THE ANTIMAXIMUM PRINCIPLE AND THE EXISTENCE OF A SOLUTION FOR THE GENERALIZED $p$–LAPLACE EQUATIONS WITH INDEFINITE WEIGHT

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Abstract. This paper treats the antimaximum principle and the existence of a solution for quasilinear elliptic equation $-\text{div} (a(x, |\nabla u|) \nabla u) = \lambda m(x)|u|^{p-2}u + h(x)$ in $\Omega$ under the Neumann boundary condition. Here, a map $a(x, |y|)y$ on $\overline{\Omega} \times \mathbb{R}^N$ is strictly monotone in the second variable and satisfies certain regularity conditions. This equation contains the $p$-Laplacian problem as a special case.

1. Introduction

In this paper, we study the antimaximum principle (AMP) and consider the existence of a solution for the following quasilinear elliptic equation:

$$(P; \lambda, m, h) \begin{cases} -\text{div} (a(x, |\nabla u|) \nabla u) = \lambda m(x)|u|^{p-2}u + h(x) \quad \text{in} \; \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \; \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary $\partial \Omega$, $\nu$ denotes the outward unit normal vector on $\partial \Omega$, $a$ is a positive function on $\overline{\Omega} \times (0, +\infty)$, $\lambda \in \mathbb{R}$, $1 < p < \infty$, $m \in L^\infty(\Omega)$ and $h \in L^\infty(\Omega)_+$. Here, we set a map $A(x,y) := a(x, |y|)y$ for $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$ and, then $A$ is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption $(A)$). The equation $(P; \lambda, m, h)$ contains the corresponding $p$-Laplacian problem as a special case. However, in general, we do not suppose that the operator $A$ is $(p-1)$-homogeneous in the second variable.

Throughout this paper, we assume that

$$|\{m > 0\}| := |\{x \in \Omega; m(x) > 0\}| > 0 \quad (1.1)$$

where $|X|$ denotes the Lebesgue measure of a measurable set $X$. In this paper, we deal with the following four cases concerning the weight function $m \in L^\infty(\Omega)$ under (1.1):

(i) $m \not\equiv 0$ and $m(x) \geq 0$ for a.e. $x \in \Omega$;  
(ii) $\int_{\Omega} m \, dx > 0$ and $|m < 0| > 0$;  
(iii) $\int_{\Omega} m \, dx = 0$;  
(iv) $\int_{\Omega} m \, dx < 0$.


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Here, we say that \( u \in W^{1,p}(\Omega) \) is a (weak) solution of \((P;\lambda,m,h)\) if

\[
\int_{\Omega} A(x,\nabla u) \nabla \phi \, dx = \lambda \int_{\Omega} m|u|^{p-2}u \phi \, dx + \int_{\Omega} h \phi \, dx
\]

for all \( \phi \in W^{1,p}(\Omega) \).

In [8], the study of AMP started by Clement and Peletier. They proved that there exists \( \delta > 0 \) for every \( \lambda \in (\lambda_1,\lambda_1 + \delta) \) such that any solution is negative in \( \Omega \) for \(-\Delta u = \lambda u + h \) in \( \Omega \) under the Dirichlet or Neumann boundary condition, where \( \lambda_1 \) denotes the first eigenvalue of \(-\Delta\). This situation is called as that “AMP holds at right of \( \lambda_1 \)”. Although the above \( \delta \) depends on \( h \) in general, they presented also the existence of such \( \delta \) independent of \( h \) in the case of \( N = 1 \) under the Neumann boundary condition. When we can take \( \delta \) independent of \( h \), we say that “AMP holds uniformly at right of \( \lambda_1 \)”. The AMP was extended in [16] to the case having the (indefinite) weight. Moreover, many authors have studied the AMP for the Laplace equation and other equations (cf. [2], [3], [5], [6], [10], [11], [22]). In the case of the \( p \)-Laplacean, Godoy et al ([13] and [14]) presented the several results concerning AMP for \(-\Delta_p u = \lambda m|u|^{p-2}u + h \) in \( \Omega \) under the Dirichlet and Neumann boundary conditions. First purpose of this paper is to prove similar results to one of [14] for the generalized \( p \)-Laplace equation \((P;\lambda,m,h)\).

On the other hand, it is obvious that the AMP has no effect if a solution does not exist. However, there are few existence results of a solution to our equation (and also the \( p \)-Laplace equation). For example, if \( \lambda < 0 \) and \( m \equiv 1 \) holds, then the standard argument guarantees the existence of a solution. In [14], it is shown that the equation \(-\Delta_p u = m|u|^{p-2}u + h \) in \( \Omega \) has a unique positive solution provided \( 0 < \lambda < \lambda^*(m) \), \( \int_{\Omega} m \, dx < 0 \) and \( 0 \not\equiv h \in L^\infty(\Omega)_+ \), where \( \lambda^*(m) \) is the principal eigenvalue defined in Section 2.1. To the Laplace problems under the Dirichlet boundary condition, the existence results are well known (cf. [1]).

Therefore, second purpose is to show that \((P;\lambda,m,h)\) has at least one solution under some condition to \( \lambda \) by variational methods. In particular, in the case where \( A \) is asymptotically \((p-1)\) homogeneous (see the condition \((AH)\) in Section 4.3), \((P;\lambda,m,h)\) has at least one solution if \( \lambda \) exists between the principal eigenvalue and the second eigenvalue (Theorem 6 and see Remark 7).

Throughout this paper, we assume that the map \( A \) satisfies the following assumption (\(A\):

\(\text{(A)}\) \( A(x,y) = a(x,|y|)y \), where \( a(x,t) > 0 \) for all \( (x,t) \in \overline{\Omega} \times (0,+\infty) \) and

\(\text{(i)}\) \( A \in C^0(\overline{\Omega} \times \mathbb{R}^N,\mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}),\mathbb{R}^N) \);

\(\text{(ii)}\) there exists \( C_1 > 0 \) such that

\[ |D_\gamma A(x,y)| \leq C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\} ; \]

\(\text{(iii)}\) there exists \( C_0 > 0 \) such that

\[ D_\gamma A(x,y) \xi \cdot \xi \geq C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega}, \ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N ; \]
(iv) there exists $C_2 > 0$ such that
\[ |D_x A(x, y)| \leq C_2 (1 + |y|^{p-1}) \quad \text{for every } x \in \overline{\Omega}, \ y \in \mathbb{R}^N \setminus \{0\}; \]

(v) there exist $C_3 > 0$ and $1 \geq t_0 > 0$ such that
\[ |D_x A(x, y)| \leq C_3 |y|^{p-1} (-\log |y|) \]
for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

Throughout this paper, we assume $C_0 \leq p - 1 \leq C_1$ because we can take such desired $C_0$ and $C_1$ anew if necessary.

A similar hypothesis to (A) is considered in the study of quasilinear elliptic problems (cf. [21, Example 2.2.] and [9], [20], [19]). It is easily seen that many examples as in the above references satisfy the condition (AH). In particular, for $A(x, y) = |y|^{p-2} y$, that is, $\text{div} A(x, \nabla u)$ stands for the usual $p$-Laplacian $\Delta_p u$, we can take $C_0 = C_1 = p - 1$ in (A). Conversely, in the case where $C_0 = C_1 = p - 1$ holds in (A), by the inequalities in Remark 1 (ii) and (iii) in Section 2, we see $a(x, t) = |t|^{p-2}$ whence $A(x, y) = |y|^{p-2} y$.

In section 2.1, we recall several results concerning the weighted eigenvalue problems for the $p$-Laplacian. Then, in Section 3, we show that the AMP holds at some $\lambda$ for our equation. Finally, we present the existence results to our equation (in Section 4).

2. Preliminaries

In what follows, the norm on $W^{1,p}(\Omega)$ is $||u||^p := ||\nabla u||_p^p + ||u||_p^p$, where $||u||_q$ denotes the norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ ($1 \leq q \leq \infty$). Setting
\[ G(x, y) := \int_0^{|y|} a(x, t) t \, dt, \]
then we can easily see that
\[ \nabla_y G(x, y) = A(x, y) \quad \text{and} \quad G(x, 0) = 0 \quad (2.1) \]
for every $x \in \overline{\Omega}$.

REMARK 1. It is easily seen that the following assertions hold under condition (A):

(i) for all $x \in \overline{\Omega}$, $A(x, y)$ is maximal monotone and strictly monotone in $y$;

(ii) $|A(x, y)| \leq \frac{C_1}{p-1} |y|^{p-1}$ for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$;

(iii) $A(x, y) y \geq \frac{C_0}{p-1} |y|^p$ for every $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$;
(iv) \( G(x,y) \) is convex in \( y \) for all \( x \) and satisfies the following inequalities:

\[
A(x,y)y \geq G(x,y) \geq \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x,y) \leq \frac{C_1}{p(p-1)}|y|^p
\]

for every \((x,y) \in \overline{\Omega} \times \mathbb{R}^N\),

where \( C_0 \) and \( C_1 \) are the positive constants in \((A)\).

**Remark 2.** Let \( m \in L^\infty(\Omega) \) and \( h \in L^\infty(\Omega)_+ \). Then, we remark the following:

(i) If \( u \in W^{1,p}(\Omega) \) is a solution of \((P; \lambda, m, h)\), then \( u \in C^{1,\alpha}(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \) and \( \partial u/\partial \nu = 0 \) on \( \partial \Omega \);

(ii) If \( u \in W^{1,p}(\Omega) \) is a non-trivial solution of \((P; \lambda, m, h)\) such that \( u \geq 0 \), then \( \min_{\overline{\Omega}} u > 0 \) holds;

*Proof.* For readers’ convenience, we give a sketch of the proof. (i): Let \( u \in W^{1,p}(\Omega) \) be a solution of \((P; \lambda, m, h)\). Then, because \( u \in L^\infty(\Omega) \) by the Moser iteration process (cf. Appendix in [19]), we see that \( u \in C^{1,\alpha}(\overline{\Omega}) \) \((0 < \alpha < 1)\) by the regularity result in [17]. Furthermore, by [7, Theorem 3], \( u \) satisfies the boundary condition

\[
0 = \frac{\partial u}{\partial \nu_A} = A(\cdot, \nabla u) \nu = a(\cdot, |\nabla u|)\frac{\partial u}{\partial \nu} \quad \text{in} \quad W^{-1/q,q}(\partial \Omega)
\]

for every \( 1 < q < \infty \) (see [7] for the definition of \( W^{-1/q,q}(\partial \Omega) \)). Since \( u \in C^{1,\alpha}(\overline{\Omega}) \) and \( a(x,t) > 0 \) for every \( t \neq 0 \), \( u \) satisfies the Neumann boundary condition, that is, \( \frac{\partial u}{\partial \nu}(x) = 0 \) for every \( x \in \partial \Omega \).

(ii): Let \( u \in W^{1,p}(\Omega) \) be a solution of \((P; \lambda, m, h)\) satisfying \( u \geq 0 \) and \( u \not\equiv 0 \). Then, we have

\[
-\text{div}A(x,\nabla u) + |\lambda|\|m\|_{\infty}u^{p-1} \geq h \geq 0 \quad \text{in} \quad \Omega.
\]

By noting that \( u \in C^{1,\alpha}(\overline{\Omega}) \) \((0 < \alpha < 1)\) by (i), we have \( u(x) > 0 \) for every \( x \in \Omega \) by Theorem B in [19, Appendix]. Due to the strong maximum principle (see Theorem A in [19, Appendix]), we easily see that \( u(x) > 0 \) for every \( x \in \partial \Omega \) (note \( \partial u/\partial \nu = 0 \) on \( \partial \Omega \) by (i)). This yields \( \min_{\overline{\Omega}} u > 0 \) because of \( u \in C^{1,\alpha}(\overline{\Omega}) \) by (i).

2.1. The weighted eigenvalue problems for the \( p \)-Laplacian

The following lemmas can be easily shown by way of contradiction. Here, we omit the proofs.

**Lemma 1.** ([14, Lemma 2.3.]) Assume \( \int_{\Omega} m \, dx < 0 \). Then, there exists a constant \( c > 0 \) such that \( \int_{\Omega} |\nabla u|^p \, dx \geq c\|u\|_p^p \) for every \( u \in W^{1,p}(\Omega) \) with \( \int_{\Omega} m \|u\|^p \, dx > 0 \).
Lemma 2. ([14, Lemma 2.8.]) Assume that $\int_\Omega m\,dx \neq 0$ and $\xi > 0$. Then, there exists a constant $b(m, \xi) > 0$ such that

$$\int_\Omega |\nabla u|^p\,dx - \xi \int_\Omega m|u|^p\,dx \geq b(m, \xi) \int_\Omega |u|^p\,dx$$

for every $u \in B(m) := \{u \in W^{1,p}(\Omega); \int_\Omega m|u|^p\,dx \leq 0\}$.

Lemma 3. Assume that $m \geq 0$ in $\Omega$. Then, for every $\xi > 0$ there exists $d(\xi) > 0$ such that

$$\int_\Omega |\nabla u|^p\,dx + \xi \int_\Omega m|u|^p\,dx \geq d(\xi) \int_\Omega |u|^p\,dx$$

for every $u \in W^{1,p}(\Omega)$.

Lemma 4. Let $N < p$, $\lambda > 0$ and $\Lambda \subset \mathbb{R}$ be a compact set. Define

$$\tilde{B}(m) := \left\{u \in W^{1,p}(\Omega); \int_\Omega m|u|^p\,dx \geq 0 \text{ and } u \text{ vanishes somewhere in } \overline{\Omega}\right\}.$$

Then, there exists a $C > 0$ such that for every $\varepsilon \in \Lambda$, the following inequality holds:

$$\int_\Omega |\nabla u|^p\,dx + \lambda \int_\Omega (m+\varepsilon)|u|^p\,dx \geq C\|u\|_p^p$$

for every $u \in \tilde{B}(m+\varepsilon)$.

Proof. By way of contradiction, we suppose that there exist $\{\varepsilon_n\} \subset \Lambda$ and $u_n \in \tilde{B}(m+\varepsilon_n)$ such that

$$\int_\Omega |\nabla u_n|^p\,dx + \lambda \int_\Omega (m+\varepsilon_n)|u_n|^p\,dx < \frac{1}{n}\|u_n\|_p^p.$$

Set $v_n := u_n/\|u_n\|_p$. Then, because we have

$$\int_\Omega |\nabla v_n|^p\,dx \leq \int_\Omega |\nabla u_n|^p\,dx + \lambda \int_\Omega (m+\varepsilon_n)|v_n|^p\,dx < \frac{1}{n}$$

(2.3)

by $v_n \in \tilde{B}(m+\varepsilon_n)$, we may assume that $v_n$ weakly converges to some $v_0$ in $W^{1,p}(\Omega)$ and $v_n(x)$ converges to $v_0(x)$ uniformly in $x \in \overline{\Omega}$ (note $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact).

