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OSCILLATION OF SECOND ORDER NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. In this paper, we present some new sufficient conditions for the oscillations of all solutions of a second order retarded differential equations with impulses.These results extend the known results for the differential equations without impulses.An example is provided to illustrate our result.

1. Introduction

In this paper we are concerned with the oscillatory behaviour of solutions of impulsive differential equations with deviating arguments

$$\begin{cases} \left[r(t)|u'(t)|^{\alpha-1}u'(t)\right]' + p(t)f\big(u(\tau(t))\big) = 0, \quad t \neq \theta_k, \\ \Delta\left[r(t)|u'(t)|^{\alpha-1}u'(t)\right]\Big|_{t=\theta_k} + b_k h\big(u(\tau(\theta_k))\big) = 0, \quad t \in [t_0,\infty),\ k \in \mathbb{N}, \end{cases} \tag{1}$$

where

$$\Delta[z(t)]\Big|_{t=\theta} = z(\theta^+) - z(\theta^-)$$

in which $z(\theta^{\mp}) := \lim\limits_{t\to\theta^{\mp}} z(t)$. For convenience we define $z(\theta) = z(\theta^-)$.

Through out this paper we assumed the following conditions to hold:

(H_1) $\alpha > 0$, $r \in C([t_0,\infty))$, $r(t) > 0$, $p \in C([t_0,\infty))$, $p(t) > 0$;

(H_2) $R(t) = \displaystyle\int_{t_0}^{t} \frac{ds}{r^{\frac{1}{\alpha}}(s)} \to \infty$ as $t \to \infty$;

(H_3) $r \in C'([t_0,\infty))$, $\tau(t) \leqslant t$, $\tau'(t) > 0$, $\tau(t) \to \infty$ as $t \to \infty$;

(H_4) $\{\theta_k\}$ is a fixed strictly increasing unbounded sequence of positive real numbers and $\{b_k\}$ is a sequence of positive real numbers;

(H_5) $f \in C(\mathbb{R})$, $h \in C(\mathbb{R})$, $xf(x) > 0$, $f'(x) \geqslant 0$, $xh(x) > 0$ for $x \neq 0$, $f \in C'(R_D)$, where $R_D = (-\infty,-D) \cup (D,\infty)$, $D > 0$;

(H_6) for any given $c_1 > 0$ there exists $c_2 > 0$ such that $|h(x)| \geqslant c_2|f(x)|$ for all $|x| \geqslant c_1$.

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By a solution of equation (1) on an interval $J \subset R_+$, we mean a continuous function $u(t)$ which is defined on J such that $u'(t)$, $(r(t)|u'(t)|^{\alpha-1}u'(t)) \in PLC(J)$ and which satisfies equation (1), where $PLC(J)$ denotes the set of all real-valued function $g(t)$ defined on J such that $g(t)$ is continuous on (θ_k, θ_{k+1}), $g(\theta_k^{\pm})$ exists and $g(\theta_k) = g(\theta_k^-)$ for each $k \geqslant k_0$. We consider only those solutions $u(t)$ of equation (1) which satisfy $\sup\{|u(t)| : t \geqslant T_u\} > 0$ for all $T_u \geqslant t_0$. It will be assumed that equation (1) has a solutions which are nontrivial for large t. Such a solution of equation (1) is called oscillatory if it has no last zero, and nonoscillatory otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

Recently, the theory of impulsive differential equations has been intensively studied by many authors since such equations are mathematical approaches for simulation of process and phenomena observed in control theory,physics, chemistry, population dynamics, biotechnology, economics, etc.There are many papers have devoted to the oscillation criteria of second order differential equations with impulses, see for example [1, 2, 3, 5, 6, 13, 16, 17], and the references cited therein.

In this paper our aim is to extend the results established in [22] to the impulsive differential equations (1). An example is provided to illustrate the main result.

2. Main results

In this section,we obtain some new oscillation criteria for the solutions of equation (1).

