixed point theorem.

THEOREM 2.2. *Assume that* (H2) *holds with*

$$\frac{\alpha_1+\alpha_2}{2}t_0+\frac{\beta_1}{t_0}<1. \tag{2.8}$$

Then problem (1.1) *has at least one solution.*

Proof. We show that T has a fixed point in a set of the form

$$B=\left\{u\in C^1[0,t_0]: u(0)=0 \text{ and } \|u\|_{C^1[0,t_0]}\leqslant R\right\}$$

with a suitable $R>0.$ First note that, for $u\in B,$ $u(0)=0$ and so

$$u(t)=\int_0^{t_0}u'(s)ds,\quad 0\leqslant t\leqslant t_0.$$

Then,

$$\|u\|_\infty\leqslant t_0\left\|u'\right\|_\infty\leqslant \left\|u'\right\|_\infty$$

and hence

$$\|u\|_{C^1[0,t_0]} = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty\right\} = \left\|u'\right\|_\infty.$$

In addition, B is a closed convex subset of $C^1[0,t_0]$. Now, since

$$\int_0^{t_0} |G_t(t,s)|\, ds \leqslant \frac{t_0}{2}$$

we see that

$$\begin{aligned}\left|(Tu)'(t)\right| &= \left|\int_0^{t_0} G_t(t,s) f\left(s,u(s),u'(s)\right) ds + \frac{1}{t_0} g(u(\eta))\right| \\ &\leqslant \frac{t_0}{2} \max_{0\leqslant t\leqslant t_0} \left|f\left(s,u(s),u'(s)\right)\right| + \frac{1}{t_0}|g(u(\eta))| \\ &\leqslant \frac{t_0}{2}\left(\alpha_1 \|u\|_\infty + \alpha_2 \left\|u'\right\|_\infty + \alpha_3\right) + \frac{1}{t_0}\left(\beta_1 \|u\|_\infty + \beta_2\right).\end{aligned}$$

Then, we have for $u \in B$,

$$\|Tu\|_{C^1[0,t_0]} = \max_{0\leqslant t\leqslant t_0} \left|(Tu)'(t)\right| \leqslant \left(\frac{\alpha_1+\alpha_2}{2} t_0 + \frac{\beta_1}{t_0}\right) \|u\|_{C^1[0,t_0]} + \frac{\alpha_3}{2} t_0 + \frac{\beta_2}{t_0}.$$

If $\|u\|_{C^1[0,t_0]} \leqslant R$, then

$$\|Tu\|_{C^1[0,t_0]} \leqslant \left(\frac{\alpha_1+\alpha_2}{2} t_0 + \frac{\beta_1}{t_0}\right) R + \frac{\alpha_3}{2} t_0 + \frac{\beta_2}{t_0}$$

and if

$$\left(\frac{\alpha_1+\alpha_2}{2} t_0 + \frac{\beta_1}{t_0}\right) R + \frac{\alpha_3}{2} t_0 + \frac{\beta_2}{t_0} \leqslant R, \tag{2.9}$$

then T maps B into itself. A number $R > 0$ satisfying (2.9) exists in view of (2.8). Furthermore, from the Arzèla-Ascoli Theorem, we have that T is a completely continuous operator in B. This fact can be justified as follows: under the assumption of continuity of f and g, the operator T is continuous from $C^1[0,t_0]$ to $C^2[0,t_0]$ and maps any bounded set M of $C^1[0,t_0]$ into a bounded set $T(M)$ of $C^2[0,t_0]$. Furthermore, the embedding of $C^2[0,t_0]$ into $C^1[0,t_0]$ is compact according to the Arzèla-Ascoli theorem (see, e.g., [41], pp. 15-18). Consequently, $T(M)$ is relatively compact in $C^1[0,t_0]$. Then, T is a completely continuous operator as we claimed. Hence, T has a fixed point by Schauder's fixed point theorem.

2.3. Application of Boyd-Wong's fixed point theorem

First, we recall the Boyd-Wong Contraction Principle (see, e.g., [7]):

THEOREM 2.3. (Boyd-Wong Contraction Principle) *Let X be a complete metric space and suppose $T : X \to X$ satisfies:*

$$d(Tx,Ty) \leqslant \Psi(d(x,y)) \quad \textit{for each } x,y \in X,$$

where $\Psi : [0,\infty) \to [0,\infty), 0 \leqslant \Psi(t) < t$ *for* $t > 0$ *and* Ψ *is upper semicontinuous from the right, that is,* $r_j \searrow r \geqslant 0$ *implies* $\limsup_{j\to\infty} \Psi(r_j) \leqslant \Psi(r)$*. Then* T *has a unique fixed point* x^* *and* $\{T^n(x)\}$ *converges to* x^* *for each* $x \in X$*.*

In this section, instead of the Lipschitz condition on f from (H1), we shall consider more generally conditions of Boyd-Wong type, namely:

(H3) there exist $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ upper semicontinuous from the right and non-decreasing, and $c > 0$ such that

$$\begin{cases} |f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant \psi_1(|u-\overline{u}|) + \psi_2(|v-\overline{v}|), \\ |g(u) - g(\overline{u})| \leqslant c\,|u-\overline{u}|, \end{cases} \tag{2.10}$$

for $t \in [0,t_0]$ and $u, \overline{u}, v, \overline{v} \in \mathbb{R}$.

THEOREM 2.4. *If* f, g *satisfy satisfy* (H3) *and*

$$\Psi(t) := \max\left\{ \frac{t_0^2}{8}(\psi_1+\psi_2)(t) + ct, \frac{t_0}{2}(\psi_1+\psi_2)(t) + \frac{c}{t_0}t \right\} < t, \tag{2.11}$$

then problem (1.1) *has a unique solution. Moreover, this solution can be obtained as limit of the sequence of succesive approximations.*

Proof. Using (H3) we have:

$$\begin{aligned} |T(u)(t) - T(\overline{u})(t)| &= \left| \int_0^{t_0} G(t,s) f\left(s, u(s), u'(s)\right) ds + \frac{t}{t_0} g(u(\eta)) \right. \\ &\qquad \left. - \int_0^{t_0} G(t,s) f\left(s, \overline{u}(s), \overline{u}'(s)\right) ds - \frac{t}{t_0} g(\overline{u}(\eta)) \right| \\ &\leqslant \int_0^{t_0} |G(t,s)| \cdot \left| f\left(s, u(s), u'(s)\right) - f\left(s, \overline{u}(s), \overline{u}'(s)\right) \right| ds \\ &\qquad + \frac{t}{t_0} \left| g(u(\eta) - g(\overline{u}(\eta)) \right| \\ &\leqslant \int_0^{t_0} |G(t,s)| \cdot \left(\psi_1\left(|u(s) - \overline{u}(s)|\right) + \psi_2\left(\left|u'(s) - \overline{u}'(s)\right|\right)\right) ds \\ &\qquad + \frac{t}{t_0} \cdot c\,|u(\eta) - \overline{u}(\eta)|. \end{aligned}$$