Moreover, it is easily seen that $v_0$ vanishes somewhere in $\overline{\Omega}$ because $v_n$ vanishes somewhere in $\overline{\Omega}$ by $u_n \in \tilde{B}(m+\varepsilon_n)$. Since $\Lambda$ is compact, we may assume that $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$ for some $\varepsilon_0 \in \Lambda$ by choosing a subsequence. Consequently, $v_0 \in \tilde{B}(m+\varepsilon_0)$ holds. On the other hand, by taking the limit inferior in (2.3), we have $\int_\Omega |\nabla v_0|^p\,dx \leq 0$.

This implies that $v_0$ is a constant function such that $\|v_0\|_p = 1$. This contradicts to the fact that $v_0$ vanishes somewhere in $\overline{\Omega}$.

Now, we state several known results relative to the following weighted eigenvalue problems for the $p$-Laplacian:

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.$$
We say that $\lambda \in \mathbb{R}$ is an eigenvalue of (2.4) if the equation (2.4) has a non-trivial solution. First, we recall the following principal eigenvalue $\lambda^*(m)$ which plays an important role for studying AMP.

$$\lambda^*(m) := \inf \left\{ \int_{\Omega} |u|^p dx \, ; \, u \in W^{1,p}(\Omega) \text{ and } \int_{\Omega} m|u|^p dx = 1 \right\}. \tag{2.5}$$

**Proposition 1.** ([14, Proposition 2.2.]) The following assertions hold;

(i) If $\int_{\Omega} m dx \geq 0$ holds, then $\lambda^*(m) = 0$;

(ii) If $\int_{\Omega} m dx < 0$ holds, then $\lambda^*(m) > 0$ is a simple eigenvalue and it admits a positive eigenfunction. In addition, the interval $(0, \lambda^*(m))$ contains no eigenvalues of (2.4).

Moreover, we recall a second value $\overline{\lambda}(m)$ defined by

$$\overline{\lambda}(m) := \inf \left\{ \int_{\Omega} |u|^p dx \, ; \, u \in W^{1,p}(\Omega), \int_{\Omega} m|u|^p dx = 1 \right\}. \tag{2.6}$$

In the case of $N < p$ (note that $W^{1,p}(\Omega)$ is compactly imbedded into $C(\overline{\Omega})$), we introduce $\tilde{\lambda}(m)$ as follows:

$$\tilde{\lambda}(m) := \inf \left\{ \int_{\Omega} |u|^p dx \, ; \, u \in W^{1,p}(\Omega), \int_{\Omega} m|u|^p dx = 1 \right\}. \tag{2.7}$$

It is easily shown that $\overline{\lambda}(m) = \tilde{\lambda}(m)$ (see section 3 in [14]). Concerning $\lambda^*(m)$ and $\overline{\lambda}(m)$, the following result is shown in [14].

**Lemma 5.** ([14, Lemma 3.1.]) If $p \leq N$, then $\lambda^*(m) = \overline{\lambda}(m)$. If $p > N$, then $\lambda^*(m) < \overline{\lambda}(m)$. Moreover, if $p > N$, then $(\lambda^*(m), \overline{\lambda}(m))$ has no eigenvalues of (2.4).

To prove lemma above, we need the following lemma proved by the same argument as in [13, Claim 4.1] or [5, Lemma 3.1.] (Note that Lemma 4 guarantees the boundedness of a minimizing sequence of $\tilde{\lambda}(m)$). Here, we omit the proof.

**Lemma 6.** Assume that $p > N$. Then, $\tilde{\lambda}(m)$ is attained. Furthermore, a minimizer for $\tilde{\lambda}(m)$ vanishes at exactly one point in $\overline{\Omega}$.

**Lemma 7.** Let $N < p$. Then, $\tilde{\lambda}(m + \varepsilon') < \tilde{\lambda}(m + \varepsilon) < \tilde{\lambda}(m)$ for every $\varepsilon' > \varepsilon > 0$ holds. Moreover, $\lim_{\varepsilon \to +0} \tilde{\lambda}(m + \varepsilon) = \tilde{\lambda}(m)$.

**Proof.** We choose a minimizer $u$ for $\tilde{\lambda}(m)$ because Lemma 6 guarantees the existence of it. Then, for every $\varepsilon > 0$, we have

$$\tilde{\lambda}(m + \varepsilon) \leq \frac{\int_{\Omega} |u|^p dx}{\int_{\Omega} (m + \varepsilon)|u|^p dx} < \frac{\int_{\Omega} |u|^p dx}{\int_{\Omega} m|u|^p dx} = \int_{\Omega} |\nabla u|^p dx = \tilde{\lambda}(m)$$
by the definition of $\tilde{\lambda}(m + \epsilon)$. By applying the same argument to a minimizer for $\tilde{\lambda}(m + \epsilon)$, we obtain $\tilde{\lambda}(m + \epsilon') < \tilde{\lambda}(m + \epsilon)$ for $\epsilon' > \epsilon > 0$.

Now, we shall prove

$$\lim_{\epsilon \to +0} \tilde{\lambda}(m + \epsilon) = \tilde{\lambda}(m).$$

Let $\{\epsilon_n\}$ be any sequence such that $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$. Because we know $\limsup_{n \to \infty} \tilde{\lambda}(m + \epsilon_n) \leq \tilde{\lambda}(m)$ by the first assertion, it suffices only to prove $\liminf_{n \to \infty} \tilde{\lambda}(m + \epsilon_n) \geq \tilde{\lambda}(m)$. We take a minimizer $u_n$ for $\tilde{\lambda}(m + \epsilon_n)$. Then, it follows from Lemma 4 with $\lambda = 1$ and $\Lambda = [0, \sup_n \epsilon_n]$ that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$ (note that $\|\nabla u_n\|^p = \tilde{\lambda}(m + \epsilon_n) < \tilde{\lambda}(m)$). Thus, we may assume, by choosing a subsequence, that there exists $u_0 \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ and $u_n \to u_0$ in $C(\overline{\Omega})$. Then, we see that $\int_{\Omega} m|u_0|^p \, dx = 1$ and $u_0$ vanishes somewhere in $\overline{\Omega}$ since $u_n(x)$ converges to $u_0(x)$ uniformly in $x \in \overline{\Omega}$. Hence, by the definition of $\tilde{\lambda}(m)$, we obtain

$$\liminf_{n \to \infty} \tilde{\lambda}(m + \epsilon_n) = \liminf_{n \to \infty} \int_{\Omega} |\nabla u_n|^p \, dx \geq \int_{\Omega} |\nabla u_0|^p \, dx \geq \tilde{\lambda}(m),$$

whence our claim is shown. Because $\{\epsilon_n\}$ is an arbitrary sequence, our conclusion is proved.

Finally, we recall the second eigenvalue of (2.4). The following result is shown in [4] (Although they handle the asymmetry case, it is sufficient to consider the case of $m \equiv n$ in this paper).

$\tilde{J}(u) := \int_{\Omega} |\nabla u|^p \, dx$ for $u \in W^{1,p}(\Omega), \quad \tilde{J} := J_{|S(m)}$ \hfill (2.8)

$S(m) := \left\{ u \in W^{1,p}(\Omega); \int_{\Omega} m|u|^p \, dx = 1 \right\}$, \hfill (2.9)

$\Sigma(m) := \{ \gamma \in C([0, 1], S(m)); \gamma(0) \in P \cap S(m), \gamma(1) \in (-P) \cap S(m) \}$, \hfill (2.10)

$c(m) := \inf_{\gamma \in \Sigma(m)} \max_{t \in [0, 1]} \tilde{J}(\gamma(t))$, \hfill (2.11)

where $P := \{ u \in W^{1,p}(\Omega); u(x) \geq 0 \text{ for a.e. } x \in \Omega \}$.

**Lemma 8.** ([4, Theorem 3.2.]) We have: $c(m)$ is an eigenvalue of (2.4) which satisfies $\lambda^+(m) < c(m)$. Moreover, there is no eigenvalues of (2.4) between $\lambda^+(m)$ and $c(m)$.

**Remark 3.** we remark that $\tilde{\lambda}(m) < c(m)$ (if $N < p$). Indeed, if $\tilde{\lambda}(m) \geq c(m)$ under $N < p$, then it contradicts to the fact that $(\lambda^+(m), \tilde{\lambda}(m))$ (note $\tilde{\lambda}(m) = \overline{\lambda}(m)$) contains no eigenvalues of (2.4) by Lemma 5 since $c(m)$ is an eigenvalue of (2.4) and $\lambda^+(m) < c(m)$.

**Lemma 9.** For every $\epsilon' > \epsilon > 0$, we have $c(m + \epsilon') < c(m + \epsilon) < c(m)$. In addition, $\lim_{\epsilon \to +0} c(m + \epsilon) = c(m)$ holds.
Proof. First, we shall prove that \( c(m + \varepsilon) < c(m) \) for \( \varepsilon > 0 \). Because \( c(m) \) is an eigenvalue, we can choose a solution \( u \in W^{1,p}(\Omega) \) with \( \|u\| = 1 \) for \( -\Delta_p u = c(m)m|u|^{p-2}u \) in \( \Omega \), \( \partial u/\partial \nu = 0 \) on \( \partial \Omega \). Then, we note that \( u \) is a sign-changing function because any eigenfunction corresponding to \( \lambda \) other than the principal eigenvalue changes sign (refer to [15, Proposition 4.3.] or see Proposition 2 with \( C_0 = C_1 = p - 1 \) and \( h \equiv 0 \)). Thus, we have \( 0 < \|\nabla u\|_p = c(m)\int_{\Omega}mu_+^p dx \) by taking \( \pm u_\pm \) as test function. Set a continuous path \( \gamma \) by

\[
\gamma_0(t) := \frac{(1-t)u_+ - tu_-}{((1-t)p\int_{\Omega}mu_+^p dx + t^p\int_{\Omega}mu_-^p dx)^{1/p}} \quad \text{for } t \in [0,1].
\]

Then, it is easily seen that \( \gamma_0 \in \Sigma(m) \) and

\[
c(m) = \|\nabla \gamma_0(t)\|_p^p > \frac{\|\nabla \gamma_0(t)\|_p^p}{1 + \varepsilon\|\gamma_0(t)\|_p^p} = \frac{\|\nabla \gamma_0(t)\|_p^p}{\int_{\Omega}(m + \varepsilon)|\gamma_0(t)|^p dx}
\]

for every \( t \in [0,1] \) (note \( \|\gamma_0(t)\|_p > 0 \)). By setting

\[
\gamma_\varepsilon := \frac{\gamma_0}{(\int_{\Omega}(m + \varepsilon)|\gamma_0|^p dx)^{1/p}} \in \Sigma(m + \varepsilon),
\]

we obtain \( c(m) > c(m + \varepsilon) \) by the definition of \( c(m + \varepsilon) \). By considering \( m + \varepsilon \) and \((m + \varepsilon) + (\varepsilon' - \varepsilon)\), we obtain \( c(m + \varepsilon) > c(m + \varepsilon') \) for \( \varepsilon' > \varepsilon \) due to the above assertion.

Let \( \{\varepsilon_n\} \) be any sequence such that \( \varepsilon_n \to 0 \) and \( \lim_{n \to \infty} \varepsilon_n = 0 \). By the first assertion, we can get \( \limsup_{n \to \infty} c(m + \varepsilon_n) \leq c(m) \). Now, we shall prove

\[
\lambda_0 := \liminf_{n \to \infty} c(m + \varepsilon_n) \geq c(m).
\]

We may put \( \lambda_0 = \lim_{n \to \infty} c(m + \varepsilon_n) \geq 0 \) by choosing a subsequence (note \( c(m + \varepsilon_n) > \lambda^*(m + \varepsilon_n) \geq 0 \)). By the same reason as in the first part, there exists a sign-changing solution \( u_n \in W^{1,p}(\Omega) \) with \( \|u_n\| = 1 \) for \( -\Delta_p u_n = c(m + \varepsilon_n)(m + \varepsilon_n)|u_n|^{p-2}u_n \) in \( \Omega \), \( \partial u_n/\partial \nu = 0 \) on \( \partial \Omega \). In addition, by the standard argument (refer to Proposition 3) and the boundedness of \( \|u_n\| \), we may assume that \( u_n \) converges to some \( u_0 \) in \( C^1(\Omega) \) by choosing a subsequence. Hence, \( u_0 \) is a solution with \( \|u_0\| = 1 \) of \( -\Delta_p u = \lambda_0 m|u|^{p-2}u \) in \( \Omega \), \( \partial u/\partial \nu = 0 \) on \( \partial \Omega \). This means that \( u_0 \) is an eigenfunction corresponding to \( \lambda_0 \) with weight \( m \). Because \( u_n \) changes sign and \( u_n \to u_0 \) in \( C^1(\Omega) \), \( u_0 \) vanishes somewhere in \( \overline{\Omega} \). Hence, we can see that \( \lambda_0 \neq 0 \) and \( \lambda_0 \neq \lambda^*(m) \) because any eigenfunction corresponding to the principal eigenvalue (that is, 0 or \( \lambda^*(m) \)) is positive or negative in \( \overline{\Omega} \) (see [15, Proposition 4.2.]). This implies that \( \lambda_0 \geq c(m) \) by Lemma 8, and so \( \liminf_{n \to \infty} c(m + \varepsilon_n) \geq c(m) \). As a result, our conclusion is shown since \( \{\varepsilon_n\} \) is an arbitrary sequence.

**Lemma 10.** Assume \( \int_{\Omega} m dx > 0 \). Then,

\[
X(m) := \left\{ u \in W^{1,p}(\Omega); \int_{\Omega} m|u|^{p-2}u dx = 0 \text{ and } \int_{\Omega} m|u|^p dx = 1 \right\} \neq \emptyset \quad (2.12)
\]
and
\[ c(m) \geq \lambda_{X(m)} := \inf_{u \in X(m)} \int_{\Omega} |\nabla u|^p \, dx > 0 \]  
(2.13)
hold, where $c(m)$ is the second eigenvalue defined by (2.11).