THEOREM 1. *Let there exists a constant $k > 0$ such that*

$$\frac{f'(x)}{|f(x)|^{1-\frac{1}{\alpha}}} \geqslant k \text{ for all } x \in R_D. \tag{2}$$

If

$$\lim_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{r(s)} \int_{s}^{\infty} p(z)dz\right]^{\frac{1}{\alpha}} ds = \infty, \tag{3}$$

and there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$, such that $\rho'(t) \geqslant 0$ and

$$\limsup_{t\to\infty} \left[\int_{t_0}^{t} \left(p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\right)ds + \sum_{t_0 \leqslant \theta_k < t} c_2 b_k \rho^{\alpha}(\theta_k)\right] = \infty, \tag{4}$$

where $\mu = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{k}\right)^{\alpha}$, then the impulsive differential equation (1) is oscillatory.

Proof. The proof is based on the arguments developed in [22]. Suppose that there exist a nonoscillatory solution $u(t)$ of equation (1). We may assume that $u(t)$ is eventually positive.The case $u(t)$ eventually negative is similar, so that we can omit it.

Since $\tau(t) \to \infty$ as $t \to \infty$, there exist a positive real number T such that $x(\tau(t)) > 0$ for all $t > T$. From equation (1), we have

$$\left[r(t)|u'(t)|^{\alpha-1}u'(t)\right]' = -p(t)f\big(u(\tau(t))\big) \leqslant 0.$$

Hence the function $r(t)|u'(t)|^{\alpha-1}u'(t)$ is nonincreasing on each interval (θ_k,θ_{k+1}) whenever $\theta_k \geqslant T$.

If $t=\theta_k$, then

$$r(\theta_k^+)|u'(\theta_k^+)|^{\alpha-1}u'(\theta_k^+)-r(\theta_k)|u'(\theta_k)|u'(\theta_k)=-b_kh(x(\tau(\theta_k)))\leqslant 0.$$

Thus $r(t)|u'(t)|^{\alpha-1}u'(t)$ is nonincreasing in (T,∞). We claim that $u'(t)$ is eventually positive. In fact, if $u'(t^*)\leqslant 0$ for some $t^*\geqslant T$, then

$$r(t)|u'(t)|^{\alpha-1}u'(t)\leqslant r(t^*)|u'(t^*)|^{\alpha-1}u'(t^*)\leqslant 0 \text{ for } t\geqslant t^*.$$

Dividing the last inequality by $r(t)$ and integrating the resulting inequality from t^* to t we have

$$u(t)-u(t^*)\leqslant r^{\frac{1}{\alpha}}(t^*)u'(t)\int_{t^*}^{t}\frac{1}{r^{\frac{1}{\alpha}}(s)}ds. \tag{5}$$

Letting $t\to\infty$ and using the hypothesis (H_2) in (5), we see that $u(t)$ must be eventually negative, which is a contradiction. Therefore our claim is true.

Define

$$w(t)=r(t)\frac{(\rho(t)u'(t))^{\alpha}}{f(u(\tau(t)))},\quad t\neq\theta_k. \tag{6}$$

Then $w(t)>0$. Differentiating $w(t)$, we get

$$\begin{aligned}w'(t)=\frac{\rho^{\alpha}(t)(r(t)(u'(t))^{\alpha})'}{f(u(\tau(t)))}&+\frac{\alpha r(t)}{\rho(t)}(\rho(t)u'(t))^{\alpha}\rho'(t)\\&-\frac{r(t)(\rho(t)u'(t))^{\alpha}}{f^2(u(\tau(t)))}f'(u(\tau(t)))u'(\tau(t))\tau'(t),\end{aligned}$$

where $t\neq\theta_k$. Since $r(t)(u'(t))^{\alpha}$ is decreasing and using equation (1), we have

$$w'(t)=\alpha\frac{\rho'(t)}{\rho(t)}w(t)-p(t)\rho^{\alpha}(t)-\frac{(w(t))^{1+\frac{1}{\alpha}}f'(u(\tau(t)))\tau'(t)}{[f(u(\tau(t)))]^{1-\frac{1}{\alpha}}\rho(t)[r(\tau(t))]^{\frac{1}{\alpha}}}, \tag{7}$$

$$\Delta w(t)\Big|_{t=\theta_k}=\frac{\rho^{\alpha}(\theta_k)}{f(u(\tau(\theta_k)))}[-b_kh(u(\tau(\theta_k)))]. \tag{8}$$

