Differential
Equations
& Applications
Volume 4, Number 4 (2012), 615–635
doi:10.7153/dea-04-35

EXISTENCE OF POSITIVE SOLUTIONS FOR A SINGULAR PROBLEMS OF CAFFARELLI–KOHN–NIRENBERG–LIN TYPE

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(*Communicated by Dongsheng Kang*)

Abstract. In this paper, we study the existence of nontrivial critical points of the functional

$$J_{\lambda,\mu}(u,v) = \frac{1}{p}\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k u\big|^p - \lambda f(x)\big||x|^{-(a+k)}u\big|^p)dx + \frac{1}{q}\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k v\big|^q - \mu g(x)\big||x|^{-(a+k)}v\big|^q)dx - \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|u|^{\alpha+1}|v|^{\beta+1}dx,$$

related to the Caffarelli-Kohn-Nirenberg inequality and its higher order variant by Lin.

1. Introduction

The aim of this paper is to establish the existence of nontrivial solutions to the functional

$$\begin{aligned} J_{\lambda,\mu}(u,v) = &\frac{1}{p}\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k u\big|^p - \lambda f(x)\big||x|^{-(a+k)}u\big|^p)dx \\ &+ \frac{1}{q}\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k v\big|^q - \mu g(x)\big||x|^{-(a+k)}v\big|^q)dx \\ &- \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|u|^{\alpha+1}|v|^{\beta+1}dx, \end{aligned} \tag{1.1}$$

in an appropriate function space. Here $p,q>1$, $\lambda>0$, $\mu>0$ and

$$\nabla^k := \begin{cases} (-\Delta)^{\frac{k}{2}}, & \text{if } k \text{ is even,} \\ \nabla(-\Delta)^{\frac{k-1}{2}}, & \text{if } k \text{ is odd,} \end{cases}$$
$$0 \leqslant a < b < a+k < \min\left\{\frac{N}{p},\frac{N}{q}\right\}. \tag{1.2}$$

We will assume that f and g satisfies

$$f,g \geqslant 0, \quad f \in L^{\frac{N}{p_0}}(\mathbb{R}^N,|x|^{-N}) \quad \text{and} \quad g \in L^{\frac{N}{q_0}}(\mathbb{R}^N,|x|^{-N}) \tag{1.3}$$

Mathematics subject classification (2010): 35A15, 35B33, 35J70.
Keywords and phrases: Nehari manifold, Caffarelli-Kohn-Nirenberg inequality, sign-changing weight function.

with

$$p_0 = p(a+k-b) \quad \text{and} \quad q_0 = q(a+k-b), \tag{1.4}$$

and c satisfies

$$\begin{aligned} &h \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad h(x) > 0 \quad \text{for some } x, \\ &h(0) \leqslant 0, \quad h(\infty) := \lim_{|x| \to \infty} h(x) \leqslant 0. \end{aligned} \tag{1.5}$$

To solve problem (1.1) variationally, we need some inequalities. The well known Caffarelli-Kohn-Nirenberg inequality in [9, 18], is characterized by

$$\left(\int_{\mathbb{R}^N} |x|^{-b\xi} |u|^\xi dx \right)^{\frac{p}{\xi}} \leqslant C_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx \quad \text{for all} \quad u \in C_0^\infty(\mathbb{R}^N), \tag{1.6}$$

where

$$1 < p < N, \quad -\infty < a < \frac{N-p}{p}, \quad a \leqslant b \leqslant a+1, \quad \xi = \frac{Np}{N-p(1+a-b)}.$$

Now, by the Sobolev interpolation in [13] and iteration of the Caffarelli-Kohn-Nirenberg inequality, we have (See the Appendix A in [15])

$$C \| |x|^{-b} u \|_{p^*} \leqslant \sum_{|\alpha|=k} \| |x|^{-a} D^\alpha u \|_p, \tag{1.7}$$

for some positive constant C, for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$1 < p, \ 0 \leqslant a \leqslant b \leqslant a+k < \frac{N}{p}, \ \text{ and } \ p^* = \frac{Np}{N-p(a+k-b)}.$$