**Proof.** Since $c(m) > 0$ is an eigenvalue of (2.4), there exists a non-trivial solution $u$ for $-\Delta_p u = c(m) m |u|^{p-2} u$ in $\Omega$, $\partial u / \partial v = 0$ on $\partial \Omega$. Then, by taking $\phi \equiv 1$ or $u$ as test function, we have $\int_\Omega m |u|^{p-2} u \, dx = 0$ (note $c(m) > 0$) and $\int_\Omega m |u|^p \, dx > 0$ because $u$ changes sign (so $\| \nabla u \|_p > 0$, cf. [14, Proposition 2.4] or Proposition 2 with $C_0 = C_1 = p - 1$), and hence $u/(\int_\Omega m |u|^p \, dx)^{1/p}$ belongs to $X(m)$. As a result, it is easily seen that $c(m) \geq \lambda_{X(m)}$ holds.

Now, we shall prove $\lambda_{X(m)} > 0$ by way of contradiction. So, we assume that there exists a $\{u_n\} \subset X(m)$ such that $\| \nabla u_n \|_p \to 0$ as $n \to \infty$. Then, since Lemma 2 (with $\int_\Omega (-m) |u_n|^p \, dx = -1$) guarantees the boundedness of $\| u_n \|$, by choosing a subsequence if necessary, we may suppose that $u_n$ strongly converges to some constant function $u_0$ in $W^{1,p}(\Omega)$ (note $\| \nabla u_0 \|_p = 0$). Hence, $u_0 = 1/(\int_\Omega m \, dx)^{1/p}$ holds because of $\int_\Omega m |u_0|^p \, dx = 1$. On the other hand, by taking the limit in the equality

\[ 0 = \int_\Omega m |u_n|^{p-2} u_n \, dx = u_0 \int_\Omega m |u_n|^{p-2} u_n \, dx = \int_\Omega m |u_n|^{p-2} u_n u_0 \, dx, \]

we obtain $\int_\Omega m |u_0|^p \, dx = 0$. This is a contradiction.

** Remark 4.** In the case of $p \geq 2$ and $m \equiv 1$, it is proved that $c(m) = \lambda_{X(m)}$ holds (see [12, Theorem 6.2.29]).

### 2.2. Elementary results

Here, we define a positive constant $A_p$ by

\[ A_p := \frac{C_1}{p-1} \left( \frac{C_1}{C_0} \right)^{p-1} \geq 1, \]

(2.14)
which is equal to 1 in the case of $A(x,y) = |y|^{p-2} y$ (i.e. the special case of the $p$-Laplacian) because we can choose $C_0 = C_1 = p - 1$.

**Lemma 11.** Let $\varepsilon > 0$. For every $u$, $\phi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega)$ with $u \geq 0$ and $\phi \geq 0$ in $\Omega$, we have

\[ \int_\Omega A(x, \nabla u) \nabla \left( \frac{\phi^p}{(u + \varepsilon)^{p-1}} \right) \, dx \leq A_p \| \nabla \phi \|_p^p. \]

**Proof.** Let $\varepsilon > 0$ and let $u$, $\phi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega)$ satisfy $u \geq 0$ and $\phi \geq 0$ in $\Omega$. Then, we have

\[ A(x, \nabla u) \nabla \left( \frac{\phi^p}{(u + \varepsilon)^{p-1}} \right) \]
\[
= p \left( \frac{\phi}{u + \varepsilon} \right)^{p-1} A(x, \nabla u) \nabla \phi - (p-1) \left( \frac{\phi}{u + \varepsilon} \right)^p A(x, \nabla u) \nabla u \\
\leq \frac{pC_1}{p-1} \left( \frac{\phi}{u + \varepsilon} \right)^{p-1} |\nabla u|^{p-1} |\nabla \phi| - C_0 \left( \frac{\phi}{u + \varepsilon} \right)^p |\nabla u|^p \\
= \left\{ \left( \frac{pC_0}{p-1} \right)^{1/p} \frac{\phi}{u + \varepsilon} \right\}^{p-1} \left( \frac{p}{p-1} \right)^{1/p} C_1 C_0^{(1-p)/p} |\nabla \phi| \\
- C_0 \left( \frac{\phi}{u + \varepsilon} \right)^p |\nabla u|^p \\
\leq A_p |\nabla \phi|^p \text{ in } \Omega
\]

by (ii) and (iii) in Remark 1 and Young’s inequality.

**Proposition 2.** Let \( h \in L^\infty(\Omega)_+ \). If one of the following holds, then the equation \((P; \lambda, m, h)\) has no solutions \( u \not\equiv 0 \) such that \( u(x) \geq 0 \) for a.e. \( x \in \Omega \):

(i) \( m \geq 0 \) in \( \Omega \) and \( \lambda > 0 \) if \( h \not\equiv 0 \) or \( \lambda \neq 0 \) if \( h \equiv 0 \);

(ii) \( \int_\Omega m dx > 0, |m < 0| > 0 \) and \( \lambda \not\in [-A_p \lambda^*(m), 0] \);

(iii) \( \int_\Omega m dx = 0 \) and \( \lambda \neq 0 \) if \( h \equiv 0 \) or \( \lambda \in \mathbb{R} \) if \( h \not\equiv 0 \);

(iv) \( \int_\Omega m dx < 0 \) and \( \lambda \not\in [0, A_p \lambda^*(m)] \).

**Proof.** Let \( u \) be a non negative solution of \((P; \lambda, m, h)\) with \( u \not\equiv 0 \). Then, \( u \in C^{1,\alpha}(\overline{\Omega}) \) (some \( 0 < \alpha < 1 \)) and \( \min_{\overline{\Omega}} u > 0 \) by Remark 2.

(i): By taking \( \phi \equiv 1 \) as test function, we have

\[
0 = \lambda \int_\Omega m u^{p-1} dx + \int_\Omega h dx.
\]

In the case of \( h \not\equiv 0 \), this yields \( \lambda < 0 \) because of

\[
\lambda \int_\Omega m u^{p-1} dx = - \int_\Omega h dx < 0 \text{ and } \int_\Omega m u^{p-1} dx > 0.
\]

In the case of \( h \equiv 0 \), then we see that \( \lambda = 0 \) occurs.

(ii) \sim (iv): By Lemma 11, we obtain

\[
A_p \|\nabla \phi\|_p^p \geq \int_\Omega A(x, \nabla u) \nabla \left( \frac{\phi^p}{(u + \varepsilon)^{p-1}} \right) dx \\
= \lambda \int_\Omega m \left( \frac{u}{u + \varepsilon} \right)^{p-1} \phi^p dx + \int_\Omega h \frac{\phi^p}{(u + \varepsilon)^{p-1}} dx \\
\geq \lambda \int_\Omega m \left( \frac{u}{u + \varepsilon} \right)^{p-1} \phi^p dx
\]
for every \( \varepsilon > 0 \) and \( \varphi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega) \) satisfying \( \varphi \geq 0 \) in \( \Omega \). Thus, by \( \varepsilon \downarrow 0 \) (note \( u > 0 \) in \( \overline{\Omega} \)), we have \( A_p \| \nabla \varphi \|^p \geq \lambda \int_{\Omega} m \varphi^p \, dx \) for every \( \varphi \in W^{1,p}(\Omega) \cap C^1(\Omega) \cap L^\infty(\Omega) \) satisfying \( \varphi \geq 0 \) in \( \Omega \). By combining the above inequality and an argument as in [14, Proposition 2.4.], we can easily prove our assertion (note \( \lambda m = (-\lambda)(-m) \)).

**Proposition 3.** Let \( f_n: \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying

\[
|f_n(x,t)| \leq D(1 + |t|^{r-1}) \quad \text{for every} \ x \in \Omega, \ t \in \mathbb{R}
\]

with some positive constant \( D \) independent of \( n \) and \( r \in [p, p^*) \), where \( p^* = \infty \) if \( N \leq p \), \( p^* = pN/(N-p) \) if \( N > p \). Assume that \( A_n: \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is a map satisfying (A) (i), (ii), (iii) and (iv) with positive constants \( C_1, C'_1 \) and \( C_2 \) independent of \( n \). If \( u_n \) is a solution for

\[
-\text{div} A_n(x,\nabla u) = f_n(x,u) \quad \text{in} \ \Omega, \quad \partial u / \partial \nu = 0 \quad \text{on} \ \partial \Omega
\]

and \( \{u_n\} \) is bounded in \( W^{1,p}(\Omega) \), then there exist a subsequence \( \{u_{n_l}\} \) of \( \{u_n\} \) and \( u_0 \in C^1(\overline{\Omega}) \) such that \( u_{n_l} \to u_0 \) in \( C^1(\overline{\Omega}) \) as \( l \to \infty \).

**Proof.** Since \( \{u_n\} \) is bounded in \( W^{1,p}(\Omega) \), we may assume that \( u_n \) weakly converges to some \( u_0 \) in \( W^{1,p}(\Omega) \) by choosing a subsequence.

We can show that there exists a \( C > 0 \) depend only \( |\Omega|, \ p, \ N, \ D, \ C'_1, \ C_2 \) and the embedding constant of \( W^{1,p}(\Omega) \) to \( L^{\overline{p}}(\Omega) \) such that

\[
\|u_n\|_\infty \leq C \max\{1, \|u_n\|((\overline{p}-p)/(\overline{p}-r))\}
\]

by the Moser iteration process (refer to Theorem C in [19]), where \( \overline{p} = p^* \) if \( N > p \) and \( \overline{p} = r \) is any constant if \( N \leq p \). Since \( D, \ C'_1 \) and \( C'_0 \) are independent of \( n \), \( \|u_n\|_\infty \) is bounded. Therefore, the regularity result in [17] guarantees that there exist \( \gamma \in (0, 1) \) and \( M > 0 \) independent of \( n \) such that \( u_n \in C^{1,\gamma}(\overline{\Omega}) \) and \( \|u_n\|_{C^{1,\gamma}(\overline{\Omega})} \leq M \) (where we use the fact that \( C'_2 \) is independent of \( n \) also). Since the inclusion of \( C^{1,\gamma}(\overline{\Omega}) \) to \( C^1(\overline{\Omega}) \) is compact, \( u_n \) converges \( u_0 \) in \( C^1(\overline{\Omega}) \) (note that \( u_n \to u_0 \) in \( W^{1,p}(\Omega) \)).

### 3. Antimaximum principle

In this section, we assume that \( \int_{\Omega} m \, dx > 0 \) without loss of generality by noting \( \lambda m = (-\lambda)(-m) \).

**Theorem 1.** Assume \( \int_{\Omega} m \, dx > 0 \) (resp. \( \int_{\Omega} m \, dx = 0 \)). Then, for any \( 0 \neq h \in L^\infty(\Omega)_+ \) there exists \( \delta = \delta(h) > 0 \) such that any solution \( u \) of \( (P; \lambda, m, h) \) satisfies \( u < 0 \) in \( \overline{\Omega} \) provided \( 0 < \lambda < \delta \) (resp. \( 0 < |\lambda| < \delta \)).
Proof. Because of $\lambda m = (-\lambda)(-m)$, it is sufficient to prove that for any $0 \neq h \in L^\infty(\Omega)_+$ there exists $\delta = \delta(h) > 0$ such that any solution $u$ of $(P; \lambda, m, h)$ satisfies $u < 0$ in $\overline{\Omega}$ provided $0 < \lambda < \delta$. By way of contradiction, we may assume that there exist $0 \neq h \in L^\infty(\Omega)_+$, $\{\lambda_n\}$ and a solution $u_n \in W^{1,p}(\Omega)$ of $(P; \lambda_n, m, h)$ such that

Moreover, we note that $\|u_n\|$ is bounded if $\|u_n\|_p$ is bounded by the following inequality

$$
\frac{C_0}{p-1} \|\nabla u_n\|^p_p \leq \int_\Omega A(x, \nabla u_n)\nabla u_n \, dx = \lambda_n \int_\Omega m |u_n|^p + \int_\Omega h u_n \, dx
\leq \lambda_n \|m\|_\infty \|u_n\|^p_p + \|h\|_\infty \|u_n\|_1,
$$

(3.1)

where we use (iii) in Remark 1. Hence, by applying Proposition 3 to $A_n(x, y) = A(x, y)$ or $A_n(x, y) := A(x, y\|u_n\|_p)/\|u_n\|_p^{-1}$, we see that $u_n$ or $u_n/\|u_n\|_p$ has a convergent subsequence in $C^1(\overline{\Omega})$ in the case where $\|u_n\|_p$ is bounded or not, respectively. Therefore, by the same argument as in [14, Theorem 3.2.], we can obtain a contradiction.

It follows from the following proposition that we can not take such $\delta$ independent of $h$ as in Theorem 1.

**Proposition 4.** Assume that $N \geq p$ and $\int_\Omega m \, dx \geq 0$. Then, for any $\varepsilon > 0$ there exists $0 \neq h \in L^\infty(\Omega)_+$ such that for any $\lambda \geq \varepsilon$ the equation $(P; \lambda, m, h)$ has no solution $u$ satisfying $u \leq 0$ in $\overline{\Omega}$ and $\{x \in \Omega_m; u(x) = 0\} = 0$, where $\Omega_m := \{x \in \Omega; m(x) \neq 0\}$.