Let us assume that $u(t)$ is bounded. Then there exists positive constants k_1 and k_2 such that for all $t\geqslant t_0$,

$$k_2\leqslant\tau(t)\leqslant k_1 \text{ and } k_2\leqslant u(\tau(t))\leqslant k_1.$$

Integrating equation (1) from t to ∞, we obtain

$$r(t)(u'(t))^{\alpha}|_t^{\infty}+\int_t^{\infty}p(s)f(u(\tau(s)))ds=0.$$

Since $r(t)(u'(t))^{\alpha}$ is positive and nonincreasing, we have

$$r(t)(u'(t))^{\alpha}\geqslant\int_t^{\infty}p(s)f(u(\tau(s)))ds.$$

Integrating the last inequality from t_0 to t, we obtain

$$k_1 \geqslant u(t) \geqslant f_0^{\frac{1}{\alpha}} \int_{t_0}^{t} \left[\frac{1}{r(s)} \int_s^\infty p(z)dz\right]^{\frac{1}{\alpha}} ds,$$

where $f_0 = \min_{u\in[k_1,k_2]} f(u)$. Letting $t \to \infty$, the last inequality contradicts (3). Therefore, we conclude that $u(t) \to \infty$ as $t \to \infty$. Thus $u(\tau(t)) > 0$ for all large t enough. Now from (2),we have

$$\frac{f'\big(u(\tau(t))\big)}{\big[f(u(\tau(t)))\big]^{1-\frac{1}{\alpha}}} \geqslant k.$$

Equation (7) implies

$$w'(t) \leqslant \alpha\frac{\rho'(t)}{\rho(t)}w(t) - p(t)\rho^\alpha(t) - k\frac{(w(t))^{1+\frac{1}{\alpha}}\tau'(t)}{(r(\tau(t)))^{\frac{1}{\alpha}}\rho(t)}, \quad t \neq \theta_k. \tag{9}$$

By using the inequality,

$$Ax - Bx^{1+\frac{1}{\alpha}} \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} A^{\alpha+1}B^{-\alpha},\ A \geqslant 0,\ B > 0,\ x \geqslant 0 \tag{10}$$

we get from (9) that

$$w'(t) \leqslant -\left[\rho^\alpha(t)p(t) - \mu\frac{r(\tau(t))(\rho'(t))^{\alpha+1}}{\rho(t)(\tau'(t))^\alpha}\right], \quad t \neq \theta_k. \tag{11}$$

In view of (8) and

$$\int_{t_1}^{t} w'(s)ds = w(t) - w(t_1) - \sum_{t_1\leqslant\theta_k<t} \Delta w(\theta_k), \tag{12}$$

if we integrate (11) from t_1 to t, then using condition (H_6), we obtain

$$w(t) \leqslant w(t_1) - \int_{t_1}^{t} \left[p(s)\rho^\alpha(s) - \mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^\alpha\rho(s)}\right]ds - \sum_{t_1\leqslant\theta_k<t} c_2b_k\rho^\alpha(\theta_k).$$

Taking liminf,in the last inequality we obtain that $w(t) \to -\infty$ inview of (4), which contradicts the fact that $w(t) > 0$. This completes the proof. $\square$

COROLLARY 2. *Assume that conditions* (2) *and* (3) *are satisfied. If there exists a differentiable positive function* ρ *such that* $\rho'(t) > 0$ *for all* $t \geqslant t_0$,

$$\int_{t_0}^{\infty} \frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^\alpha\rho(s)}ds + \sum_{t_0\leqslant\theta_k<\infty} c_2b_k\rho^\alpha(\theta_k) = \infty, \tag{13}$$

and

$$\lim_{t\to\infty} \frac{\rho^{\alpha+1}(t)p(t)(\tau'(t))^\alpha}{r(\tau(t))(\rho'(t))^{\alpha+1}} > \mu, \tag{14}$$

then the impulsive delay differential equation (1) *is oscillatory.*

Proof. From the assumption (13), it follows that there exists $\varepsilon > 0$ such that for all large t

$$\frac{\rho^{\alpha+1}(t)p(t)(\tau'(t))^{\alpha}}{r(\tau(t))(\rho'(t))^{\alpha+1}} > \mu + \varepsilon.$$