It is clear that nontrivial critical points of the above functional will correspond to solutions of a nonlinear partial differential equation of order $2k$. As a particular case, it will that the system

$$\begin{cases} \Delta(|x|^{-ap}|\Delta u|^{p-2}\Delta u) - \lambda f(x)|x|^{-(a+k)p}|u|^{p-2}u \\ \qquad\qquad = (\alpha+1)h(x)|x|^{-b(\alpha+\beta+2)}|u|^{\alpha-1}u|v|^{\beta+1}, \quad x \in \mathbb{R}^N, \\ \Delta(|x|^{-aq}|\Delta v|^{q-2}\Delta v) - \mu g(x)|x|^{-(a+k)q}|v|^{q-2}v \\ \qquad\qquad = (\beta+1)h(x)|x|^{-b(\alpha+\beta+2)}|u|^{\alpha+1}|v|^{\beta-1}v, \quad x \in \mathbb{R}^N, \end{cases} \tag{1.8}$$

has a nontrivial solution. This corresponds to the case $k = 2$.

In recent years, several authors have used the Nehari manifold to solve semilinear and quasilinear problems (see [1, 4, 8, 6, 7, 10, 17] and references therein). Brown and Zhang [5] have studied a subcritical semi-linear elliptic equation with a sign-changing weight function and a bifurcation real parameter in the case $p = 2$ and Dirichlet boundary conditions. Exploiting the relationship between the Nehari manifold and fibering maps (i.e., maps of the form $t \mapsto J_\lambda(tu)$ where J_λ is the Euler function associated with the equation), they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. Also, some authors also studied the singular problems with Hardy-Sobolev critical exponents ([2, 11, 12] the references therein).

In [15], Waliullah studied the existence of critical points for the following functional,

$$J_\lambda(u) = \int_{\mathbb{R}^N} \Big(\frac{1}{p}\big(\big||x|^{-a}\nabla^k u\big|^p - \lambda h(x)\big||x|^{-(a+k)}u\big|^p\big) - \frac{1}{q}Q(x)\big||x|^{-b}u\big|^q \Big) dx. \quad (1.9)$$

In this work, we give a variational method which is similar to the fibering method (see [15]) to prove the existence of at least two nontrivial nonnegative critical points of functional (1.1). Since the doubly critical phenomena appear in (1.1), we have to over come more difficulties caused by the critical terms.

This paper is divided into three sections, organized as follows. In Section 2, we give some notations, preliminaries, properties of the Nehari manifold and set up the variational framework of the problem. In Section 3, we discuss the case $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$ and show how the behavior of the manifold as $\lambda \to \lambda_1(a)^-$, $\mu \to \mu_1(b)^-$. In Section 4, we discuss the case $\lambda > \lambda_1(a)$, $\mu > \mu_1(b)$ and obtain a new interpretation of δ, σ.

2. Preliminaries

We define the Sobolev spaces $Y_p = D_a^{k,p}(\mathbb{R}^N)$ and $Y_q = D_a^{k,q}(\mathbb{R}^N)$, which are the completion of $C_0^\infty(\mathbb{R}^N)$ equipped with norms

$$||u||_p = \Big(\int_{\mathbb{R}^N} \big||x|^{-a}\nabla^k u\big|^p dx\Big)^{\frac{1}{p}}, \quad ||v||_q = \Big(\int_{\mathbb{R}^N} \big||x|^{-a}\nabla^k v\big|^q dx\Big)^{\frac{1}{q}}.$$

Then we define $W = Y_p \times Y_q$ and for $(u,v) \in W$,

$$||(u,v)|| = ||u||_p^p + ||v||_q^q.$$

The paper is organized in the following manner. We first show that

$$\lambda_1 = \inf_{u\in Y_p,\ u\neq 0} \frac{\int_{\mathbb{R}^N} \big||x|^{-a}\nabla^k u\big|^p dx}{\int_{\mathbb{R}^N} f(x)\big||x|^{-(a+k)}u\big|^p dx} \quad (2.1)$$

is strictly positive and attained.