**Proof.** By using Lemma 11 instead of [14, Lemma 2.5.] as in the argument of [14, Theorem 3.5.], we shall give the proof. Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for any $0 \neq h \in L^\infty(\Omega)_+$, there exist $\lambda_h \geq \varepsilon_0$ and $u_h$ being a solution of $(P; \lambda_h, m, h)$ with $u_h \leq 0$ in $\overline{\Omega}$ and $\{x \in \Omega_m; u_h(x) = 0\} = 0$. Fix $0 < \delta < \varepsilon_0/A_p$, where $A_p$ is the positive constant defined by (2.14). Because we know $\lambda(m) = \lambda^*(m) = 0$ in the case of $N \geq p$ by Lemma 5, there exists $\varphi \in W^{1,p}(\Omega)$ such that $\varphi = 0$ on some (open) ball $B \subset \Omega$,

$$
\int_\Omega m |\varphi|^p \, dx = 1 \quad \text{and} \quad \int_\Omega |\nabla \varphi|^p \, dx < \delta.
$$

By considering $|\varphi|$ instead of $\varphi$, we may assume that $\varphi \geq 0$ in $\Omega$. Here, we choose $h \in C^0(\Omega)$ such that $h \geq 0$, $h \neq 0$ and $\text{supp} h \subset B$. By the above contradictory hypothesis, we can obtain $\lambda_h \geq \varepsilon_0$ and $u_h \in W^{1,p}(\Omega)$ being a solution of $(P; \lambda_h, m, h)$ with $u_h \leq 0$ in $\overline{\Omega}$ and $\{x \in \Omega_m; u_h(x) = 0\} = 0$. Set $v = -u_h$, then $v$ is non negative solution of

$$
-\text{div} A(x, \nabla v) = \lambda_h m v^{p-1} - h \quad \text{in} \ \Omega, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \ \partial \Omega
$$

since $A$ is odd in the second variable. Let $\varphi_M := \max \{\varphi, M\} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ for $M > 0$. Then, for this $\varphi_M$ and $v$, the inequality as in Lemma 11 holds because we
see that $p(\varphi_M/(v+\varepsilon))^{p-1}\nabla \varphi_M - (p-1)(\varphi_M/(v+\varepsilon))p \nabla v \in L^p(\Omega)^N$ (note $v = -u_h \in C^1(\overline{\Omega})$ (see Remark 2)). Thus, we obtain

$$A_p \|\nabla \varphi_M\|_p^p \geq \int_{\Omega} A(x, \nabla v) \nabla \left( \frac{\varphi_M^p}{(v+\varepsilon)^{p-1}} \right) dx = \lambda_h \int_{\Omega_m} m \left( \frac{v}{v+\varepsilon} \right)^{p-1} \varphi_M^p dx$$

for every $\varepsilon > 0$ and $M > 0$ by supp $h \cap$ supp $\varphi_M = \text{supp} h \cap$ supp $\varphi = \emptyset$. Because $v > 0$ a.e. on $\Omega_m$, by taking $\varepsilon \downarrow 0$ and $M \to \infty$ in the above inequality, we obtain

$$\varepsilon_0 \leq \lambda_h = \lambda_h \int_{\Omega} m \varphi^p dx = \lambda_h \int_{\Omega_m} m \varphi^p dx \leq A_p \|\nabla \varphi\|_p^p < A_p \delta < \varepsilon_0.$$ 

This is a contradiction.

**Remark 5.** For the usual $p$-Laplace equation under the Dirichlet boundary condition, it is known that AMP holds at right of the principal eigenvalue $\lambda_1(m)$ and at left of $-\lambda_1(-m)$ (see [14]). However, in generall, we do not know wherther AMP holds near $\pm \lambda_1(\pm m)$ or not for the equation

$$-\text{div} A(x, \nabla u) = \lambda m |u|^{p-2}u + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \quad (3.2)$$

A major cause is that $C_0 \lambda_1(m)/(p-1) < A_p \lambda_1(m)$ occurs in the case of $C_0 < C_1$ because $\lambda_1(m)$ is positive. On the other hand, by the same argument as in the proof of Proposition 2, we can prove that equation (3.2) has no positive solutions provided $\lambda \not\in [-A_p\lambda_1(-m), A_p\lambda_1(m)]$ and $0 \neq h \in L^{\infty}(\Omega)_+.$

**3.1. The case of $N < p$**

**Theorem 2.** Assume that $N < p$ and $0 \neq h \in L^{\infty}(\Omega)_+$. Then, the following assertions hold:

(i) Suppose $\int_{\Omega} m dx > 0$ and $\lambda$ satisfies $0 < \lambda \leq C_0 \bar{\lambda}(m)/(p-1)$. Then, any solution $u$ of $(P; \lambda, m, h)$ satisfies $u < 0$ in $\Omega$.

In addition, if $|\{m < 0\}| > 0$ and $(C_1/C_0)^p \lambda^*(-m) > \bar{\lambda}(-m)$, then the same conclusion holds for every $\lambda$ satisfying

$$- \frac{C_0}{p-1} \bar{\lambda}(-m) \leq \lambda < -A_p \lambda^*(-m);$$

(ii) Suppose $\int_{\Omega} m dx = 0$ and $\lambda$ satisfies

$$0 < \lambda \leq \frac{C_0 \bar{\lambda}(m)}{p-1} \quad \text{or} \quad - \frac{C_0 \bar{\lambda}(-m)}{p-1} \leq \lambda < 0.$$

Then, any solution $u$ of $(P; \lambda, m, h)$ satisfies $u < 0$ in $\Omega$. 
Proof. By $\lambda m = (-\lambda)(-m)$ and $\lambda^*(m) = 0$ if $\int_{\Omega} m \, dx \geq 0$, it is sufficient to prove that any solution of $(P; \lambda, m, h)$ is negative in $\overline{\Omega}$ under the hypothesis that $0 \neq h \in L^\infty(\Omega)_+$, $A_p \lambda^*(m) < \lambda \leq C_0 \tilde{\lambda}(m)/(p - 1)$ with any $m \in L^\infty(\Omega)$ with (1.1). By way of contradiction, we shall prove our assertion above. So, we assume that there exist $m \in L^\infty(\Omega)$ with (1.1), $0 \neq h \in L^\infty(\Omega)_+$, $\lambda \in (A_p \lambda^*(m), C_0 \tilde{\lambda}(m)/(p - 1)]$ and $u \in W^{1,p}(\Omega)$ being a solution of $(P; \lambda, m, h)$ with $u \geq 0$ somewhere in $\overline{\Omega}$. By taking $\varphi = -u$ as test function, we have

$$\frac{C_0}{p - 1} \|\nabla u_-\|_p^p \leq \int_{\Omega} A(x, \nabla u)(-\nabla u_-) \, dx$$

$$= \lambda \int_{\Omega} mu_- \, dx - \int_{\Omega} hu_- \, dx$$

$$\leq \lambda \int_{\Omega} mu_- \, dx.$$  \hspace{1cm} (3.3)

Then, we can see that $\int_{\Omega} mu_- \, dx > 0$. Indeed, if $\int_{\Omega} mu_- \, dx = 0$ (note $\lambda > 0$), then $\nabla u_- \equiv 0$ and $\int_{\Omega} hu_- \, dx = 0$ holds, whence $u_- \equiv 0$. Thus, $u \geq 0$ and $u \neq 0$ by $h \neq 0$. This contradicts to Proposition 2 because of $\lambda > A_p \lambda^*(m)$.

As a result, we can get a contradiction easily by the following inequality (obtained by (3.3))

$$\frac{\|\nabla u_-\|_p^p}{\int_{\Omega} mu_- \, dx} \leq \frac{p - 1}{C_0} \lambda \leq \tilde{\lambda}(m),$$

the definition of $\tilde{\lambda}(m)$, Lemma 6 and a similar argument to [5, Theorem 2.1.].

**Theorem 3.** Assume that $N < p$ and $0 \neq h \in L^\infty(\Omega)_+$. Then, the following assertions hold:

(i) Let $\int_{\Omega} m \, dx > 0$. Then, there exists $\delta = \delta(h) > 0$ for every $\lambda$ satisfying

$$C_0 \tilde{\lambda}(m)/(p - 1) < \lambda < C_0 \tilde{\lambda}(m)/(p - 1) + \delta$$

such that any solution $u$ of $(P; \lambda, m, h)$ satisfies $u < 0$ in $\overline{\Omega}$.

In addition, if $|\{m < 0\}| > 0$ and $(C_1/C_0)^p \tilde{\lambda}^*(-m) < \tilde{\lambda}(-m)$, then there exists $\delta' = \delta'(h) > 0$ such that the same conclusion holds for every $\lambda$ satisfying

$$- \frac{C_0}{p - 1} \tilde{\lambda}(-m) - \delta' < \lambda < - \frac{C_0}{p - 1} \tilde{\lambda}(-m);$$

(ii) Let $\int_{\Omega} m \, dx = 0$. Then, there exists $\delta = \delta(h) > 0$ for every $\lambda$ satisfying

$$\frac{C_0 \tilde{\lambda}(m)}{p - 1} < \lambda < \frac{C_0 \tilde{\lambda}(m)}{p - 1} + \delta \text{ or } - \frac{C_0 \tilde{\lambda}(-m)}{p - 1} - \delta' < \lambda < - \frac{C_0 \tilde{\lambda}(-m)}{p - 1}$$

such that any solution $u$ of $(P; \lambda, m, h)$ satisfies $u < 0$ in $\overline{\Omega}$.
Proof. By the same reason as in the proof of Theorem 2, it is sufficient to prove that for any $m \in L^\infty(\Omega)$ with (1.1) and $0 \neq h \in L^\infty(\Omega)_+$, there exists a $\delta > 0$ such that any solution of $(P; \lambda, m, h)$ is negative in $\overline{\Omega}$ if

$$C_0 \frac{\tilde{\lambda}(m)}{p - 1} < \lambda < C_0 \frac{\tilde{\lambda}(m)}{p - 1} + \delta$$

under the hypothesis $A_p \lambda^*(m) < C_0 \tilde{\lambda}(m)/(p - 1)$ (note $A_p \lambda^*(m) < C_0 \tilde{\lambda}(m)/(p - 1)$ if and only if $(C_1/C_0)^p \lambda^*(m) < \tilde{\lambda}(m)$), where $A_p$ is the positive constant defined by (2.14). Thus, by way of contradiction, we assume that there exist $m \in L^\infty(\Omega)$ with (1.1), $0 \neq h \in L^\infty(\Omega)_+$, $\{\lambda_n\}$ and $\{u_n\} \subset W^{1,p}(\Omega)$ such that $\lambda_n \downarrow C_0 \tilde{\lambda}(m)/(p - 1)$ and $u_n$ is a solution of $(P; \lambda_n, m, h)$ satisfying $u_n \geq 0$ somewhere in $\overline{\Omega}$.

If $\|u_n\|_p$ is bounded, then we can obtain a subsequence $\{u_{n_k}\}$ convergent to some $u_0$ in $C^1(\overline{\Omega})$ by Proposition 3 with $A_n = A$. This implies that $u_0$ is a solution of $(P; \lambda, m, h)$ with $u_0 \geq 0$ somewhere in $\Omega$ for $\lambda = C_0 \tilde{\lambda}(m)/(p - 1)$. This contradicts to Theorem 2.

Thus, we may assume that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence if necessary. Set $v_n := u_n/\|u_n\|_p$. Then, by a similar inequality to (3.1), we can get the boundedness of $\|v_n\|$. So, we may suppose, by choosing a subsequence, that there exists $v \in W^{1,p}(\Omega)$ such that $v_n \to v$ in $W^{1,p}(\Omega)$ and $v_n(x) \to v(x)$ uniformly in $x \in \overline{\Omega}$. We note that $v \geq 0$ somewhere in $\overline{\Omega}$ because $v_n \geq 0$ somewhere in $\overline{\Omega}$. Moreover, we can obtain (note $\lambda_n \to C_0 \tilde{\lambda}(m)/(p - 1)$):

$$\|\nabla v_+\|_p^p \leq \tilde{\lambda}(m) \int_{\Omega} mv_+^p \, dx \quad \text{and} \quad \|\nabla v_-\|_p^p \leq \tilde{\lambda}(m) \int_{\Omega} mv_-^p \, dx$$

(3.4)

by taking the limit inferior in the following inequalities.

$$\frac{C_0}{p - 1} \|\nabla v_+\|_p^p \leq \int_{\Omega} A(x, \nabla u_n) \frac{\nabla u_{n+}}{\|u_n\|_p^p} \, dx = \lambda_n \int_{\Omega} mv_{n+}^p \, dx + \int_{\Omega} h \frac{v_{n+}}{\|u_n\|_p^p} \, dx,$$

$$\frac{C_0}{p - 1} \|\nabla v_-\|_p^p \leq \int_{\Omega} A(x, \nabla u_n) \frac{-\nabla u_{n-}}{\|u_n\|_p^p} \, dx = \lambda_n \int_{\Omega} mv_{n-}^p \, dx - \int_{\Omega} h \frac{v_{n-}}{\|u_n\|_p^p} \, dx,$$

where we use (iii) in Remark 1.

Here, we shall consider by dividing into three cases:

(a) $\int_{\Omega} mv_+^p \, dx > 0$; \hspace{10pt} (b) $\int_{\Omega} mv_+^p \, dx = 0$ and $\int_{\Omega} mv_-^p \, dx = 0$;

(c) $\int_{\Omega} mv_+^p \, dx = 0$ and $\int_{\Omega} mv_-^p \, dx > 0$.

Case (a): If $v_+ > 0$ in $\overline{\Omega}$, then $v_+$ is positive in $\overline{\Omega}$ for sufficiently large $n$ because $v = v_+ > 0$ in $\overline{\Omega}$ and $v_n(x) \to v(x)$ uniformly in $x \in \overline{\Omega}$. This means that $u_n$ is a positive solution of $(P; \lambda_n, m, h)$ for sufficiently large $n$. This contradicts to Proposition 2 because of $\lambda_n > C_0 \tilde{\lambda}(m)/(p - 1) > A_p \lambda^*(m)$. So, we suppose that $v_+$ vanishes somewhere in $\overline{\Omega}$. Then, it follows from (3.4) that $v_+/(\int_{\Omega} mv_+^p dx)^{1/p}$ is a minimizer for $\tilde{\lambda}(m)$. Thus $v_+$ vanishes at exactly one point $x_0 \in \overline{\Omega}$ by Lemma 6, whence $v = v_+$ occurs. Now we shall prove that

$$\frac{C_0 \tilde{\lambda}(m)}{p - 1} \int_{\Omega} m \varphi^p \, dx \leq A_p \|\nabla \varphi\|_p^p \quad \text{for every} \ \varphi \in W^{1,p}(\Omega) \ \text{with} \ \varphi \geq 0.$$
If (3.5) is shown, then we have a contradiction because we can choose some \( \phi \in W^{1,p}(\Omega) \) with \( \phi \geq 0 \), \( \int_{\Omega} m \phi^p \, dx = 1 \) and \( \| \nabla \phi \|_p < \lambda^*(m) + \delta \) for \( \delta > 0 \) satisfying \( A_p \delta < C_0 \tilde{\lambda}(m)/(p-1) - A_p \lambda^*(m) \) (note \( C_0 \tilde{\lambda}(m)/(p-1) > A_p \lambda^*(m) \)).