From Theorem 1, we have

$$w'(t) \leqslant -\frac{r(\tau(t))(\rho'(t))^{\alpha+1}}{\rho(t)(\tau'(t))^{\alpha}}\Big[\mu - \frac{\rho^{\alpha}(t)p(t)\rho(t)(\tau'(t))^{\alpha}}{r(\tau(t))(\rho'(t))^{\alpha+1}}\Big], \quad t \neq \theta_k, \tag{15}$$

$$\Delta w(t)\big|_{t=\theta_k} = -\frac{\rho^{\alpha}(\theta_k)b_k h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))}. \tag{16}$$

Integrating (15) from t_1 to t, using (12) and (16),we obtain

$$w(t) \leqslant w(t_1) - \int_{t_1}^{t} \frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{\rho(s)(\tau'(s))^{\alpha}}\Big[\mu - \frac{\rho^{\alpha}(s)p(s)\rho(s)(\tau'(s))^{\alpha}}{r(\tau(s))(\rho'(s))^{\alpha+1}}\Big]ds - \sum_{t_1 \leqslant \theta_k < t} \frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))},$$

that is,

$$w(t) \leqslant w(t_1) - \Big[\varepsilon \int_{t_1}^{t} \frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{\rho(s)(\tau'(s))^{\alpha}} ds + \sum_{t_1 \leqslant \theta_k < t} c_2 b_k \rho^{\alpha}(\theta_k)\Big],$$

where we have used (H_6).Taking limit as $t \to \infty$ in the last inequality, we obtain a contradiction with $w(t) > 0$. This completes the proof. □

Next, let us introduce the class of functions P defined as in [4, 18, 19],which will be extensively used in the sequel.

Let

$$D_0 = \{(t,s) \colon t > s \geqslant t_0\} \quad \text{and} \quad D = \{(t,s) \colon t \geqslant s \geqslant t_0\}.$$

We say that the function $H \in C(D,R)$ belongs to the class P denoted by $H \in P$, if

(F_1) $H(t,t) = 0$ for $t \geqslant t_0$ and $H(t,s) > 0$ on D_0;

(F_2) $\frac{\partial H(t,s)}{\partial s} \leqslant 0$ for all $(t,s) \in D$.

Suppose that $\lambda : D_0 \to \mathbb{R}$ is a continuous function such that

(F_3) $\alpha\frac{\rho'(s)}{\rho(s)}H(t,s) = -\lambda(t,s)(H(t,s))^{\frac{\alpha}{\alpha+1}}$ for all $(t,s) \in D_0$, where λ is a locally integrable function.

THEOREM 3. *Assume conditions* (2) *and* (3) *hold. If there exists a positive continuous differentiable function* ρ *and* $H \in P$ *such that*

$$\limsup_{t\to\infty}\Big[\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\Big(p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big)ds \\ +\frac{1}{H(t,t_0)}\sum_{t_0\leqslant\theta_k<t}H(t,\theta_k)c_2b_k\rho^{\alpha}(\theta_k)\Big]=\infty, \quad (17)$$

then the impulsive differential equation (1) *is oscillatory.*

Proof. Let $u(t)$ be a nonoscillatory solution of equation (1).Then we may assume that $u(t)>0$ eventually,since the proof for the case $u(t)<0$ is similar.Proceeding as in the proof of Theorem 1,we have the following

$$w'(t)\leqslant-\Big[\rho^{\alpha}(t)p(t)-\mu\frac{r(\tau(t))(\rho'(t))^{\alpha+1}}{\rho(t)(\tau'(t))^{\alpha}}\Big],\quad t\neq\theta_k.$$

$$\Delta w(t)\Big|_{t=\theta_k}=-\frac{\rho^{\alpha}(\theta_k)b_kh(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))}. \quad (18)$$

Multiplying the last inequality by $H(t,s)$, we obtain

$$H(t,s)w'(t)\leqslant-H(t,s)\Big[\rho^{\alpha}(t)p(t)-\mu\frac{r(\tau(t))(\rho'(t))^{\alpha+1}}{\rho(t)(\tau'(t))^{\alpha}}\Big],\quad t\neq\theta_k. \quad (19)$$