Taking the supremum and using (2.7), we obtain

$$\begin{aligned} &\|T(u) - T(\overline{u})\|_\infty \\ &\quad \leqslant \left[\psi_1\left(\|u-\overline{u}\|_\infty\right) + \psi_2\left(\left\|u'-\overline{u}'\right\|_\infty\right)\right] \int_0^{t_0} |G(t,s)|\, ds + c\,\|u-\overline{u}\|_\infty \\ &\quad \leqslant \frac{t_0^2}{8}(\psi_1+\psi_2)\left(\|u-\overline{u}\|_{C^1[0,t_0]}\right) + c\,\|u-\overline{u}\|_{C^1[0,t_0]}. \end{aligned}$$

Similarly, we obtain that

$$\left\|(Tu)' - (T\overline{u})'\right\|_\infty \leqslant \frac{t_0}{2}(\psi_1+\psi_2)\left(\|u-\overline{u}\|_{C^1[0,t_0]}\right) + \frac{c}{t_0}\|u-\overline{u}\|_{C^1[0,t_0]}.$$

Therefore

$$\|Tu-T\overline{u}\|_{C^1[0,t_0]} \leqslant \max\left\{\frac{t_0^2}{8}(\psi_1+\psi_2)\left(\|u-\overline{u}\|_{C^1[0,t_0]}\right) + c\|u-\overline{u}\|_{C^1[0,t_0]},\right.$$
$$\left.\frac{t_0}{2}(\psi_1+\psi_2)\left(\|u-\overline{u}\|_{C^1[0,t_0]}\right) + \frac{c}{t_0}\|u-\overline{u}\|_{C^1[0,t_0]}\right\}.$$

Since

$$\Psi(t) := \max\left\{\frac{t_0^2}{8}(\psi_1+\psi_2)(t)+ct, \frac{t_0}{2}(\psi_1+\psi_2)(t)+\frac{c}{t_0}t\right\} < t, \quad \text{for all } t>0,$$

then Boyd-Wong's Contraction Principle can be applied and T has a unique fixed point.

REMARK 2.1. Theorem 2.4 is a generalization of Theorem 2.1. Indeed, for

$$\psi_1(t) = at \quad \text{and} \quad \psi_2(t) = bt,$$

condition (H3) becomes (H1) and (2.11) is satisfied if and only if (2.6) holds.

REMARK 2.2. This type of results could be obtained using conditions of the type

$$|f(t,u,v)| \leqslant \psi_1(|u|)+\psi_2(|v|)+\alpha_3 \text{ and } |g(u)| \leqslant \beta_1|u|+\beta_2,$$

for $\psi_i : [0,\infty) \to [0,\infty)$, $(i = 1,2)$ upper semicontinuous from the right and nondecreasing and $\alpha_3, \beta_1, \beta_2 > 0$. Moreover, similar arguments to those from Section 2.3 could be used for the treatment of the systems from Section 3.

3. Existence results for systems

We next deal with the three-point boundary value problem for second order differential systems of the type:

$$\begin{cases} u''(t) = f(t,u(t),v(t)), \\ v''(t) = g(t,u(t),v(t)), \quad 0<t<t_0, \\ u(0) = 0,\ u(t_0) = \phi(u(\eta),v(\eta)), \\ v(0) = 0,\ v(t_0) = \psi(u(\eta),v(\eta)). \end{cases} \tag{3.1}$$

Problem (3.1) is equivalent to the following integral system in $C[0,t_0]^2 := C[0,t_0] \times C[0,t_0]$

$$\begin{cases} u(t) = \int_0^{t_0} G(t,s)f(s,u(s),v(s))\,ds + \frac{t}{t_0}\phi(u(\eta),v(\eta)), \\ v(t) = \int_0^{t_0} G(t,s)g(s,u(s),v(s))\,ds + \frac{t}{t_0}\psi(u(\eta),v(\eta)). \end{cases}$$

This can be viewed as a fixed point problem in $C[0,t_0]^2$

$$\begin{cases} u(t) = T_1(u(t),v(t)), \\ v(t) = T_2(u(t),v(t)), \end{cases}$$

for a completely continuous operator $T = (T_1,T_2)$, $T : C[0,t_0]^2 \to C[0,t_0]^2$, where T_1,T_2 are given by

$$\begin{cases} T_1(u,v)(t) = \int_0^{t_0} G(t,s)f(s,u(s),v(s))\,ds + \frac{t}{t_0}\phi(u(\eta),v(\eta)), \\ T_2(u,v)(t) = \int_0^{t_0} G(t,s)g(s,u(s),v(s))\,ds + \frac{t}{t_0}\psi(u(\eta),v(\eta)). \end{cases}$$

3.1. Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here the existence of solutions to problem (3.1) is established by means of Perov's fixed point theorem. For this, we assume global Lipschitz conditions, that is

$$\begin{cases} |f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant a_1|u-\overline{u}| + b_1|v-\overline{v}|, \\ |g(t,u,v) - g(t,\overline{u},\overline{v})| \leqslant a_2|u-\overline{u}| + b_2|v-\overline{v}|, \\ |\phi(u,v) - \phi(\overline{u},\overline{v})| \leqslant c_1|u-\overline{u}| + d_1|v-\overline{v}|, \\ |\psi(u,v) - \psi(\overline{u},\overline{v})| \leqslant c_2|u-\overline{u}| + d_2|v-\overline{v}|, \end{cases} \tag{3.2}$$

for $t \in [0,t_0]$, $u,v,\overline{u},\overline{v} \in \mathbf{R}$ and some $a_1,b_1,a_2,b_2,c_1,d_1,c_2,d_2 > 0$.

THEOREM 3.1. *Assume that condition (3.2) holds. If matrix*

$$M := \begin{bmatrix} \frac{a_1}{8}t_0^2 + c_1 & \frac{b_1}{8}t_0^2 + d_1, \\ \frac{a_2}{8}t_0^2 + c_2 & \frac{b_2}{8}t_0^2 + d_2, \end{bmatrix} \tag{3.3}$$

converges to zero, then problem (3.1) has a unique solution in $C[0,t_0]^2$.

Proof. We shall apply Perov's fixed point theorem in $C[0,t_0]^2$ endowed with the vector-valued norm $\|\cdot\|_{C[0,t_0]}$ defined by

$$\|x\|_{C[0,t_0]} = \begin{bmatrix} \|u\|_\infty, \\ \|v\|_\infty, \end{bmatrix}$$

for $x = (u,v)$, $u,v \in C[0,t_0]$ and

$$\|u\|_\infty = \max_{0\leqslant t\leqslant t_0} |u(t)|.$$

We have to prove that T is a generalized contraction, more exactly that

$$\|T(x) - T(\overline{x})\|_{C[0,t_0]} \leqslant M\,\|x-\overline{x}\|_{C[0,t_0]}$$

for all $x = (u,v)$, $\overline{x} = (\overline{u},\overline{v}) \in C[0,t_0]^2$ and some matrix M converging to zero.