To prove (3.5), we fix \( \varepsilon > 0 \) and \( \phi \in C^1(\overline{\Omega}) \) with \( \phi \geq 0 \). For sufficiently large \( n \), we have \( v_n + \varepsilon \geq \varepsilon/2 \) in \( \overline{\Omega} \), and hence \( u_n + \varepsilon \| u_n \|_p \geq \varepsilon \| u_n \|_p /2 > 0 \) in \( \overline{\Omega} \) since \( v_n \) converges to \( v = v^+ \) uniformly in \( \overline{\Omega} \). Thus, Lemma 11 yields the following inequality (note \( u_n \in C^1(\overline{\Omega}) \)):

\[
A_p \| \nabla \phi \|_p \geq \int_{\Omega} A(x, \nabla u_n) \nabla \left( \frac{\phi^p}{(u_n + \varepsilon \| u_n \|_p)^{p-1}} \right) \, dx
\]

\[
= \lambda_n \int_{\Omega} m \left( \frac{u_n}{u_n + \varepsilon \| u_n \|_p} \right)^{p-1} \phi^p \, dx + \int_{\Omega} h \left( \frac{\phi^p}{(u_n + \varepsilon \| u_n \|_p)^{p-1}} \right) \, dx
\]

\[
\geq \lambda_n \int_{\Omega} m \left( \frac{v_n}{v_n + \varepsilon} \right)^{p-1} \phi^p \, dx.
\]

Hence, by taking the limit in the above inequality, we have

\[
\frac{C_0 \tilde{\lambda}(m)}{p-1} \int_{\Omega} m \left( \frac{v}{v + \varepsilon} \right)^{p-1} \phi^p \, dx \leq A_p \| \nabla \phi \|_p.
\]

Moreover, by taking \( \varepsilon \downarrow 0 \), we can get (3.5) since \( C^1(\overline{\Omega}) \) is dense in \( W^{1,p}(\Omega) \) and \( v(x) > 0 \) if \( x \neq x_0 \).

Case (b): In this case, it follows from (3.4) that \( \nabla v = 0 \) holds, and so \( v \) is a constant function with \( \| v \|_p = 1 \). Because \( v \geq 0 \) somewhere in \( \overline{\Omega} \), we see \( v = 1/|\Omega|^{1/p} \).

Then, by the same reason as in the first part of the case (a), we have a contradiction.

Case (c): In this case, we can see that \( v \) is non positive in \( \overline{\Omega} \) (that is, \( v = -v_- \)) since \( \nabla v_+ \equiv 0 \) by (3.4) and \( \int_{\Omega} m v_-^p \, dx = 0 < \int_{\Omega} m v_+^p \, dx \).

If \( v = -v_- \) does not vanish in \( \overline{\Omega} \), then \( u_n < 0 \) in \( \overline{\Omega} \) for sufficiently large \( n \). This yields a contradiction because \( u_n \geq 0 \) somewhere in \( \overline{\Omega} \).

Thus, we may assume that \( v_- \) vanishes somewhere in \( \overline{\Omega} \). Then, (3.4) implies that \( v_- / (\int_{\Omega} m v_+^p \, dx)^{1/p} \) is a minimizer for \( \tilde{\lambda}(m) \). By considering

\[
\int_{\Omega} A(x, -\nabla u_n) \nabla \left( \frac{\phi^p}{(-u_n + \varepsilon \| u_n \|_p)^{p-1}} \right) \, dx \leq A_p \| \nabla \phi \|_p
\]

instead of (3.6) as in the proof of case (a), we have the same inequality (3.5) for every \( \phi \in W^{1,p}(\Omega) \) with \( \phi \geq 0 \) (note that \( A \) is odd in the second variable and \( -u_n(x) + \varepsilon \| u_n \|_p \to \infty \) uniformly in \( x \in \overline{\Omega} \)). As a result, we can get a contradiction by the same reason as in the last part of the case (a).

4. Existence of a solution

4.1. Existence of a positive solution

**Theorem 4.** Let \( 0 \neq h \in L^\infty(\Omega)_+ \). If one of the following cases holds, then \( (P; \lambda, m, h) \) has a positive solution:
(i) \( m \geq 0 \) in \( \Omega \) and \( \lambda < 0 \);

(ii) \( \int_{\Omega} mdx > 0, \ |\{m < 0\}| > 0 \) and \( 0 > \lambda > -C_0 \lambda^*(m)/(p-1) \);

(iii) \( \int_{\Omega} mdx < 0 \) and \( 0 < \lambda < C_0 \lambda^*(m)/(p-1) \),

where \( \lambda^*(m) \) is the principal eigenvalue obtained by (2.5).

To prove the existence of a positive solution, we define a \( C^1 \) functional \( I_\lambda^+ \) on \( W^{1,p}(\Omega) \) as follows:

\[
I_\lambda^+(u) := \int_{\Omega} G(x, \nabla u) \, dx - \frac{\lambda}{p} \int_{\Omega} mu^p_+ \, dx - \int_{\Omega} h u_+ \, dx + \frac{1}{p} \|u_-\|^p_p \tag{4.1}
\]

for \( \lambda \in \mathbb{R} \) and \( u \in W^{1,p}(\Omega) \), where \( G(x, y) := \int_0^{\|y\|} a(x, t) \, dt \) (see (2.1) for details).

**Remark 6.** We remark that non-trivial critical points of \( I_\lambda^+ \) correspond to positive solutions for \( (P; \lambda, m, h) \). Indeed, if \( u \) is a critical point of \( I_\lambda^+ \), then we have

\[
\frac{C_0}{p-1} \|\nabla u_-\|^p_p + \|u_-\|^p_p \leq \int_{\Omega} A(x, \nabla u)(-\nabla u_-) \, dx + \int_{\Omega} h u_- \, dx + \|u_-\|^p_p = 0
\]

by taking \(-u_-\) as test function. Thus, \( u_- \equiv 0 \), and hence \( u \geq 0 \). As a result,

\[
\int_{\Omega} A(x, \nabla u) \nabla \varphi \, dx = \lambda \int_{\Omega} mu^{p-1} \varphi \, dx + \int_{\Omega} h \varphi \, dx
\]

holds for every \( \varphi \in W^{1,p}(\Omega) \). Because of \( u \not\equiv 0 \), \( u \) is a positive solution of \( (P; \lambda, m, h) \) by Remark 2.

**Lemma 12.** Let \( 0 \neq h \in L^\infty(\Omega)_+ \). If either \( m \geq 0 \) and \( \lambda < 0 \) or \( \int_{\Omega} mdx < 0 \) and \( 0 < \lambda < C_0 \lambda^*(m)/(p-1) \) holds, then \( I_\lambda^+ \) is bounded from below, coercive and weakly lower semi-continuous (w.l.s.c.) on \( W^{1,p}(\Omega) \).

**Proof.** Note that \( \Phi(u) := \int_{\Omega} G(x, \nabla u) \, dx \) is w.l.s.c. on \( W^{1,p}(\Omega) \) (cf. [18, Theorem 1.2]) because \( \Phi \) is convex and continuous on \( W^{1,p}(\Omega) \). Thus, \( I_\lambda^+ \) is also w.l.s.c. on \( W^{1,p}(\Omega) \) since the inclusion of \( W^{1,p}(\Omega) \) to \( L^p(\Omega) \) is compact.

Now, we prove that \( I_\lambda^+ \) is bounded from below and coercive on \( W^{1,p}(\Omega) \).

Case of \( m \geq 0 \) and \( \lambda < 0 \): By Lemma 3 and (2.2), we can obtain

\[
I_\lambda^+(u) \geq \frac{C_0}{p(p-1)} \|\nabla u\|^p_p + \frac{\lambda}{p} \int_{\Omega} mu^p_+ \, dx - \|h\|_{\infty} \|u\|_1 + \frac{1}{p} \|u_-\|^p_p \\
\geq \frac{C_0}{p(p-1)} \|u_-\|^p_p + \frac{C_0}{2p(p-1)} \left( \|\nabla u_+\|^p_p + \frac{2(p-1)|\lambda|}{C_0} \int_{\Omega} mu^p_+ \, dx \right) \\
+ \frac{C_0}{2p(p-1)} \|\nabla u_+\|^p_p - \|h\|_{\infty} \|u\|_p |\Omega|(p-1)/p \\
\geq \frac{C_0}{p(p-1)} \|u_-\|^p_p + \frac{C_0 \min\{d(\xi), 1\}}{2p(p-1)} \|u_+\|^p_p - \|h\|_{\infty} \|u\|_p |\Omega|(p-1)/p
\]
for every $u \in W^{1,p}(\Omega)$ (note $p - 1 \geq C_0$), where $d(\xi) > 0$ is a constant obtained by Lemma 3 with $\xi = 2(p - 1)|\lambda|/C_0$. This implies that $I_\lambda^+$ is bounded from below, coercive because of $p > 1$.

Case of $\int_{\Omega} mdx < 0$ and $0 < \lambda < C_0 \lambda^*(m)/(p - 1)$: For every $u \in W^{1,p}(\Omega)$ with $\int_{\Omega} mu_+^p dx > 0$, we have

\[
I_\lambda^+(u) \geq \frac{1}{p} \left( \frac{C_0}{p} - \frac{\lambda}{\lambda^*(m)} \right) \|\nabla u_+\|^p_p + \frac{C_0}{p(p - 1)} \|\nabla u_-\|^p_p - \|h\|_\infty \|u\|_1 + \frac{1}{p} \|u_-\|^p_p
\]

\[
\geq \frac{1}{2p} \left( \frac{C_0}{p} - \frac{\lambda}{\lambda^*(m)} \right) \|\nabla u_+\|^p_p + \frac{c}{2p} \left( \frac{C_0}{p} - \frac{\lambda}{\lambda^*(m)} \right) \|u_+\|^p_p
\]

\[
+ \frac{C_0}{p(p - 1)} \|u_-\|^p_p - \|h\|_\infty \|u\|^p_{p|\Omega|^{(p - 1)/p}} \tag{4.2}
\]

by (2.2), the definition of $\lambda^*(m)$, Lemma 1 and $C_0 \leq p - 1$, where $c$ is a positive constant independent of $u$ obtained in Lemma 1. Next, we deal with $u \in W^{1,p}(\Omega)$ with $\int_{\Omega} mu_+^p dx \leq 0$. Take a $\delta$ such that $0 < \delta < \lambda$. Then, we obtain for any $u \in W^{1,p}(\Omega)$ with $\int_{\Omega} mu_+^p dx \leq 0$

\[
I_\lambda^+(u) \geq \frac{C_0}{2p(p - 1)} \left( \|\nabla u_+\|^p_p - \frac{2(p - 1)\delta}{C_0} \int_{\Omega} mu_+^p dx \right) + \frac{C_0}{2p(p - 1)} \|\nabla u_+\|^p_p
\]

\[
+ \frac{\delta - \lambda}{p} \int_{\Omega} mu_+^p dx + \frac{C_0}{p(p - 1)} \|u_-\|^p_p - \|h\|_\infty \|u\|_1
\]

\[
\geq \frac{C_0 b(m, \xi)}{2p(p - 1)} \|u_+\|^p_p + \frac{C_0}{2p(p - 1)} \|\nabla u_+\|^p_p
\]

\[
+ \frac{C_0}{p(p - 1)} \|u_-\|^p_p - \|h\|_\infty \|u\|^p_{p|\Omega|^{(p - 1)/p}} \tag{4.3}
\]

by (2.2) and Lemma 2, where $b(m, \xi)$ is a positive constant obtained in Lemma 2 with $\xi = 2(p - 1)\delta/C_0$. Consequently, our conclusion follows from (4.2) and (4.3).

**Proof of Theorem 4.** By the properties of $I_\lambda^+$ obtained as in Lemma 12, $I_\lambda^+$ has a global minimizer in all cases as in Theorem 4 (cf. [18, Theorem 1.1.]), where, we use $\lambda m = (-\lambda)(-m)$ in the case (ii). Thus, we see that $(P; \lambda, m, h)$ has a positive solution by Remark 6.

### 4.2. Other existence results for the general case

**Theorem 5.** Assume $0 \neq h \in L^\infty(\Omega)_+$. If one of the following conditions holds, then $(P; \lambda, m, h)$ has a solution:

- (i) $m \geq 0$ in $\Omega$ and $0 < \lambda < C_0 c(m)/(p - 1)$;
- (ii) $\int_{\Omega} mdx > 0$ and $0 < \lambda < C_0 \lambda^*(m)/(p - 1)$;
- (iii) $N < p$ and $A_p \lambda^*(m) < \lambda < C_0 \lambda(m)/(p - 1)$,
where $c(m)$, $\lambda_{X(m)}$, $A_p$ and $\tilde{\lambda}(m)$ are positive constants defined by (2.11), (2.13), (2.14) and (2.7), respectively.

To show the existence of a solution, we define a $C^1$ functional $I_\lambda$ on $W^{1,p}(\Omega)$ as follows:

$$I_\lambda(u) := \int_\Omega G(x, \nabla u) \, dx - \frac{\lambda}{p} \int_\Omega m|u|^p \, dx - \int_\Omega h u \, dx$$  \hspace{1cm} (4.4)

for $\lambda \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$, where $G(x,y) := \int_0^{|y|} a(x,t) \, dt$ (see (2.1) for details). Note that critical points of $I_\lambda$ correspond to solutions of $(P; \lambda, m, h)$ (see Remark 2).

First, we shall prove that $I_\lambda$ has the mountain pass geometry.

**Lemma 13.** Assume that $h \in L^\infty(\Omega)_+$, $\int_\Omega m \, dx \neq 0$ and

$$\frac{C_1 \lambda^*(m)}{p-1} < \lambda < \frac{C_0 c(m)}{p-1}.$$  

Then, $I_\lambda$ is bounded from below on $E(m)$ defined by

$$E(m) := \left\{ u \in W^{1,p}(\Omega); \|\nabla u\|^p_p \geq c(m) \int_\Omega m|u|^p \, dx \right\}. \hspace{1cm} (4.5)$$

Furthermore, there exist $u_0$, $u_1 \in W^{1,p}(\Omega)$ such that

$$\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{E(m)} I_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma(t))$$

for every $\gamma \in \Gamma$, where

$$\Gamma := \left\{ \gamma \in C([0,1], W^{1,p}(\Omega)); \gamma(0) = u_0, \gamma(1) = u_1 \right\}.$$  

**Proof.** First, we shall prove $\inf_{E(m)} I_\lambda > -\infty$. For every $u \in W^{1,p}(\Omega)$ with

$$\int_\Omega m|u|^p \, dx \leq 0,$$

we have

$$I_\lambda(u) \geq \frac{C_0}{p(p-1)} \|\nabla u\|^p_p - \frac{\lambda}{p} \int_\Omega m|u|^p \, dx - \|h\|_\infty \|u\|_1$$

$$\geq \frac{C_0 b(m, \xi)}{p(p-1)} \|u\|^p_p - \|h\|_\infty \|u\|_p \|\Omega\|(p-1)/p > -\infty$$  \hspace{1cm} (4.6)

by (2.2), Lemma 2 and Hölder’s inequality, where $b(m, \xi)$ is a positive constant obtained in Lemma 2 with $\xi = (p-1)\tilde{\lambda}/C_0$. Thus, $I_\lambda$ is bounded from below on $B(m)$, where $B(m)$ is a set defined by

$$B(m) := \{ u \in W^{1,p}(\Omega); \int_\Omega m|u|^p \, dx \leq 0 \} \subset E(m). \hspace{1cm} (4.7)$$
Here, we choose a constant $\delta$ such that $1 > \delta > \lambda(p - 1)/(c(m)C_0)$.