Integrating the inequality (19) from t_1 to t, using (12) and (18),we obtain

$$\int_{t_1}^{t}H(t,s)w'(s)ds\leqslant-\int_{t_1}^{t}H(t,s)\Big[p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big]ds \\ -\sum_{t_1\leqslant\theta_k<t}H(t,\theta_k)\frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))},$$

that is,

$$\int_{t_1}^{t}H(t,s)\Big[p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big]ds \\ +\sum_{t_1\leqslant\theta_k<t}H(t,\theta_k)\frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))} \\ \leqslant H(t,t_1)w(t_1)\leqslant H(t,t_0)w(t_1),$$

where we have used (F_1) and (F_2). Therefore

$$\int_{t_0}^{t}H(t,s)\Big[p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big]ds \\ +\sum_{t_1\leqslant\theta_k<t}H(t,\theta_k)\frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))} \\ =\int_{t_0}^{t_1}H(t,s)\Big[p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big]ds$$

$$
\begin{aligned}
&+\int_{t_1}^{t} H(t,s)\Big[p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big]ds\\
&+\sum_{t_1\leqslant\theta_k<t} H(t,\theta_k)\frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))}\\
\leqslant{}& H(t,t_0)\int_{t_0}^{t_1} p(s)\rho^{\alpha}(s)ds+H(t,t_0)w(t_1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\Big[p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big]ds\\
&\qquad+\frac{1}{H(t,t_0)}\sum_{t_0\leqslant\theta_k<t} H(t,\theta_k)c_2b_k\rho^{\alpha}(\theta_k)\\
&\qquad\qquad\leqslant\int_{t_0}^{t_1} p(s)\rho^{\alpha}(s)ds+w(t_1). \qquad (20)
\end{aligned}
$$

Taking limsup in (20) we obtain a contradiction with (17). This completes the proof.

THEOREM 4. *Assume conditions* (2) *and* (3) *hold. If there exists a positive continuous differentiable function* $\rho(t)$ *satisfying* (F_3) *and* $H\in P$ *with*

$$
\begin{aligned}
\limsup_{t\to\infty}\Big[&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Big(H(t,s)p(s)\rho^{\alpha}(s)-\mu_1\frac{r(\tau(s))(\rho'(s))^{\alpha}}{(\tau'(s))^{\alpha}}(\lambda(t,s))^{\alpha+1}\Big)ds\\
&+\frac{1}{H(t,t_0)}\sum_{t_1\leqslant\theta_k<t} H(t,\theta_k)c_2b_k\rho^{\alpha}(\theta_k)\Big]=\infty, \qquad (21)
\end{aligned}
$$

where $\mu_1=\frac{1}{(\alpha+1)^{\alpha+1}}(\frac{\alpha}{k})^{\alpha}$ *hold, then the impulsive differential equation* (1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 1 we obtain the following

$$
\begin{aligned}
w'(t)&\leqslant\alpha\frac{\rho'(t)}{\rho(t)}w(t)-p(t)\rho^{\alpha}(t)-k\frac{(w(t))^{1+\frac{1}{\alpha}}\tau'(t)}{(r(\tau(t)))^{\frac{1}{\alpha}}\rho(t)}, \quad t\neq\theta_k,\\
\Delta w(t)\big|_{t=\theta_k}&=-\frac{\rho^{\alpha}(\theta_k)b_kh(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))}.
\end{aligned}
$$

Multiplying the last inequality by $H(t,s)$ and integrating from t_1 to t, we get

$$
\begin{aligned}
\int_{t_1}^{t} H(t,s)w'(s)ds&\leqslant\int_{t_1}^{t}\alpha H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds-\int_{t_1}^{t} H(t,s)p(s)\rho^{\alpha}(s)ds\\
&-\int_{t_1}^{t} kH(t,s)\frac{(w(s))^{1+\frac{1}{\alpha}}\tau'(s)}{(r(\tau(s)))^{\frac{1}{\alpha}}\rho(s)}ds-\sum_{t_1\leqslant\theta_k<t} H(t,\theta_k)\frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))}.
\end{aligned}
$$