Let $(u,v),(\overline{u},\overline{v})$ be any two elements of $C[0,t_0]^2$. We have that

$$\begin{aligned}
&|T_1(u,v)(t)-T_1(\overline{u},\overline{v})(t)| \\
&\quad = \Big| \int_0^{t_0} G(t,s) f(s,u(s),v(s))\,ds + \frac{t}{t_0}\phi(u(\eta),v(\eta)) \\
&\qquad - \int_0^{t_0} G(t,s) f(s,\overline{u}(s),\overline{v}(s))\,ds - \frac{t}{t_0}\phi(\overline{u}(\eta),\overline{v}(\eta)) \Big| \\
&\quad \leqslant \int_0^{t_0} |G(t,s)| \cdot \left| f\left(s,u(s),u'(s)\right) - f\left(s,\overline{u}(s),\overline{v}(s)\right)\right| ds \\
&\qquad + \frac{t}{t_0} |\phi(u(\eta),v(\eta)) - \phi(\overline{u}(\eta),\overline{v}(\eta))| \\
&\quad \leqslant \int_0^{t_0} |G(t,s)| \cdot (a_1 |u(s)-\overline{u}(s)| + b_1 |v(s)-\overline{v}(s)|) ds \\
&\qquad + \frac{t}{t_0} \cdot (c_1 |u(\eta)-\overline{u}(\eta)| + d_1 |v(\eta)-\overline{v}(\eta)|).
\end{aligned}$$

Therefore, since $t < t_0$,

$$\|T_1(u,v)-T_1(\overline{u},\overline{v})\|_\infty \leqslant \left(\frac{a_1}{8}t_0^2 + c_1\right) \|u-\overline{u}\|_\infty + \left(\frac{b_1}{8}t_0^2 + d_1\right) \|v-\overline{v}\|_\infty. \tag{3.4}$$

Similarly,

$$\|T_2(u,v)-T_2(\overline{u},\overline{v})\|_\infty \leqslant \left(\frac{a_2}{8}t_0^2 + c_2\right) \|u-\overline{u}\|_\infty + \left(\frac{b_2}{8}t_0^2 + d_2\right) \|v-\overline{v}\|_\infty. \tag{3.5}$$

Now, (3.4) and (3.5) can be put together and be rewritten equivalently as

$$\begin{bmatrix} \|T_1(u,v)-T_1(\overline{u},\overline{v})\|_\infty \\ \|T_2(u,v)-T_2(\overline{u},\overline{v})\|_\infty \end{bmatrix} \leqslant M \begin{bmatrix} \|u-\overline{u}\|_\infty \\ \|v-\overline{v}\|_\infty \end{bmatrix}. \tag{3.6}$$

Then (3.6) is equivalent to

$$\|T(x)-T(\overline{x})\|_{C[0,t_0]} \leqslant M \|x-\overline{x}\|_{C[0,t_0]},$$

where $x=(u,v),\overline{x}=(\overline{u},\overline{v}) \in C[0,t_0]^2$. The result follows now from Perov's fixed point theorem.

3.2. Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here the existence of solutions to problem (3.1) is established by means of Schauder's fixed point theorem in case that f,g satisfy instead of the Lipschitz condition the more relaxed condition of growth at most linear, that is

$$\begin{cases} |f(t,u,v)| \leqslant \widetilde{a}_1 |u| + \widetilde{b}_1 |v| + \widetilde{c}_1, \\ |g(t,u,v)| \leqslant \widetilde{a}_2 |u| + \widetilde{b}_2 |v| + \widetilde{c}_2, \\ |\phi(u,v)| \leqslant \widetilde{a}_{01} |u| + \widetilde{b}_{01} |v| + \widetilde{c}_{01}, \\ |\psi(u,v)| \leqslant \widetilde{a}_{02} |u| + \widetilde{b}_{02} |v| + \widetilde{c}_{02}, \end{cases} \tag{3.7}$$

for all $t \in [0,t_0]$, $u,v \in \mathbb{R}$ and some $\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i, \widetilde{a}_{0i}, \widetilde{b}_{0i}, \widetilde{c}_{0i} > 0, i = 1,2$.

THEOREM 3.2. *If f,g satisfy conditions* (3.7) *and matrix*

$$M_S := \begin{bmatrix} \frac{\widetilde{a}_1}{8}t_0^2 + \widetilde{a}_{01} & \frac{\widetilde{b}_1}{8}t_0^2 + \widetilde{b}_{01}, \\ \frac{\widetilde{a}_2}{8}t_0^2 + \widetilde{a}_{02} & \frac{\widetilde{b}_2}{8}t_0^2 + \widetilde{b}_{02} \end{bmatrix} \tag{3.8}$$

converges to zero, then problem (3.1) *has at least one solution in $C[0,t_0]^2$.*

Proof. In order to apply Schauder's fixed point theorem we look for a nonempty, bounded, closed and convex subset B of $C[0,t_0]^2$, so that $T(B) \subset B$.

Let u,v be any two elements of $C[0,t_0]$. We have that

$$\begin{aligned} |T_1(u,v)(t)| &= \left| \int_0^{t_0} G(t,s) f(s,u(s),v(s))\,ds + \frac{t}{t_0}\phi(u(\eta),v(\eta)) \right| \\ &\leqslant \int_0^{t_0} |G(t,s)| \cdot |f(s,u(s),v(s))|\,ds + \frac{t}{t_0} \cdot |\phi(u(\eta),v(\eta))|\,ds \\ &\leqslant \int_0^{t_0} |G(t,s)| \cdot (\widetilde{a}_1 |u(s)| + \widetilde{b}_1 |v(s)| + \widetilde{c}_1) ds \\ &\quad + \frac{t}{t_0}(\widetilde{a}_{01} |u(s)| + \widetilde{b}_{01} |v(s)| + \widetilde{c}_{01}). \end{aligned}$$

Therefore, since $t < t_0$, we obtain

$$\|T_1(u,v)\|_\infty \leqslant \left(\frac{\widetilde{a}_1}{8}t_0^2 + \widetilde{a}_{01}\right)\|u\|_\infty + \left(\frac{\widetilde{b}_1}{8}t_0^2 + \widetilde{b}_{01}\right)\|v\|_\infty + \widetilde{d}_1, \tag{3.9}$$

where $\widetilde{d}_1 = \frac{\widetilde{c}_1}{8}t_0^2 + \widetilde{c}_{01}$. Similarly,

$$\|T_2(u,v)\|_\infty \leqslant \left(\frac{\widetilde{a}_2}{8}t_0^2 + \widetilde{a}_{02}\right)\|u\|_\infty + \left(\frac{\widetilde{b}_2}{8}t_0^2 + \widetilde{b}_{02}\right)\|v\|_\infty + \widetilde{d}_2, \tag{3.10}$$

where $\widetilde{d}_2 = \frac{\widetilde{c}_2}{8}t_0^2 + \widetilde{c}_{02}$. Now, (3.9) and (3.10) can be put together and be rewritten equivalently as

$$\begin{bmatrix} \|T_1(u,v)\|_\infty \\ \|T_2(u,v)\|_\infty \end{bmatrix} \leqslant M_S \begin{bmatrix} \|u\|_\infty \\ \|v\|_\infty \end{bmatrix} + \begin{bmatrix} \widetilde{d}_1 \\ \widetilde{d}_2 \end{bmatrix}. \tag{3.11}$$

Next, we look for two positive numbers R_1, R_2 such that if $\|u\|_\infty \leqslant R_1, \|v\|_\infty \leqslant R_2$, then $\|T_1(u,v)\|_\infty \leqslant R_1, \|T_2(u,v)\|_\infty \leqslant R_2$. To this end it is sufficient that

$$\begin{cases} \left(\frac{\widetilde{a}_1}{8}t_0^2 + \widetilde{a}_{01}\right)R_1 + \left(\frac{\widetilde{b}_1}{8}t_0^2 + \widetilde{b}_{01}\right)R_2 + \widetilde{d}_1 \leqslant R_1, \\ \left(\frac{\widetilde{a}_2}{8}t_0^2 + \widetilde{a}_{02}\right)R_1 + \left(\frac{\widetilde{b}_2}{8}t_0^2 + \widetilde{b}_{02}\right)R_2 + \widetilde{d}_2 \leqslant R_2, \end{cases}$$

or equivalently

$$M_S \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \widetilde{d}_1 \\ \widetilde{d}_2 \end{bmatrix} \leqslant \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \geqslant (I - M_S)^{-1} \begin{bmatrix} \widetilde{d}_1 \\ \widetilde{d}_2 \end{bmatrix}.$$

Notice that $I - M_S$ is invertible and its inverse $(I - M_S)^{-1}$ has nonnegative elements since M_S converges to zero. Thus, if

$$B = \left\{ (u, v) \in C[0, t_0]^2 : \|u\|_\infty \leqslant R_1, \|v\|_\infty \leqslant R_2 \right\}$$

then $T(B) \subset B$ and Schauder's fixed point theorem can be applied.