Let $m \geq 0$ in $\Omega$. In this case, for every $u \in E(m)$, we have

$$
I_\lambda(u) \geq \frac{C_0(1 - \delta)}{p(p - 1)} \|\nabla u\|_p^p + \frac{1}{p} \left( \frac{C_0\delta c(m)}{p - 1} - \lambda \right) \int_{\Omega} m |u|^p \, dx - \|h\|_{\infty} \|u\|_{p,\Omega}^{(p-1)/p}
$$

$$
= \frac{C_0(1 - \delta)}{p(p - 1)} \left\{ \|\nabla u\|_p^p + \frac{p - 1}{C_0(1 - \delta)} \left( \frac{C_0\delta c(m)}{p - 1} - \lambda \right) \int_{\Omega} m |u|^p \, dx \right\}
$$

$$
- \|h\|_{\infty} \|u\|_{p,\Omega}^{(p-1)/p}
$$

$$
\geq d(\xi') \frac{C_0(1 - \delta)}{p(p - 1)} \|u\|_p^p - \|h\|_{\infty} \|u\|_{p,\Omega}^{(p-1)/p} > -\infty
$$

(4.8)

by (2.2), the definition of $E(m)$ and Lemma 3, where $d(\xi') > 0$ is a constant obtained in Lemma 3 with

$$
\xi' = \frac{p - 1}{C_0(1 - \delta)} \left( \frac{C_0\delta c(m)}{p - 1} - \lambda \right).
$$

Similarly, in the other cases (that is, $m$ changes sign), for every $u \in E(m)$ with

$$
\int_{\Omega} m |u|^p \, dx > 0,
$$

we obtain

$$
I_\lambda(u) \geq \frac{C_0(1 - \delta)}{p(p - 1)} \|\nabla u\|_p^p + \frac{1}{p} \left( \frac{C_0\delta c(m)}{p - 1} - \lambda \right) \int_{\Omega} m |u|^p \, dx - \|h\|_{\infty} \|u\|_{p,\Omega}^{(p-1)/p}
$$

$$
= \frac{C_0(1 - \delta)}{p(p - 1)} \left\{ \|\nabla u\|_p^p + \frac{p - 1}{C_0(1 - \delta)} \left( \frac{C_0\delta c(m)}{p - 1} - \lambda \right) \int_{\Omega} (-m) |u|^p \, dx \right\}
$$

$$
- \|h\|_{\infty} \|u\|_{p,\Omega}^{(p-1)/p}
$$

$$
\geq b(-m, \xi') \frac{C_0(1 - \delta)}{p - 1} \|u\|_p^p - \|h\|_{\infty} \|u\|_{p,\Omega}^{(p-1)/p} > -\infty
$$

(4.9)

by (2.2), the definition of $E(m)$, Lemma 2 (note $\int_{\Omega} (-m) |u|^p \, dx < 0$) and Hölder’s inequality, where $b(-m, \xi')$ is a positive constant obtained in Lemma 2 with

$$
\xi' = \frac{p - 1}{C_0(1 - \delta)} \left( \frac{C_0\delta c(m)}{p - 1} - \lambda \right).
$$

Consequently, we see that $I_\lambda$ is bounded from below on $E(m)$ by (4.6) and (4.8) or (4.9).

Fix a positive constant $\varepsilon$ such that $C_1(\lambda^*(m) + \varepsilon)/(p - 1) < \lambda$. Then, by the definition of $\lambda^*(m)$, we can choose a non negative function $v_0 \in W^{1,p}(\Omega)$ (note that we can use $|v_0|$ instead of $v_0$ if necessary) such that

$$
\int_{\Omega} m v_0^p \, dx = 1 \quad \text{and} \quad \|\nabla v_0\|_p^p < \lambda^*(m) + \varepsilon.
$$
Then, for sufficiently large $T > 0$, we have

$$I_\lambda(\pm T v_0) \leq \frac{C_1 T^p}{p(p-1)} \|\nabla v_0\|_p^p - \frac{\lambda T^p}{p} + T \int_\Omega h v_0 \, dx$$

$$< -\frac{T^p}{p} \left( \lambda - \frac{C_1}{p-1} (\lambda^* + \epsilon) \right) + T \int_\Omega h v_0 \, dx < \inf_{E(m)} I_\lambda$$  \hfill (4.10)

by (2.2), $\lambda - C_1 (\lambda^* + \epsilon)/(p - 1) > 0$ and $p > 1$. Hence, we set $u_0 := T v_0$ and $u_1 := -T v_0$ for $T > 0$ satisfying (4.10).

Now, we shall prove

$$\max_{t \in [0,1]} I_\lambda(\gamma(t)) \geq \inf_{E(m)} I_\lambda$$

for every $\gamma \in \Gamma$.

Fix any $\gamma \in \Gamma$. If $\gamma([0,1]) \cap B(m) \neq \emptyset$, then

$$\max_{t \in [0,1]} I_\lambda(\gamma(t)) \geq \inf_{B(m)} I_\lambda \geq \inf_{E(m)} I_\lambda$$

because of $B(m) \subset E(m)$ (see (4.7)). So, we may assume that $\gamma([0,1]) \cap B(m) = \emptyset$, namely $\int_\Omega m|\gamma(t)|^p \, dx > 0$ for every $t \in [0,1]$. Set

$$\tilde{\gamma}(t) := \frac{\gamma(t)}{\left( \int_\Omega m|\gamma(t)|^p \, dx \right)^{1/p}},$$

and then $\tilde{\gamma} \in \Sigma(m)$ (see (2.10) for the definition of $\Sigma(m)$). By the definition of $c(m)$, we have $\max_{t \in [0,1]} \|\nabla \tilde{\gamma}(t)\|_p^p \geq c(m)$. This implies that there exists $u_\gamma \in \gamma([0,1])$ such that

$$\|\nabla u_\gamma\|_p^p \geq c(m) \int_\Omega m|u_\gamma|^p \, dx,$$

whence $u_\gamma \in E(m)$. As a result, we obtain

$$\max_{t \in [0,1]} I_\lambda(\gamma(t)) \geq I_\lambda(u_\gamma) \geq \inf_{E(m)} I_\lambda.$$

**Lemma 14.** Assume that

$$h \in L^\infty(\Omega)_+, \quad \int_\Omega m \, dx = 0 \quad \text{and} \quad 0 < \lambda < \frac{C_0 c(m)}{p-1}.$$

Then, there exists $\epsilon_0 > 0$ such that $\lambda < C_0 (m + \epsilon_0)/(p - 1)$ and $I_\lambda$ is bounded from below on $E(m + \epsilon_0)$ defined by (4.5) with $m + \epsilon_0$. Furthermore, there exist $u_0, u_1 \in W^{1,p}(\Omega)$ such that

$$\max\{I_\lambda(u_0), I_\lambda(u_1)\} \leq \inf_{E(m + \epsilon_0)} I_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma(t))$$

for every $\gamma \in C([0,1], W^{1,p}(\Omega))$ with $\gamma(0) = u_0$ and $\gamma(1) = u_1$. 

Proof. By Lemma 9 and \( \lambda < C_0 c(m)/(p-1) \), we can choose \( \varepsilon_0 > 0 \) satisfying \( \lambda < C_0 (m + \varepsilon_0)/(p-1) \). For every \( u \in W^{1,p}(\Omega) \), we have

\[
I_\lambda(u) \geq \frac{C_0}{p(p-1)} \|\nabla u\|_p^p - \frac{\lambda}{p} \int_\Omega (m + \varepsilon_0)|u|^p dx + \frac{\varepsilon_0 \lambda}{p} \|u\|_p^p - \|h\|_\infty \|u\|_1.
\]

Thus, by the same argument as in the proof of Lemma 13 with \( m + \varepsilon_0 \) instead of \( m \), we can show that \( I_\lambda \) is bounded from below on \( E(m + \varepsilon_0) \) (note \( \varepsilon_0 \lambda > 0 \)). Moreover, by choosing a non negative function \( v_0 \in W^{1,p}(\Omega) \) such that

\[
\int_\Omega mv_0^p \, dx = 1 \quad \text{and} \quad \|\nabla v_0\|_p < \lambda^*(m) + \varepsilon = \varepsilon
\]

for \( 0 < \varepsilon < \lambda (p-1)/C_1 \), we have

\[
I_\lambda(\pm T v_0) \leq - \frac{T^p}{p} \left( \lambda - \frac{C_1 \varepsilon}{p-1} \right) + T \int_\Omega hv_0 \, dx < \inf_{E(m+\varepsilon_0)} I_\lambda
\]

for sufficiently large \( T > 0 \), where we use (2.2) in the first integral. The last assertion can be proved by the same argument as in the proof of Lemma 13 with \( m + \varepsilon_0 \) instead of \( m \).

PROOF OF THEOREM 5. By Proposition 6 in the last subsection 4.4, we will see that \( I_\lambda \) satisfies the Palais-Smale condition in all cases. Hence, the mountain pass theorem guarantees the existence of a critical point of \( I_\lambda \) since \( I_\lambda \) has the mountain pass geometry by Lemma 13 (if \( \int_\Omega mdx \neq 0 \)) or Lemma 14 (if \( \int_\Omega mdx = 0 \)), where we use \( A_p \geq C_1/(p-1) \) when \( \int_\Omega mdx < 0 \) and \( N < p \). Therefore, \( (P;\lambda, m, h) \) has at least one solution.

4.3. Asymptotically \((p-1)\) homogeneous case

In this subsection, we deal with the special case where the map \( A(x,y) \) is asymptotically \((p-1)\) homogeneous in the following sense:

\( (AH) \) there exist a positive function \( a_\infty \in C^1(\overline{\Omega}, \mathbb{R}) \) and a continuous function \( \tilde{a}(x,t) \) on \( \overline{\Omega} \times \mathbb{R} \) such that

\[
A(x,y) = a_\infty(x)|y|^{p-2}y + \tilde{a}(x,|y|)y \quad \text{for every} \ x \in \Omega, \ y \in \mathbb{R}^N,
\]

\[
\lim_{t \to +\infty} \frac{\tilde{a}(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in} \ x \in \overline{\Omega},
\]

and \( A \) satisfies the hypothesis \((A)\).

Under this hypothesis, we obtain the following existence result.

THEOREM 6. Assume that \((AH)\), \( m \in L^\infty(\Omega) \) and \( 0 \neq h \in L^\infty(\Omega)_+ \). If

\[
\lambda^*(m) \sup_{x \in \Omega} a_\infty(x) < \lambda < c(m) \inf_{x \in \Omega} a_\infty(x),
\]

then \( (P;\lambda, m, h) \) has at least one solution.
Under the hypothesis \((AH)\), we define
\[
\tilde{G}(x,y) := \int_{0}^{\|y\|} \tilde{a}(x,t) \, dt.
\]
Then, the functional \(I_\lambda\) is written by
\[
I_\lambda(u) = \frac{1}{p} \int_{\Omega} a_{\infty}|\nabla u|^p \, dx + \int_{\Omega} \tilde{G}(x,\nabla u) \, dx - \frac{\lambda}{p} \int_{\Omega} m|u|^p \, dx - \int_{\Omega} h u \, dx
\]
for \(u \in W^{1,p}(\Omega)\).

Now, we shall prove that \(I_\lambda\) has the mountain pass geometry.

**Lemma 15.** Assume that \((AH), h \in L^\infty(\Omega)_{+}, \int_{\Omega} m \, dx \neq 0 \) and \(\lambda^* (m) \sup_{x \in \Omega} a_{\infty}(x) < \lambda < c(m) \inf_{x \in \Omega} a_{\infty}(x)\).

Then, \(I_\lambda\) is bounded from below on \(E(m)\) defined by (4.5). Furthermore, there exist \(u_0, u_1 \in W^{1,p}(\Omega)\) such that
\[
\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{E(m)} I_\lambda \leq \max_{t \in [0,1]} I_\lambda(\gamma(t))
\]
for every \(\gamma \in \Gamma\), where
\[
\Gamma := \{ \gamma \in C([0,1], W^{1,p}(\Omega)) ; \gamma(0) = u_0, \gamma(1) = u_1 \}.
\]

**Proof.** By the property of the function \(\tilde{a}\) as in \((AH)\) and Young’s inequality, for every \(\varepsilon > 0\) there exist constants \(C_{\varepsilon} > 0\) and \(C_{\varepsilon}' > 0\) such that
\[
\left| \tilde{G}(x,y) \right| \leq \frac{\varepsilon}{2} |y|^p + C_{\varepsilon} |y| \leq \varepsilon |y|^p + C_{\varepsilon}'
\]
for every \(x \in \Omega\) and \(y \in \mathbb{R}^N\). Therefore, we have
\[
I_\lambda(u) \geq \frac{a - p\varepsilon}{p} \|\nabla u\|_p^p - \frac{\lambda}{p} \int_{\Omega} m|u|^p \, dx - \|h\|_{\infty} \|u\|_1 - C_{\varepsilon}'|\Omega|
\]
for every \(u \in W^{1,p}(\Omega)\), where \(a := \inf_{x \in \Omega} a_{\infty}(x)\). Here, we choose \(\varepsilon > 0\) and \(0 < \delta < 1\) such that \(\lambda < (a + p\varepsilon)\delta c(m)\). By a similar argument to Lemma 13, we can show that \(I_\lambda\) is bounded from below on \(E(m)\).