Now using the inequality (10) and simplifying,we obtain

$$\int_{t_1}^{t}\left[H(t,s)p(s)\rho^{\alpha}(s)ds-\mu_1\rho^{\alpha}(s)\frac{r(\tau(s))(\lambda(t,s))^{\alpha+1}}{(\tau'(s))^{\alpha}}\right]ds$$
$$+\sum_{t_1\leqslant\theta_k<t}H(t,\theta_k)\frac{b_k\rho^{\alpha}(\theta_k)h(u(\tau(\theta_k)))}{f(u(\tau(\theta_k)))}\leqslant H(t,t_1)w(t_1).$$

The rest of the proof is similar to that of Theorem 2.2 and hence the details are omitted.

3. Example

In this section, we present an example to illustrate the main result.

EXAMPLE 5. Consider the following second order impulsive type delay differential equation

$$\begin{cases}\left[|u'(t)|^{\alpha-1}u'(t)\right]'+\frac{1}{t}\left|u(t/2)\right|^{\beta-1}u(t/2)=0, & t\neq m^{\beta},\\ \Delta|u'(t)|^{\alpha-1}u'(t)]\Big|_{t=m^{\beta}}+\frac{1}{m^{\beta}}\left|u(m^{\beta}/2)\right|^{\beta-1}u(m^{\beta}/2)=0,\end{cases}\tag{22}$$

where α, β are positive constants such that $\beta\geqslant\alpha$ and $p\in C([1,\infty),\mathbb{R}^+)$. Here $f(u)=|u|^{\beta-1}u$. Then there exists a $k>0$ such that $f'(u)\geqslant k$ for all $u\in R_D$, k large enough. Hence (2) holds.

By choosing $\rho(t)=t^{\eta/\alpha}$ for $t\geqslant s\geqslant 1$ such that $\alpha+1<\eta<\alpha^2$ .Here

$$r(t)=1,\ p(t)=\frac{1}{t},\ \tau(t)=\frac{t}{2},\ b_k=\frac{1}{m^{\beta}},\ \theta_k=m^{\beta},\ \text{and } c_2=1.$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\left[\frac{1}{r(s)}\int_s^{\infty}p(z)dz\right]^{\frac{1}{\alpha}}ds=\lim_{t\to\infty}\int_1^t\left[\int_s^{\infty}\frac{1}{z}dz\right]^{\frac{1}{\alpha}}ds=\infty.$$

Hence condition (3) is satisfied. Now

$$\begin{aligned}&\limsup_{t\to\infty}\left[\int_{t_0}^{t}\Big(p(s)\rho^{\alpha}(s)-\mu\frac{r(\tau(s))(\rho'(s))^{\alpha+1}}{(\tau'(s))^{\alpha}\rho(s)}\Big)ds+\sum_{t_0\leqslant\theta_k<t}c_2b_k\rho^{\alpha}(\theta_k)\right]\\ &=\limsup_{t\to\infty}\left[\int_{t_0}^{t}\Big(\frac{1}{s}s^{\eta}-\mu\Big(\frac{\eta}{\alpha}\Big)^{\alpha+1}\big(s^{\frac{\eta-\alpha}{\alpha}}\big)^{\alpha+1}\frac{1}{(\frac{1}{2})^{\alpha}s^{\frac{\eta}{\alpha}}}\Big)ds+\sum_{t_0\leqslant\theta_k<t}\frac{1}{m^{\beta}}\theta_k^{\eta}\right]\\ &=\limsup_{t\to\infty}\left[\frac{1}{\eta}(t^{\eta}-1)-\mu\Big(\frac{\eta}{2}\Big)^{\alpha+1}2^{\alpha}\frac{\alpha}{\eta-\alpha^2}\big(t^{\frac{\eta-\alpha^2}{\alpha}}-1\big)+\sum_{t_0\leqslant\theta_k<t}\frac{1}{m^{\beta}}\theta_k^{\eta}\right]\\ &=\infty.\end{aligned}$$

Therefore all conditions of Theorem 1 are satisfied, and hence equation (22) is oscillatory.

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