4. Existence results on a larger interval

Next, we present existence results for the three-point boundary value problem:

$$\begin{cases} u'' = f(t, u, u'), \quad 0 < t < 1, \\ u(0) = 0, u(t_0) = g(u(\eta)), \end{cases} \tag{4.1}$$

where $0 < t_0 < \eta < 1$ and f, g are continuous functions. Problem (4.1) could be splitted into two parts, one for the subinterval $[0, t_0]$ and the other one for $[t_0, 1]$. More exactly, we look for u such as

$$u(t) = \begin{cases} v(t), & \text{if } t \in [0, t_0], \\ w(t), & \text{if } t \in [t_0, 1], \end{cases}$$

where v solves

$$\begin{cases} v'' = f(t, v, v'), \quad 0 < t < t_0, \\ v(0) = 0, v(t_0) = g(v(\eta)), \end{cases} \tag{4.2}$$

while w is a solution of

$$\begin{cases} w'' = f(t, w, w'), \quad t_0 < t < 1, \\ w(t_0) = v(t_0), \\ w'(t_0) = v'(t_0). \end{cases} \tag{4.3}$$

Problem (4.2) was already discussed in Section 2. Here we just point out that it is equivalent to a fixed point problem for the Fredholm operator $T_F : C^1[0, t_0] \to C^1[0, t_0]$,

$$(T_F v)(t) = \int_0^{t_0} G(t, s) f\left(s, v(s), v'(s)\right) ds + \frac{t}{t_0} g(v(\eta)).$$

For (4.3) we construct a Volterra integral operator $T_V : C^1[t_0, 1] \to C^1[t_0, 1]$ given by

$$(T_V w)(t) = v(t_0) + (t - t_0) v'(t_0) + \int_{t_0}^{t} \int_{t_0}^{\sigma} f\left(s, w(s), w'(s)\right) ds d\sigma. \tag{4.4}$$

Notice that w solves (4.3) if and only if w is a fixed point of the operator T_V. We shall endow $C^1[t_0,1]$ with an equivalent norm of Bielecki type:

$$\|w\|_B = \max\left\{\|w\|_\theta, \|w'\|_\theta\right\},$$

where $\|w\|_\theta = \max\limits_{t_0 \leqslant t \leqslant 1} |w(t)| e^{-\theta(t-t_0)}$ and θ is a suitable positive number. In this way, we shall guarantee the applicability of Banach's and Schauder's fixed point theorems.

4.1. Application of Banach's Contraction Principle

We assume global Lipschitz conditions, that is the existence of $a_1, b_1 > 0$ such that

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant a_1 |u-\overline{u}| + b_1 |v-\overline{v}|, \tag{4.5}$$

for all $t \in [t_0,1]$ and $u,v,\overline{u},\overline{v} \in \mathbb{R}$.

THEOREM 4.1. *If f satisfies (4.5) for some arbitrary $a_1, b_1 > 0$, then problem (4.3) has a unique solution.*

Proof. Consider $T_V : C^1[t_0,1] \to C^1[t_0,1]$ given by (4.4). We have

$$\begin{aligned}
&|(T_V w)(t) - (T_V \overline{w})(t)| \\
&\quad = \left| \int_{t_0}^{t} \int_{t_0}^{\sigma} f\left(s, w(s), w'(s)\right) ds d\sigma - \int_{t_0}^{t} \int_{t_0}^{\sigma} f\left(s, \overline{w}(s), \overline{w}'(s)\right) ds d\sigma \right| \\
&\quad \leqslant \int_{t_0}^{t} \int_{t_0}^{\sigma} \left| f\left(s, w(s), w'(s)\right) - f\left(s, \overline{w}(s), \overline{w}'(s)\right) \right| ds d\sigma \\
&\quad \leqslant \int_{t_0}^{t} \int_{t_0}^{\sigma} (a_1 |w(s) - \overline{w}(s)| + b_1 \left|w'(s) - \overline{w}'(s)\right|) ds d\sigma \\
&\quad = \int_{t_0}^{t} \int_{t_0}^{\sigma} a_1 |w(s) - \overline{w}(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} + \\
&\qquad\qquad + b_1 \left|w'(s) - \overline{w}'(s)\right| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)}) ds d\sigma \\
&\quad \leqslant a_1 \|w - \overline{w}\|_\theta \int_{t_0}^{t} \int_{t_0}^{\sigma} e^{\theta(s-t_0)} ds d\sigma + b_1 \left\|w' - \overline{w}'\right\|_\theta \int_{t_0}^{t} \int_{t_0}^{\sigma} e^{\theta(s-t_0)} ds d\sigma \\
&\quad \leqslant \frac{a_1}{\theta} \|w - \overline{w}\|_\theta \int_{t_0}^{t} (e^{\theta(\sigma-t_0)} - 1) ds + \frac{b_1}{\theta} \left\|w' - \overline{w}'\right\|_\theta \int_{t_0}^{t} (e^{\theta(\sigma-t_0)} - 1) ds \\
&\quad \leqslant \frac{a_1}{\theta^2} \|w - \overline{w}\|_\theta \cdot e^{\theta(t-t_0)} + \frac{b_1}{\theta^2} \left\|w' - \overline{w}'\right\|_\theta \cdot e^{\theta(t-t_0)}.
\end{aligned}$$

Dividing by $e^{\theta(t-t_0)}$ and taking the supremum, we obtain

$$\|T_V w - T_V \overline{w}\|_\theta \leqslant \frac{a_1}{\theta^2} \|w - \overline{w}\|_\theta + \frac{b_1}{\theta^2} \left\|w' - \overline{w}'\right\|_\theta.$$

Using the same method, we obtain

$$\left\|(T_V w)' - (T_V \overline{w})'\right\|_\theta \leqslant \frac{a_1}{\theta} \|w - \overline{w}\|_\theta + \frac{b_1}{\theta} \left\|w' - \overline{w}'\right\|_\theta.$$

Therefore, if $\theta \geqslant 1$, then

$$\|T_V w - T_V \overline{w}\|_B = \max\left\{ \|T_V w - T_V \overline{w}\|_\theta, \left\| (T_V w)' - (T_V \overline{w})' \right\|_\theta \right\}$$
$$\leqslant \frac{1}{\theta}(a_1 + b_1)\|w - \overline{w}\|_B.$$

This shows that T_V is a contraction if we choose $\theta \geqslant 1$ large enough that

$$\frac{a_1 + b_1}{\theta} < 1.$$

Thus, Banach's Contraction Principle can be applied.