Next, we shall prove the existence of desired \(u_0\) and \(u_1\). Take \(\varepsilon' > 0\) satisfying
\[
\lambda > (\overline{a} + p\varepsilon')(\lambda^*(m) + \varepsilon'),
\]
where \(\overline{a} := \sup_{x \in \Omega} a_{\infty}(x)\). Choose a function \(v_0 \in W^{1,p}(\Omega)\) such that
\[
\int_{\Omega} m|v_0|^p \, dx = 1 \quad \text{and} \quad \|\nabla v_0\|_p^p < \lambda^*(m) + \varepsilon'.
\]
Then, for sufficiently large $T > 0$, we have
\[
I_{\lambda}(\pm T v_0) \leq -\frac{T^p}{p} \{ \lambda - (\sigma + p\varepsilon') (\lambda^*(m) + \varepsilon') \} + T \int_{\Omega} |h| v_0 |dx + C'_\varepsilon |\Omega|
\]
\[
< \inf_{E(m)} I_{\lambda},
\]
where we use (4.11) with $\varepsilon = \varepsilon'$. Thus, by setting $u_0 := T v_0$ and $u_1 := -T v_0$ for such $T > 0$, our claim is shown. Finally, it follows from the same argument as in Lemma 13 that every $\gamma \in \Gamma$ links $E(m)$.

By combining the proof of Lemma 15 with one of Lemma 14, we can show the following lemma in the case of $\int_{\Omega} m dx = 0$. Here, we omit the proof.

**Lemma 16.** Assume that (AH), $h \in L^\infty(\Omega)_+$, $\int_{\Omega} m dx = 0$ and
\[
\lambda^*(m) \sup_{x \in \Omega} a_\infty(x) < \lambda < c(m) \inf_{x \in \Omega} a_\infty(x).
\]
Then, there exists $\varepsilon_0 > 0$ such that $\lambda < c(m + \varepsilon_0) \inf_{x \in \Omega} a_\infty(x)$ and $I_{\lambda}$ is bounded from below on $E(m + \varepsilon_0)$ defined by (4.5) with $m + \varepsilon_0$. Furthermore, there exist $u_0$, $u_1 \in W^{1,p}(\Omega)$ such that
\[
\max \{ I_{\lambda}(u_0), I_{\lambda}(u_1) \} < \inf_{E(m + \varepsilon_0)} I_{\lambda} \leq \max_{t \in [0,1]} I_{\lambda}(\gamma(t))
\]
for every $\gamma \in C([0,1], W^{1,p}(\Omega))$ with $\gamma(0) = u_0$ and $\gamma(1) = u_1$.

**Proof of Theorem 6.** It suffices to prove the existence of a critical point of $I_{\lambda}$ because critical points of $I_{\lambda}$ correspond to solutions of (P; $\lambda, m, h$). By Proposition 6 in the last subsection 4.4, we will see that $I_{\lambda}$ satisfies the Palais-Smale condition if $\lambda$ is not an eigenvalue of
\[
-\text{div} \left( a_\infty(x) \left| \nabla u \right|^{p-2} \nabla u \right) = \lambda m(x) |u|^{p-2} u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \tag{4.12}
\]
Hence, by admitting that $\lambda$ is not an eigenvalue of (4.12), the mountain pass theorem guarantees the existence of a critical point of $I_{\lambda}$ since $I_{\lambda}$ has the mountain pass geometry by Lemma 15 or 16 in the case of $\int_{\Omega} m dx \neq 0$ or $\int_{\Omega} m dx = 0$, respectively.

Now, we shall prove that the equation (4.12) has no non-trivial solution provided $\lambda^*(m) \sup_{x \in \Omega} a_\infty(x) < \lambda < c(m) \inf_{x \in \Omega} a_\infty(x)$ by way of contradiction. So, we assume that there exists a non-trivial solution $v \in W^{1,p}(\Omega)$ of (4.12). By taking $\pm v_\pm$ as test function, we have
\[
\inf_{x \in \Omega} a_\infty(x) \| \nabla v_\pm \|_p^p \leq \lambda \int_{\Omega} m v_\pm^p dx \leq \sup_{x \in \Omega} a_\infty(x) \| \nabla v_\pm \|_p^p. \tag{4.13}
\]
We shall show that $\int_{\Omega} m v_\pm^p dx > 0$ and $\int_{\Omega} m v_\pm^p dx > 0$. If $\int_{\Omega} m v_\pm^p dx = 0$ holds, then $v = -v_-$ or $v = c > 0$ with $c = \| v_+ \|_p$ occurs because of $\| \nabla v_+ \|_p > 0$ obtained by
(4.13). Thus, \(-v\) or \(v\) is a positive solution of (4.12) belonging to \(C^1(\overline{\Omega})\) such that 
\[\min_{\overline{\Omega}} v_\pm = \min_{\overline{\Omega}} (-v) > 0\] or 
\[\min_{\overline{\Omega}} v > 0,\]
respectively (see Remark 2 with \(h \equiv 0\)). Then, by applying an argument as in Proposition 2 (with \(h \equiv 0\)) to the equation (4.12), we obtain the inequality
\[
\lambda \int_{\Omega} m \varphi^p \, dx = \int_{\Omega} a_\infty |\nabla \psi|^{p-2} \nabla \psi \nabla \left( \frac{\varphi^p}{\psi^{p-1}} \right) \, dx 
\leq \int_{\Omega} a_\infty |\nabla \varphi|^p \, dx \leq \sup_{x \in \Omega} a_\infty(x) \|\nabla \varphi\|^p_p
\]
for every \(\varphi \in C^1(\overline{\Omega})\) with \(\psi \geq 0\), where \(\psi = v_-\) if \(v < 0\) or \(\psi = v_+\) if \(v > 0\). By the density of \(C^1(\overline{\Omega})\), we have
\[
\lambda \int_{\Omega} m \varphi^p \, dx \leq \sup_{x \in \Omega} a_\infty(x) \|\nabla \varphi\|^p_p \quad \text{for every } \varphi \in W^{1,p}(\Omega) \text{ with } \varphi \geq 0.
\]
This implies that \(\lambda \leq \lambda^*(m) \sup_{x \in \Omega} a_\infty(x)\) (refer to Proposition 2). This is a contradiction.

Similarly, if \(\int_{\Omega} m v_+^p \, dx = 0\), then we can get a contradiction since \(v = v_+\) or \(v = -c < 0\) holds. Therefore, our claim is shown. As a result, we can define a continuous path \(\gamma_0 \in \Sigma(m)\) (see (2.10) for the definition of \(\Sigma(m)\)) by
\[
\gamma_0(t) := \frac{(1-t)v_+ - tv_-}{(1-t)p \int_{\Omega} m v_+^p \, dx + t^p \int_{\Omega} m v_-^p \, dx}^{1/p}.
\]
Hence, we have a contradiction to the definition of \(c(m)\) because
\[
\tilde{J}(\gamma_0(t)) = \|\nabla \gamma_0(t)\|^p_p \leq \frac{\lambda}{\inf_{x \in \Omega} a_\infty(x)} < c(m) \quad \text{for every } t \in [0, 1]
\]
holds by (4.13), where \(\tilde{J}\) is the functional defined by (2.8).

**Remark 7.** Let \(\lambda^*(a_\infty, m)\) and \(c(a_\infty, m)\) be the principal eigenvalue or the second eigenvalue of
\[
-\text{div} \left( a_\infty(x)|\nabla u|^{p-2} \nabla u \right) = \lambda m(x)|u|^{p-2} u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \quad (4.14)
\]
respectively. Namely,
\[
\lambda^*(a_\infty, m) := \inf \left\{ \int_{\Omega} a_\infty |\nabla u|^p \, dx; \int_{\Omega} m |u|^p \, dx = 1 \right\},
\]
\[
c(a_\infty, m) := \inf_{\gamma \in \Sigma(m)} \max_{t \in [0, 1]} \int_{\Omega} a_\infty |\nabla \gamma(t)|^p \, dx.
\]
Then, in the assumption of Theorem 6, we can replace
\[
\lambda^*(m) \sup_{x \in \Omega} a_\infty(x) < \lambda < c(m) \inf_{x \in \Omega} a_\infty(x)
\]
with \(\lambda^*(a_\infty, m) < \lambda < c(a_\infty, m)\). In [23], the present author provides the existence result in the more general cases.
4.4. Palais-Smale condition

In this section, we prove that $I_\lambda$ satisfies the Palais-Smale condition under the several situation. The following result is proved in [20]. It plays an important role for our proof.

**Proposition 5.** ([20, Proposition 1]) Let $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ be the map defined by

$$\langle A(u), v \rangle = \int_\Omega A(x, \nabla u) \nabla v \, dx$$

for $u, \ v \in W$. Then, $A$ is maximal monotone, strictly monotone and has the $(S)_+$ property, that is, any sequence $\{u_n\}$ weakly convergent to $u$ with

$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0 \quad \text{strongly converges to } u.$$

First, we state the result for the Palais-Smale condition in the general case.

**Proposition 6.** Let $h \in L^\infty(\Omega)$. If one of the following cases holds, then $I_\lambda$ satisfies the Palais-Smale condition:

(i) $m \geq 0$ in $\Omega$ and $0 < \lambda < C_0 c(m)/(p - 1)$,

(ii) $\int_\Omega m \, dx > 0$ and $0 < \lambda < C_0 \lambda(m)/(p - 1)$,

(iii) $N < p$ and $A_p \lambda^*(m) < \lambda < C_0 \tilde{\lambda}(m)/(p - 1)$,

where $c(m), \lambda(m), A_p$ and $\tilde{\lambda}(m)$ are positive constants defined by (2.11), (2.13), (2.14) and (2.7), respectively.

**Proof.** Let $\{u_n\}$ be a Palais-Smale sequence of $I_\lambda$, namely,

$$I_\lambda(u_n) \to c \quad \text{and} \quad \| I'_\lambda(u_n) \|_{W^*} \to 0 \quad \text{as } n \to \infty$$

for some $c \in \mathbb{R}$. It is sufficient to prove the boundedness of $\{u_n\}$ in $W^{1,p}(\Omega)$ because the operator $A$ defined in Proposition 5 has the $(S)_+$ property and the inclusion from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ is compact. Then, by noting the following inequality

$$\frac{C_0}{p(p - 1)} \sqrt[p]{\| \nabla u_n \|_p^p} \leq \int_\Omega G(x, \nabla u_n) \, dx = I_\lambda(u_n) + \frac{\lambda}{p} \int_\Omega m \|u_n\|^p \, dx + \int_\Omega h \, u_n \, dx \leq I_\lambda(u_n) + \lambda \| m \|_{\infty} \| u_n \|_p^p / p + \| h \|_{\infty} \| u_n \|_1$$

(4.15)

by (2.2), it is sufficient to prove only the boundedness of $\{u_n\}$ in $L^p(\Omega)$. We shall prove it by contradiction. So, we may assume $\| u_n \|_p \to \infty$ by choosing a subsequence. Put $v_n := u_n / \| u_n \|_p$. Then, we may suppose that there exists a $v \in W^{1,p}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{in } W^{1,p}(\Omega) \quad \text{and hence} \quad v_n \to v \quad \text{in } L^p(\Omega)$$
since (4.15) ensures the boundedness of \( \{v_n\} \) in \( W^{1,p}(\Omega) \). By taking the limit inferior in the following inequality

\[
\frac{C_0}{p-1} \|\nabla v_{n+}\|_p^p \leq \int_{\Omega} A(x, \nabla u_n) \frac{\nabla u_{n+}}{\|u_n\|_p^p} \, dx
\]

\[
= \lambda \int_{\Omega} mv_{n+}^p \, dx + \int_{\Omega} h \frac{v_{n+}}{\|u_n\|_p^p} \, dx + \left\langle I'_\lambda(u_n), \frac{v_{n+}}{\|u_n\|_p^p} \right\rangle
\]

(where we use Remark 1 (iii) in the first inequality), we have

\[
\frac{C_0}{p-1} \|\nabla v_{n+}\|_p^p \leq \lambda \int_{\Omega} mv_{n+}^p \, dx. \tag{4.16}
\]

Similarly, we also get

\[
\frac{C_0}{p-1} \|\nabla v_-\|_p^p \leq \lambda \int_{\Omega} mv_{n-}^p \, dx. \tag{4.17}
\]

Here, we note that it is sufficient to prove the two inequalities \( \int_{\Omega} mv_{n+}^p \, dx > 0 \) and \( \int_{\Omega} mv_{n-}^p \, dx > 0 \). Indeed, if we can show the above inequalities, then we can define a continuous path \( \gamma_0 \in \Sigma(m) \) (see (2.10) for the definition of \( \Sigma(m) \)) by

\[
\gamma_0(t) := \frac{(1-t)v_+ - tv_-}{((1-t)p \int_{\Omega} mv_{n+}^p \, dx + t p \int_{\Omega} mv_{n-}^p \, dx)^{1/p}}.
\]

For this continuous path, by an easy estimate with (4.16) and (4.17), we have

\[
\tilde{J}(\gamma_0(t)) = \|\nabla \gamma_0(t)\|_p^p \leq \frac{p-1}{C_0} \lambda < c(m) \quad \text{for every } t \in [0,1],
\]

where \( \tilde{J} \) is the functional defined by (2.8). This contradicts to the definition of \( c(m) \).

So, we shall prove

\[
\int_{\Omega} mv_{n+}^p \, dx > 0 \quad \text{and} \quad \int_{\Omega} mv_{n-}^p \, dx > 0
\]

in each case of (i) \( \sim \) (iii) by noting (4.16) and (4.17).

Case (i): If \( \int_{\Omega} mv_{n+}^p \, dx = 0 \), then \( v_+ \) is a constant function by (4.16). Moreover, because of \( \int_{\Omega} m \, dx > 0 \), we see that \( v_+ \equiv 0 \), and so \( v \leq 0 \) in \( \Omega \). Then, by the equality

\[
o(1) = \left\langle I'_\lambda(u_n), \frac{1}{\|u_n\|_p^{p-1}} \right\rangle = \lambda \int_{\Omega} m|v_n|^{p-2}v_n \, dx + \int_{\Omega} h / \|u_n\|_p^{p-1} \, dx,
\]

we have

\[
\int_{\Omega} m|v|^{p-2}v \, dx = - \int_{\Omega} mv_{n+}^{p-1} = 0
\]

(note \( \lambda > 0 \)). This yields that \( m(x)v_{n+}^{p-1}(x) = 0 \) for a.e. \( x \in \Omega \) (note \( m \geq 0 \) in \( \Omega \)). Thus, \( m(x)v_{n+}^p(x) = 0 \) for a.e. \( x \in \Omega \). Therefore, (4.17) shows that \( v_- \) is a constant function, and so

\[
v = -v_- \equiv 0 \quad \text{by} \quad \int_{\Omega} mv_{n-}^p \, dx = 0.
\]
This contradicts to $\|v\|_p = 1$. Hence, we have $\int_\Omega mv_+^p \, dx > 0$. Similarly, we see that $\int_\Omega mv_-^p \, dx > 0$.