4.2. Application of Schauder's fixed point theorem

THEOREM 4.2. *If the condition* (2.2) *holds with*

$$\frac{\alpha_1 + \alpha_2}{\theta} < 1, \tag{4.6}$$

then problem (4.3) *has at least one solution.*

Proof. We show that T_V has a fixed point in $C^1[t_0, 1]$. Let

$$B_2 = \{w \in C^1[t_0,1] : w(t_0) = v(t_0), w'(t_0) = v'(t_0), \|w - v(t_0)\|_\theta \leqslant R, \|w'\|_\theta \leqslant R\}.$$

Using (2.2), we have

$$\begin{aligned}
&\left|T_V w(t) - v(t_0)\right| \\
&= \left| v(t_0) + (t - t_0)v'(t_0) + \int_{t_0}^{t}\int_{t_0}^{\sigma} f\left(s, w(s), w'(s)\right) ds d\sigma - v(t_0) \right| \\
&\leqslant |t - t_0|\left|v'(t_0)\right| + \int_{t_0}^{t}\int_{t_0}^{\sigma} \left|f\left(s, w(s), w'(s)\right)\right| ds d\sigma \\
&\leqslant |t - t_0|\left|v'(t_0)\right| + \int_{t_0}^{t}\int_{t_0}^{\sigma} \left(\alpha_1 |w(s)| + \alpha_2 \left|w'(s)\right| + \alpha_3\right) ds d\sigma \\
&\leqslant c + \int_{t_0}^{t}\int_{t_0}^{\sigma} \left(\alpha_1 |w(s) - v(t_0)| + \alpha_1 |v(t_0)| + \alpha_2 \left|w'(s)\right| + \alpha_3\right) ds d\sigma \\
&= c + \int_{t_0}^{t}\int_{t_0}^{\sigma} \Big(\alpha_1 |w(s) - v(t_0)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} + \alpha_1 |v(t_0)| \\
&\qquad + \alpha_2 \left|w'(s)\right| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} + \alpha_3\Big) ds d\sigma \\
&\leqslant c + \int_{t_0}^{t}\int_{t_0}^{\sigma} \left(\alpha_1 \|w - v(t_0)\|_\theta \cdot e^{\theta(s-t_0)} + \alpha_2 \left\|w'\right\|_\theta \cdot e^{\theta(s-t_0)} + c_1\right) ds d\sigma \\
&\leqslant c + (1 - t_0)\int_{t_0}^{t} \left(\frac{\alpha_1}{\theta} \|w - v(t_0)\|_\theta \cdot e^{\theta(s-t_0)} + \frac{\alpha_2}{\theta} \left\|w'\right\|_\theta \cdot e^{\theta(s-t_0)} + c_1\right) d\sigma
\end{aligned}$$

$$\leqslant c+(1-t_0)\left(\frac{\alpha_1}{\theta^2}\|w-v(t_0)\|_\theta\cdot e^{\theta(t-t_0)}+\frac{\alpha_2}{\theta^2}\left\|w'\right\|_\theta\cdot e^{\theta(t-t_0)}+c_2\right),$$

where $c:=(1-t_0)\left|v'(t_0)\right|, c_1:=\alpha_1\left|v(t_0)\right|+\alpha_3$ and $c_2:=c_1(1-t_0)^2$. Dividing by $e^{\theta(t-t_0)}$ and taking the supremum, we obtain

$$\|T_V w-v(t_0)\|_\theta\leqslant\widetilde{c}+(1-t_0)\left(\frac{\alpha_1}{\theta^2}\|w-v(t_0)\|_\theta+\frac{\alpha_2}{\theta^2}\left\|w'\right\|_\theta\right),$$

where $\widetilde{c}:=c+(1-t_0)c_2$. If $w\in B_2$, then

$$\|T_V w-v(t_0)\|_\theta\leqslant\widetilde{c}+(1-t_0)\frac{\alpha_1+\alpha_2}{\theta^2}R.$$

Furthermore,

$$\begin{aligned}
&\left|(T_V w)'(t)\right|\\
&\leqslant\left|v'(t_0)\right|+\left|\int_{t_0}^t(1-t_0)f\left(\sigma,w(\sigma),w'(\sigma)\right)d\sigma\right|\\
&\leqslant\left|v'(t_0)\right|+(1-t_0)\int_{t_0}^t\left(\alpha_1\left|w(\sigma)\right|+\alpha_2\left|w'(\sigma)\right|+\alpha_3\right)d\sigma\\
&\leqslant\left|v'(t_0)\right|+(1-t_0)\Big(\alpha_1\left|w(\sigma)-v(t_0)\right|\cdot e^{-\theta(\sigma-t_0)}\cdot e^{\theta(\sigma-t_0)}\\
&\qquad+\alpha_2\left|w'(\sigma)\right|\cdot e^{-\theta(\sigma-t_0)}\cdot e^{\theta(\sigma-t_0)}+\alpha_1\left|v(t_0)\right|+\alpha_3\Big)d\sigma\\
&\leqslant\left|v'(t_0)\right|+(1-t_0)\left(\alpha_1\|w-v(t_0)\|_\theta\cdot e^{\theta(\sigma-t_0)}+\alpha_2\left\|w'\right\|_\theta\cdot e^{\theta(\sigma-t_0)}+d\right)d\sigma.
\end{aligned}$$

Dividing by $e^{\theta(\sigma-t_0)}$ and taking the supremum, we have that

$$\left\|(T_V w)'\right\|_\theta\leqslant\widetilde{d}+(1-t_0)\frac{\alpha_1}{\theta}\|w-v(t_0)\|_\theta+\frac{\alpha_2}{\theta^2}\left\|w'\right\|_\theta,$$

where $\widetilde{d}:=\left|v'(t_0)\right|+(1-t_0)^2 d$ and $d:=\alpha_1\left|v(t_0)\right|+\alpha_3$. Taking

$$\|w-v(t_0)\|_\theta\leqslant R,\left\|w'\right\|_\theta\leqslant R,$$

we obtain

$$\left\|(T_V w)'\right\|_\theta\leqslant\widetilde{d}+(1-t_0)\frac{\alpha_1+\alpha_2}{\theta}R.$$

If

$$\widetilde{c}+(1-t_0)\frac{\alpha_1+\alpha_2}{\theta^2}R\leqslant R \tag{4.7}$$

and

$$\widetilde{d}+(1-t_0)\frac{\alpha_1+\alpha_2}{\theta}R\leqslant R, \tag{4.8}$$

then T_V applies B_2 into B_2. Thus a number $R>0$ with (4.7) and (4.8) exists provided that

$$\begin{cases}\frac{\alpha_1+\alpha_2}{\theta^2}<1,\\ \frac{\alpha_1+\alpha_2}{\theta}<1.\end{cases} \tag{4.9}$$

Notice that for $\theta \geqslant 1$,

$$\frac{\alpha_1+\alpha_2}{\theta^2} \leqslant \frac{\alpha_1+\alpha_2}{\theta}$$

and both conditions in (4.9) hold if

$$\frac{\alpha_1+\alpha_2}{\theta} < 1.$$

Since T_V is a completely continuous operator, from Schauder's fixed point theorem it results that T_V has at least one fixed point.

4.3. Existence and uniqueness results on $[0,1]$

Putting together the results from Section 2.1, Section 4.1 and the results from Section 2.2, Section 4.2 respectively, we obtain the following results for equations on the entire interval $[0,1]$:

THEOREM 4.3. *If* f,g *satisfy* (H1) *with* (2.6) *and condition* (4.5), *then problem* (4.1) *has a unique solution on* $[0,1]$.

THEOREM 4.4. *Assume that* (H2) *holds with* (2.8). *If, in addition, condition* (2.2) *holds with* (4.6)*, then problem* (4.1) *has at least one solution on* $[0,1]$.

5. Systems on a larger interval

Here we consider the three point boundary value problems for second order differential systems of the type:

$$\begin{cases} u''(t) = f(t,u(t),v(t)), \\ v''(t) = g(t,u(t),v(t)), \quad 0<t<1, \\ u(0)=0,\ u(t_0)=\phi(u(\eta),v(\eta)), \\ v(0)=0,\ v(t_0)=\psi(u(\eta),v(\eta)), \end{cases} \tag{5.1}$$

and we give existence and uniqueness results for ordinary differential systems of this type. These systems can be splitted into two parts, one for the subinterval $[0,t_0]$ and the other one for $[t_0,1]$, respectively. A similar algorithm was given for equations in Section 4. Systems on $[0,t_0]$ were already discussed in Section 3.

In what follows, we treat three point value problems for differential systems on $C[t_0,1]$ of the type:

$$\begin{cases} w''(t) = f(t,w(t),x(t)), \\ x''(t) = g(t,w(t),x(t)), \quad t_0<t<1, \\ w(t_0)=v(t_0),\ w'(t_0)=v'(t_0), \\ x(t_0)=x(t_0),\ x'(t_0)=v'(t_0). \end{cases} \tag{5.2}$$

Problem (3.1) is equivalent with the following integral system in $C[t_0,1]^2$:

$$\begin{cases} w(t) = v(t_0)+(t-t_0)v'(t_0)+\int_{t_0}^{t}\int_{t_0}^{\sigma} f(s,w(s),x(s))\,ds d\sigma, \\ x(t) = v(t_0)+(t-t_0)v'(t_0)+\int_{t_0}^{t}\int_{t_0}^{\sigma} g(s,w(s),x(s))\,ds d\sigma. \end{cases}$$

This can be viewed as a fixed point problem in $C[t_0,1]^2$ for a completely continuous operator $T=(T_1,T_2)$, $T:C[t_0,1]^2 \to C[t_0,1]^2$, where

$$\begin{cases} w(t)=T_1(w(t),x(t)) \\ x(t)=T_2(w(t),x(t)) \end{cases}$$

and T_1,T_2 respectively are given by:

$$\begin{cases} T_1(u,v)(t)=v(t_0)+(t-t_0)v'(t_0)+\int_{t_0}^{t}\int_{t_0}^{\sigma} f(s,w(s),x(s))\,dsd\sigma, \\ T_2(u,v)(t)=v(t_0)+(t-t_0)v'(t_0)+\int_{t_0}^{t}\int_{t_0}^{\sigma} g(s,w(s),x(s))\,dsd\sigma. \end{cases}$$

5.1. Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here we show that the existence of solutions to problem (5.2) is established by means of Perov's fixed point theorem. For this, we assume global Lipschitz conditions, that is there exist $a_1,b_1,c_1,d_1>0$ such that:

$$\begin{cases} |f(t,w,x)-f(t,\overline{w},\overline{x})| \leqslant a_1|w-\overline{w}|+b_1|x-\overline{x}|, \\ |g(t,w,x)-g(t,\overline{w},\overline{x})| \leqslant c_1|w-\overline{w}|+d_1|x-\overline{x}|, \end{cases} \tag{5.3}$$

for $t\in[t_0,1]$ and $w,x,\overline{w},\overline{x}\in\mathbb{R}$.

THEOREM 5.1. *Assume that the conditions* (5.3) *hold. If matrix*

$$M_\theta := \begin{bmatrix} \frac{a_1}{\theta^2} & \frac{b_1}{\theta^2} \\ \frac{c_1}{\theta^2} & \frac{d_1}{\theta^2} \end{bmatrix} \tag{5.4}$$

converges to zero, then problem (5.2) *has a unique solution in* $C[t_0,1]^2$.

Proof. We shall apply Perov's fixed point theorem in $C[t_0,1]^2$ endowed with the vector norm $\|\cdot\|_B$ defined by

$$\|y\|_B=\max\{\|w\|_\theta,\|x\|_\theta\}$$

and

$$\|w\|_\theta=\max_{t_0\leqslant t\leqslant 1}|w(t)|\cdot e^{-\theta(t-t_0)},$$

where $y=(w,x)\in C[0,t_0]^2$. We have to prove that T is a generalized contraction, more exactly that

$$\|T(y)-T(\overline{y})\|_{C[t_0,1]} \leqslant M_\theta \|y-\overline{y}\|_{C[t_0,1]}$$

for all $y=(w,x),\overline{y}=(\overline{w},\overline{x})\in C[t_0,1]^2$ and some matrix M_θ converging to zero for a large enough θ.

Let $(w,x),(\overline{w},\overline{x})$ be any elements of $C[t_0,1]^2$. We have that

$$\left|T_1(w,x)(t)-T_1(\overline{w},\overline{x})(t)\right|$$

$$
\begin{aligned}
&= \left| \int_{t_0}^{t} \int_{t_0}^{\sigma} f(s, w(s), x(s))\, ds d\sigma - \int_{t_0}^{t} \int_{t_0}^{\sigma} f(s, \overline{w}(s), \overline{x}(s))\, ds d\sigma \right| \\
&\leqslant \int_{t_0}^{t} \int_{t_0}^{\sigma} |f(s, w(s), x(s)) - f(s, \overline{w}(s), \overline{x}(s))|\, ds d\sigma \\
&\leqslant \int_{t_0}^{t} \int_{t_0}^{\sigma} (a_1 |w(s) - \overline{w}(s)| + b_1 |x(s) - \overline{x}(s)|) ds d\sigma \\
&= \int_{t_0}^{t} \int_{t_0}^{\sigma} (a_1 |w(s) - \overline{w}(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)} + \\
&\qquad + b_1 |x(s) - \overline{x}(s)| \cdot e^{-\theta(s-t_0)} \cdot e^{\theta(s-t_0)}) ds d\sigma \\
&\leqslant a_1 \|w - \overline{w}\|_\theta \int_{t_0}^{t} \int_{t_0}^{\sigma} e^{\theta(s-t_0)} ds d\sigma + b_1 \|x - \overline{x}\|_\theta \int_{t_0}^{t} \int_{t_0}^{\sigma} e^{\theta(s-t_0)} ds d\sigma \\
&\leqslant \frac{a_1}{\theta} \|w - \overline{w}\|_\theta \int_{t_0}^{t} (e^{\theta(\sigma-t_0)} - 1) ds + \frac{b_1}{\theta} \|x - \overline{x}\|_\theta \int_{t_0}^{t} (e^{\theta(\sigma-t_0)} - 1) ds \\
&\leqslant \frac{a_1}{\theta^2} \|w - \overline{w}\|_\theta \cdot e^{\theta(t-t_0)} + \frac{b_1}{\theta^2} \|x - \overline{x}\|_\theta \cdot e^{\theta(t-t_0)}.
\end{aligned}
$$