Case (ii): First, let $\int_\Omega mv_-^p \, dx = 0$ occur. Then, by the same argument as in case (i), we have $v \leq 0$ in $\Omega$ and

$$\int_\Omega m|v|^{p-2}v \, dx = - \int_\Omega mv_-^{p-1} = 0.$$ 

If $\int_\Omega mv_-^p \, dx > 0$ holds, then $v_-/ (\int_\Omega mv_-^p \, dx)^{1/p} \in X(m)$ and we have

$$\frac{\|\nabla v_\rho\|_p^p}{\int_\Omega mv_-^p \, dx} \leq (p - 1) \frac{\lambda}{C_0} \lambda_X(m)$$

by (4.17). This contradicts to the definition of $\lambda_X(m)$.

On the other hand, if $\int_\Omega mv_-^p \, dx = 0$, then $v_-$ is a constant function by (4.17). Hence we obtain a contradiction in this case also since

$$0 = \int_\Omega mv_-^p \, dx = v_-^p \int_\Omega m \, dx = \frac{1}{|\Omega|} \int_\Omega m \, dx > 0$$

(note $\|v\|_p = 1$ and also that $v_-$ is a constant function). Consequently, we have shown $\int_\Omega mv_-^p \, dx > 0$.

Similarly, we can prove that $\int_\Omega mv_+^p \, dx > 0$ by a similar argument above with $v_+$ instead of $v_-$. 

Case (iii): We consider by dividing into the following three cases:

(a) $\int_\Omega mv_+^p \, dx = 0 = \int_\Omega mv_-^p \, dx$;

(b) $\int_\Omega mv_+^p \, dx > 0 = \int_\Omega mv_-^p \, dx$;

(c) $\int_\Omega mv_+^p \, dx > 0 < \int_\Omega mv_-^p \, dx$.

In the case of (a), it follows from (4.16) and (4.17) that $v$ is a constant function. Thus, $v = 1/|\Omega|^{1/p}$ or $v = -1/|\Omega|^{1/p}$ occurs. First, we shall deal with the case of $v = 1/|\Omega|^{1/p} > 0$. Thus, we may assume that $u_n \geq \|u_n\|_p/2|\Omega|^{1/p}$ in $\overline{\Omega}$ for sufficiently large $n$ (note $N < p$ and so $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact). So, we obtain

$$\|\nabla (1/u_n^{p-1})\|_p \leq \frac{2^p(p - 1)}{|\Omega|\|u_n\|_p^{p-1}} \|\nabla v_n\|_p$$

and

$$\| (1/u_n^{p-1})\|_p \leq \frac{2^p(p - 1)|\Omega|}{\|u_n\|_p^{p-1}}.$$ (4.18)

and so $1/u_n^{p-1} \in W^{1,p}(\Omega)$ for such sufficiently large $n$. Here, we fix any $\varphi \in C^1(\overline{\Omega})$ such that $\varphi > 0$ in $\Omega$. By taking the limit in the following inequality

$$A_p\|\nabla \varphi\|_p^p \geq \int_\Omega A(x, \nabla u_n)\nabla \left( \frac{\varphi^p}{u_n^{p-1}} \right) \, dx$$

$$= \lambda \int_\Omega m\varphi \, dx + \int_\Omega \frac{h \varphi^p}{u_n^{p-1}} \, dx + \langle f, \varphi^p / u_n^{p-1} \rangle$$
(note \(\|\varphi^p / u_n^{p-1}\| = o(1)\) by (4.18)), where the first inequality is shown by Proposition 2, we have
\[
A_p \|\nabla \varphi\|_p^p \geq \lambda \int_{\Omega} m \varphi^p \, dx
\]
for every \(\varphi \in C^1(\overline{\Omega})\) with \(\varphi \geq 0\) in \(\Omega\). Since \(C^1(\overline{\Omega})\) is dense in \(W^{1,p}(\Omega)\), we obtain
\[
A_p \|\nabla \varphi\|_p^p \geq \lambda \int_{\Omega} m \varphi^p \, dx
\]
for every \(\varphi \in W^{1,p}(\Omega)\) with \(\varphi \geq 0\) in \(\Omega\). Because we can choose \(\varphi_k \in W^{1,p}(\Omega)\) such that \(\varphi_k \geq 0\) in \(\Omega\), \(\int_{\Omega} m \varphi_k^p \, dx = 1\) and \(\|\nabla \varphi_k\|_p^p < \lambda^*(m) + 1/k\) (we consider \(|\varphi_k|\) instead of \(\varphi_k\) if necessary), we have a contradiction.

In the case of \(v = -1/|\Omega|^{1/p} < 0\) also, we have a contradiction by using \(-u_n\) instead of \(u_n\) as in the above argument (note that \(A\) is odd in the second variable).

In the case of (b), it is easily seen that \(v = v_+ \geq 0\) holds by (4.17) and
\[
\int_{\Omega} mv_+^p \, dx = 0 < \int_{\Omega} mv_n^p \, dx.
\]
Since we obtain
\[
\frac{\|\nabla v_+\|_p^p}{\int_{\Omega} mv_+^p \, dx} \leq (p - 1) \frac{\lambda}{C_0} < \tilde{\lambda}(m)
\]
by (4.16) and \(\int_{\Omega} mv_n^p \, dx > 0\), it follows that \(v_+\) has no zero points in \(\overline{\Omega}\) from the definition of \(\tilde{\lambda}(m)\). This means that \(v > 0\) in \(\Omega\). Thus, we may assume that \(u_n \geq \delta \|u_n\|_p^p/2\) in \(\overline{\Omega}\) for sufficiently large \(n\), where \(\delta = \min_{\Omega} v(x)\) because the inclusion of \(W^{1,p}(\Omega)\) to \(C(\overline{\Omega})\) is compact. So, we can get a contradiction by the same argument as in the case of (b) under \(v > 0\).

In the case of (c), we see that \(v < 0\) in \(\overline{\Omega}\) by a similar argument to the case of (b). This yields a contradiction by a similar argument to the case of (a) under \(v < 0\).

To deal with the case of \((AH)\), we prepare the following result.

**Lemma 17.** Assume \((AH)\) and let \(\{u_n\} \subset W^{1,p}(\Omega)\) be a Palais-Smale sequence for \(I_\lambda\) with \(\|u_n\|_p \to \infty\) as \(n \to \infty\). Then, \(v_n := u_n/\|u_n\|_p\) has a subsequence strongly convergent to a solution \(v\) for
\[
- \text{div} \left( a_\infty(x)|\nabla v|^{p-2} \nabla v \right) = \lambda m|v|^{p-2}v \quad \text{in } \Omega, \quad \frac{\partial v}{\partial v} = 0 \quad \text{on } \partial \Omega, \tag{4.19}
\]
where \(a_\infty\) is the positive function as in \((AH)\).

**Proof.** By the same argument as in the proof of Proposition 6, we can show the boundedness of \(\|v_n\|\) and obtain the inequality
\[
\frac{C_0}{p-1} \|\nabla v_n\|_p^p \leq \lambda \int_{\Omega} m|v_n|^p \, dx + o(1) \quad \text{as } n \to \infty. \tag{4.20}
\]
Therefore, we obtain above inequality, we can get since \( \| \) by (4.20), because the \( p \)-Laplace operator has the \((S)_+\) property. To obtain (4.21), we shall get the following

\[
\lim_{n \to \infty} \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \, dx = 0 \tag{4.21}
\]

because the \( p \)-Laplace operator has the \((S)_+\) property. To obtain (4.21), we shall get the following

\[
\lim_{n \to \infty} \left| \frac{1}{\| u_n \|_p^{p-1}} \int_\Omega \tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla (v_n - v) \, dx \right| = 0, \tag{4.22}
\]

where \( \tilde{a} \) is the function as in \((AH)\). Here, we fix an any \( \varepsilon > 0 \). By the property of the function \( \tilde{a} \), there exist \( R > 0 \) and \( C > 0 \) such that

\[
|\tilde{a}(x, t)| \leq \varepsilon |t|^{p-2} \text{ if } |t| \geq R \text{ and } |\tilde{a}(x, t)| \leq C \text{ if } |t| \leq R. \tag{4.23}
\]

Therefore, we obtain

\[
\left| \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n \nabla (v_n - v) \, dx}{\| u_n \|_p^{p-1}} \right| \leq \int_{|\nabla u_n| \geq R} \varepsilon (|\nabla v_n|^p + |\nabla v_n|^{p-1} |\nabla v|) \, dx + \int_{|\nabla u_n| \leq R} \frac{C|\nabla u_n|}{\| u_n \|_p^{p-1}} (|\nabla v_n| + |\nabla v|) \, dx
\]

\[
\leq \varepsilon (|\nabla v_n|^p + |\nabla v_n|^{p-1} |\nabla v|_p) + RC (|\nabla v_n|_p + |\nabla v|_p) \frac{|\Omega|^{(p-1)/p}}{\| u_n \|_p^{p-1}}
\]

\[
\leq \frac{2\varepsilon \lambda |(p-1)m|_\infty}{C_0} + o(1) + RC \left( \frac{2\lambda |(p-1)m|_\infty}{C_0} + o(1) \right)^{1/p} \frac{|\Omega|^{(p-1)/p}}{\| u_n \|_p^{p-1}}
\]

by (4.20). \( \| v_n \|_p = 1 \) and Hölder’s inequality. Thus, by taking the limit superior in the above inequality, we can get

\[
\limsup_{n \to \infty} \left| \frac{1}{\| u_n \|_p^{p-1}} \int_\Omega \tilde{a}(x, \nabla u_n) \nabla u_n \nabla (v_n - v) \, dx \right| \leq \frac{2\varepsilon \lambda |(p-1)m|_\infty}{C_0}
\]

since \( \| u_n \|_p \to \infty \) as \( n \to \infty \). This implies (4.22) because \( \varepsilon > 0 \) is arbitrary. By taking the limit in the following and noting (4.22)

\[
o(1) = \langle \mathcal{L}^*(u_n), v_n - v \rangle / \| u_n \|_p^{p-1}
\]

\[
= \int a_c |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \, dx + \int_\Omega \frac{\tilde{a}(x, |\nabla u_n|) \nabla u_n}{\| u_n \|_p^{p-1}} \nabla (v_n - v) \, dx
\]

\[
- \lambda \int_\Omega m |v_n|^{p-2} v_n (v_n - v) \, dx - \int_\Omega \frac{h}{\| u_n \|_p^{p-1}} (v_n - v) \, dx,
\]
we have the inequality (4.21) (note \( \inf_{\Omega} a > 0 \)), whence \( v_n \to v \) in \( W^{1,p}(\Omega) \).

Finally, we shall show that \( v \) is a solution for (4.19). So, we fix any \( \varphi \in W^{1,p}(\Omega) \). Then, by considering \( \varphi \) instead of \( (v_n - v) \) in (4.24), we have the following inequality for every \( \varepsilon > 0 \):

\[
\left| \int_{\Omega} \tilde{a}(x,|\nabla u_n|) \nabla u_n \nabla \varphi \, dx \right| \leq \varepsilon \| \nabla v_n \|_p^{p-1} \| \nabla \varphi \|_p + \frac{C R \| \nabla \varphi \|_p |\Omega|^{(p-1)/p}}{\| u_n \|_p^{p-1}} \| \nabla \varphi \|_p + \left( \frac{\lambda (p-1)}{C_0} \right)^{(p-1)/p} \| \nabla \varphi \|_p + \frac{C R \| \nabla \varphi \|_p |\Omega|^{(p-1)/p}}{\| u_n \|_p^{p-1}} \| \nabla \varphi \|_p.
\]

This gives

\[
\lim_{n \to \infty} \int_{\Omega} \tilde{a}(x,|\nabla u_n|) \nabla u_n \nabla \varphi \, dx = 0 \quad (4.25)
\]

since \( \varepsilon > 0 \) is arbitrary. By taking the limit in

\[
o(1) = \frac{\langle I'_\lambda(u_n), \varphi \rangle}{\| u_n \|_p^{p-1}},
\]

we have

\[
\int_{\Omega} a_\infty |\nabla \varphi|^{p-2} \nabla \varphi \, dx = \lambda \int_{\Omega} m |\nabla \varphi|^{p-2} \varphi \, dx
\]

by (4.25), \( v_n \to v \) in \( W^{1,p}(\Omega) \) and \( \| u_n \|_p \to \infty \) as \( n \to \infty \). Because \( \varphi \) is any function in \( W^{1,p}(\Omega) \), our conclusion is shown.

Now, we state the result in the case of \( (AH) \).

**Proposition 7.** Assume \( (AH) \) and \( \lambda \) is not an eigenvalue of

\[
- \text{div} \left( a_\infty(x)|\nabla u|^{p-2} \nabla u \right) = \lambda m |\nabla u|^{p-2} u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \quad (4.26)
\]

where \( a_\infty \) is the positive function as in \( (AH) \). Then, \( I_\lambda \) satisfies the Palais-Smale condition.

**Proof.** Let \( \{u_n\} \) be a Palais-Smale sequence of \( I_\lambda \), namely,

\[
I_\lambda(u_n) \to c \quad \text{and} \quad \| I'_\lambda(u_n) \|_{W^*} \to 0 \quad \text{as } n \to \infty
\]

for some \( c \in \mathbb{R} \). It is sufficient to prove only the boundedness of \( \| u_n \|_p \) by the same reason as in the proof of Proposition 6. So, by contradiction, we may suppose that \( \| u_n \|_p \to \infty \) as \( n \to \infty \) by choosing a subsequence. Set \( v_n := u_n/\| u_n \|_p \). Then, it follows from Lemma 17 that \( \{v_n\} \) has a subsequence strongly convergent to a non-trivial solution \( v \) for (4.26) with \( \| v \|_p = 1 \). This is a contradiction because \( \lambda \) is not an eigenvalue of (4.26).
**Remark 8.** Concerning the existence of a solution, under the Dirichlet boundary condition also, we can similar results to the ones as in section 4 by using several constants corresponding to the Dirichlet problem.

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**References**


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