We obtain

$$
\|T_1(w,x) - T_1(\overline{w}, \overline{x})\|_\theta \leqslant \frac{a_1}{\theta^2} \|w - \overline{w}\|_\theta + \frac{b_1}{\theta^2} \|x - \overline{x}\|_\theta .
$$

Similarly

$$
\|T_2(w,x) - T_2(\overline{w}, \overline{x})\|_\theta \leqslant \frac{c_1}{\theta^2} \|w - \overline{w}\|_\theta + \frac{d_1}{\theta^2} \|x - \overline{x}\|_\theta .
$$

This can be rewritten equivalently as

$$
\begin{bmatrix} \|T_1(w,x) - T_1(\overline{w}, \overline{x})\|_\theta \\ \|T_2(w,x) - T_2(\overline{w}, \overline{x})\|_\theta \end{bmatrix} \leqslant M_\theta \begin{bmatrix} \|w - \overline{w}\|_\theta \\ \|x - \overline{x}\|_\theta \end{bmatrix}. \tag{5.5}
$$

Then (5.5) is equivalent to

$$
\|T(y) - T(\overline{y})\|_{C[t_0,1]} \leqslant M_\theta \|y - \overline{y}\|_{C[t_0,1]}
$$

for all $y = (w,x), \overline{y} = (\overline{w}, \overline{x}) \in C[t_0,1]^2$. The result follows now from Perov's fixed point theorem.

5.2. Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here we show that the existence of solutions to problem (5.2) is established by means of Schauder's fixed point theorem in case that f, g satisfy instead of the Lipschitz conditions, the more relaxed conditions of growth at most linear, that is

$$
\begin{cases} |f(t,w,x)| \leqslant \widetilde{a}_1 |w| + \widetilde{b}_1 |x| + \widetilde{c}_1, \\ |g(t,w,x)| \leqslant \widetilde{a}_2 |w| + \widetilde{b}_2 |x| + \widetilde{c}_2, \end{cases} \tag{5.6}
$$

for $t \in [t_0, 1]$, $w, x \in [-R, R]$ and some $\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i > 0, i = \overline{1,2}$.

THEOREM 5.2. *If f,g satisfy conditions (5.6) and*

$$M_{\theta}' := \begin{bmatrix} \frac{\widetilde{a}_1}{\theta^2} & \frac{\widetilde{b}_1}{\theta^2}, \\ \frac{\widetilde{a}_2}{\theta^2} & \frac{\widetilde{b}_2}{\theta^2}, \end{bmatrix} \tag{5.7}$$

converges to zero for a large enough θ, then problem (5.2) has at least one solution in $C[t_0,1]^2$.

Proof. In order to apply Schauder's fixed point theorem we look for a nonempty, bounded, closed and convex subset B of $C[t_0,1]^2$, so that $T(B) \subset B$.

Let w,x be any elements of $C[t_0,1]$. We have that

$$\begin{aligned} |T_1(w,x)(t)| &= \left| v(t_0) + (t-t_0)v'(t_0) + \int_{t_0}^{t}\int_{t_0}^{\sigma} f(s,w(s),x(s))\,dsd\sigma \right| \\ &\leqslant |v(t_0)| + |t-t_0|\cdot\left|v'(t_0)\right| \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{\sigma}\left(\widetilde{a}_1|w(s)| + \widetilde{b}_1|x(s)| + \widetilde{c}_1\right)dsd\sigma \\ &\leqslant \widetilde{c}_0 + \frac{\widetilde{a}_1}{\theta^2}\|w\|_\theta \cdot e^{(t-t_0)} + \frac{\widetilde{b}_1}{\theta^2}\|x\|_\theta \cdot e^{(t-t_0)} + \widetilde{c}_{01}, \end{aligned}$$

where $\widetilde{c}_{01} = \widetilde{c}_1 \int_{t_0}^{t}\int_{t_0}^{\sigma} dsd\sigma$ and $\widetilde{c}_0 := |v(t_0)| + |t-t_0|\cdot|v'(t_0)|$. We obtain

$$\|T_1(w,x)\|_\theta \leqslant \frac{\widetilde{a}_1}{\theta^2}\|w\|_\theta + \frac{\widetilde{b}_1}{\theta^2}\|x\|_\theta + \widetilde{d}_{01}, \tag{5.8}$$

where $\widetilde{d}_{01} := \widetilde{c}_0 + \widetilde{c}_{01}$. Similarly,

$$\|T_2(w,x)\| \leqslant \frac{\widetilde{a}_2}{\theta^2}\|w\| + \frac{\widetilde{b}_2}{\theta^2}\|x\| + \widetilde{d}_{02}, \tag{5.9}$$

where $\widetilde{d}_{02} := \widetilde{c}_0 + \widetilde{c}_{02}$. (5.8) and (5.9) can be put together and be rewritten equivalently as

$$\begin{bmatrix} \|T_1(w,x)\|_\theta \\ \|T_2(w,x)\|_\theta \end{bmatrix} \leqslant M_\theta' \begin{bmatrix} \|w\|_\theta \\ \|x\|_\theta \end{bmatrix} + \begin{bmatrix} \widetilde{d}_{01} \\ \widetilde{d}_{02} \end{bmatrix}.$$

Next, we look for two positive numbers R_1,R_2 such that if $\|w\|_\theta \leqslant R_1, \|x\|_\theta \leqslant R_2$, then

$$\|T_1(w,x)\|_\theta \leqslant R_1, \|T_2(w,x)\|_\theta \leqslant R_2.$$

To this end it is sufficient that

$$\begin{cases} \frac{\widetilde{a}_1}{\theta^2}R_1 + \frac{\widetilde{b}_1}{\theta^2}R_2 + \widetilde{d}_{01} \leqslant R_1, \\ \frac{\widetilde{a}_2}{\theta^2}R_1 + \frac{\widetilde{b}_2}{\theta^2}R_2 + \widetilde{d}_{02} \leqslant R_2, \end{cases}$$

or equivalently

$$M_\theta' \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \widetilde{d}_{01} \\ \widetilde{d}_{02} \end{bmatrix} \leqslant \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \geqslant \left(I - M'_\theta\right)^{-1} \begin{bmatrix} \widetilde{d}_{01} \\ \widetilde{d}_{02} \end{bmatrix}.$$

Notice that $I - M'_\theta$ is invertible and its inverse $\left(I - M'_\theta\right)^{-1}$ has nonnegative elements since M'_θ converges to zero for a large enough θ. Thus, if

$$B = \left\{(w,x) \in C[t_0,1]^2 : \|w\|_\theta \leqslant R_1, \|x\|_\theta \leqslant R_2\right\},$$

then $T(B) \subset B$ and Schauder's fixed point theorem can be applied.

5.3. Existence and uniqueness results for systems on $[0,1]$

Putting together the results from Section 3.1, Section 5.1 and the results from Section 3.2, Section 5.2 respectively, we obtain the following results for systems on the entire interval $[0,1]$:

THEOREM 5.3. *Assume that conditions* (3.2) *and* (5.3) *hold. If matrices* (3.3) *and* (5.4) *are convergent to zero, then problem* (5.1) *has a unique solution in* $C[0,1]^2$.

THEOREM 5.4. *If* f,g *satisfy conditions* (3.7) *and* (5.6) *and if matrices* (3.8) *and* (5.7) *are convergent to zero, then problem* (5.1) *has at least one solution in* $C[0,1]^2$.

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REFERENCES

[1] R. P. AGARWAL, D. O. REGAN, P. J. Y. WONG, *Positive solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.

[2] R. P. AGARWAL, M. MEEHAN, D. O'REGAN, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[3] R. I. AVERY, J. HENDERSON, *Three symmetric positive solutions for a second-order boundary value problem*, Appl. Math. Lett. **13** (2000) 1–7.

[4] A. BOUCHERIF, *Differential equations with nonlocal boundary conditions*, Nonlinear Anal., **47** (2001), 2419–2430.

[5] A. BOUCHERIF, *First-order differential inclusions with nonlocal initial conditions*, Appl. Math. Letters, **15** (2002), 409–414.

[6] A. BOUCHERIF, R. PRECUP, *On the nonlocal initial value problem for first order differential equations*, Fixed Point Theory, **4** (2) (2003), 205–212.

[7] D.W. BOYD, J.S.W. WONG, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20** (1969), 458–464.

[8] L. BYSZEWSKI, V. LAKSHMIKANTHAM, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Applicable Anal., **40** (1990), 11–19.

[9] L. BYSZEWSKI, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162** (1991), 494–505.

[10] E.V. CASTELANI, TO FU MA, *Numerical soluitions for a second order three-point boundary value problem*, Communications in Applied Analysis, **11** (1) (2007), 87–95.

[11] S.H. CHEN, T. HU, L. CHEN, ET AL., *Existence results for n-point boundary value problem of second order ordinary differential equations*, J. Comput. Appl. Math., **180** (2005), 425–432.

[12] H. CHEN, P. LI, *Three-point boundary value problems for second-order ordinary differential equations in Banach spaces*, Computers and Mathematics with Applications, **56** (2008), 1852–1860.

[13] K. DEMLING, *Ordinary Differential Equations in Banach Spaces*, Springer, Berlin, 1977.

[14] P.W. ELOE, J. HENDERSON, *Positive solutions and nonlinear (k, n - k) conjugate eigenvalue problem*, Diff. Equ. Dynam. Syst., **6** (1998), 309–317.

[15] W. FENG, *On a m-point nonlinear boundary value problem*, Nonlinear Anal. TMA., **30** (6) (1997), 5369–5374.

[16] W. FENG, J.R.L. WEBB, *Solvability of a m-point boundary value problems with nonlinear growth*, J. Math. Anal. Appl. **212** (1997), 467–480.

[17] H. FENG, W. GE, M. JIANG, *Multiple positive solutions for m-point boundary-value problems with a one p-Laplacian*, Nonlinear Analysis, **68** (2008), 2269–2279.

[18] J.R. GRAEF, B. YANG, *On a nonlinear boundary-value problem for fourth-order equations*, Appl. Anal., **72** (1999), 439–448.

[19] J.R. GRAEF, B. YANG, *Existence and non-existence of positive solutions of fourth-order nonlinear boundary-value problems*, Appl. Anal., **74** (2000), 201–204.

[20] D. GUO, V. LAKSHMIKANTHAN, X. LIN, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic, Dordrecht, 1996.

[21] D. GUO, *Boundary value problems for impulsive integro-differential equation on unbounded domains in a Banach space*, Appl. Math. Comput. **99** (1999), 1–15.

[22] C.P. GUPTA, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations*, J. Math. Anal. Appl., **168** (1992), 540–551.

[23] C.P. GUPTA, S.K. NTOUYAS, P.CH. TSAMATOS, *On an m-point boundary value problem for second order ordinary differential equations*, Nonlinear Anal. TMA., **23** (11) (1994), 1427–1436.

[24] C.P. GUPTA, *Solvability of an m-point nonlinear boundary value problem for second order ordinary differential equations*, J. Math. Anal. Appl., **189** (1995), 575–584.

[25] C.P. GUPTA, S.I. TROFIMCHUK, *A sharper condition for the solvability of a three-point second order boundary value problem*, J. Math. Anal. Appl., **205** (1997), 586–597.

[26] C.P. GUPTA, S.I. TROFIMCHUK, *Existence of a solution to a three-point boundary-value problem and the spectral radius of o related linear operator*, Nonl. Anal. TMA., **34** (1998), 498–507.

[27] X. HE, W. GE, *Existence of positive solutions for the one-dimension p-Laplacian equation*, Acta Math. Sinica, **46** (4) (2003), 805–810.

[28] V.A. IL'IN, E.I. MOISEEV, *Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects*, Differential Equations, **23** (7) (1987), 803–810.

[29] V. A. IL'IN, E.I. MOISEEV, *Nonlocal boundary value problem of the second kind for a Sturm–Liouville operator*, Differential Equations, **23** (8) (1987), 979–987.

[30] L.V. KANTOROVICI, G. P. AKILOV, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[31] V. LAKSHMIKANTHAN, S. LEELA, *Nonlinear Differential Equations in Abstract Spaces*, Pergamon, Oxford, 1981.

[32] Y. LIU, *Boundary value problems for second order differential equations on unbounded domains in a Banach space*, Appl. Math. Compou., **135** (2003), 569–584.

[33] B. LIU, *Positive solutions of nonlinear four-point boundary value problem*, Appl. Math. Comput., **155** (2004) 179–203.

[34] R.Y. MA, H.Y. WANG, *On the existence of positive solutions of fourth-order ordinary differential equations*, Appl. Anal., **59** (1995), 225–231.

[35] R. MA, *Existence theorems for a second order three-point boundary value problem*, J. Math. Anal. Appl., **212** (1997), 430-442.

[36] R. Ma, *Positive solutions for a nonlinear three-point boundary value problem*, Electron. J. Differential Equations Appl. Math. Comput., **34** (1999), 1–8.

[37] R. MA, *Existence of solutions of nonlinear m-point boundary value problem*, J. Math. Anal. Appl., **256** (2001), 556–567.

[38] R. MA, *Positive solutions of a nonlinear m-point boundary value problem*, Comput. Math. Appl., **42** (2001), 755–765.

[39] M. MOSHINSKY, *Sobre los problemas de condiciones a la frontiera en una dimension de caracteristicas discontinuas*, Bol. Soc. Mat. Mexicana, **7** (1950), 1–25.

[40] S.M. NIKOLSKY, *A Course of Mathematical Analysis*, Mir Publishers, Moscow, 1981.

[41] R. PRECUP, *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht-Boston-London, 2002.

[42] R. PRECUP, *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comp. Modelling, **49** (2009), 703–708.

[43] Y. SUN, *Positive solutions of nonlinear second-order m-point boundary value problem*, Nonlinear Analysis, **61** (2005), 1283–1294.

[44] S. TIMOSHENKO, *Theory of Elastic Stability*, McGraw-Hill, NewYork, 1961.

[45] Y. Wang, C. Hou, *Existence of multiple positive solutions for one-dimensional p-Laplacian*, J. Math. Anal. Appl., **315** (2006), 144–153.

[46] Y. WANG, W. GE, *Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian*, Nonlinear Anal., **66** (2007), 1246–1256.

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