Now we state the assumptions of this paper:

(H1) let $1 < p < \alpha+1$, $1 < q < \beta+1$ and

$$d = (\alpha+1)(\beta+1) - (\alpha-p+1)(\beta-q+1) > 0;$$

(H2) let

$$\frac{N-p(a+k-b)}{p}(\alpha+1) + \frac{N-q(a+k-b)}{q}(\beta+1) < N,$$

which implies that $\alpha+1<p^*$, $\beta+1<q^*$, where

$$p^*=\frac{Np}{N-p(a+k-b)} \quad \text{and} \quad q^*=\frac{Nq}{N-q(a+k-b)}$$

are the well-known critical exponents;

(H3) the functions $f(x)$, $b(x)$, $c(x)$ are smooth functions which change sign in $\mathbb{R}^N$.

Next, define

$$F(x,u,v)=\frac{\lambda}{p}f(x)\big||x|^{-(a+k)}u\big|^p+\frac{\mu}{q}g(x)\big||x|^{-(a+k)}u\big|^q + h(x)|x|^{-b(\alpha+\beta+2)}|u|^{\alpha+1}|v|^{\beta+1}$$

and again $J_{\lambda,\mu}:W\to R$ be defined by

$$J_{\lambda,\mu}(u,v)=\frac{1}{p}\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k u\big|^p dx+\frac{1}{q}\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k v\big|^q dx-\int_{\mathbb{R}^N}F(x,u,v)dx. \tag{2.2}$$

Then we introduce the following notation: for any functional $f:W\to R$ we denote by $f'(u,v)(h_1,h_2)$ the Gatěaux derivative of f at $(u,v)\in W$ in the direction of $(h_1,h_2)\in W$, and

$$f^{(1)}(u,v)h_1=f'(u+\varepsilon h_1,v)|_{\varepsilon=0},\quad f^{(2)}(u,v)h_2=f'(u,v+\delta h_2)|_{\delta=0}.$$

Let

$$S_{\lambda,\mu}=\{(u,v)\in W;\ J'_{\lambda,\mu}(u,v)(u,v)=(J^{(1)}_{\lambda,\mu}(u,v)u,J^{(2)}_{\lambda,\mu}(u,v)v)=0\}.$$

It is clear that all critical points of $J_{\lambda,\mu}$ must lie on $S_{\lambda,\mu}$ which is known as the Nehari manifold (see [14, 16]). We will see below that local minimizers of $J_{\lambda,\mu}$ on $S_{\lambda,\mu}$ are usually critical points of $J_{\lambda,\mu}$.

We simplify the notation by using

$$L(u)=\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k u\big|^p-\lambda f(x)\big||x|^{-(a+k)}u\big|^p)dx,$$
$$R(v)=\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k v\big|^q-\mu g(x)\big||x|^{-(a+k)}v\big|^q)dx,$$
$$G(u,v)=\int_{\mathbb{R}^N}h(x)|x|^{-b(\alpha+\beta+2)}|u|^{\alpha+1}|v|^{\beta+1}dx.$$

It is easy to see that $(u,v)\in S_{\lambda,\mu}$ if and only if

$$\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k u\big|^p dx=\lambda\int_{\mathbb{R}^N}f(x)\big||x|^{-(a+k)}u\big|^p dx+(\alpha+1)G(u,v), \tag{2.3}$$
$$\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k v\big|^q=\mu\int_{\mathbb{R}^N}g(x)\big||x|^{-(a+k)}v\big|^q dx+(\beta+1)G(u,v). \tag{2.4}$$

It is useful to understand $S_{\lambda,\mu}$ in terms of the stationary points of the form

$$I(t,s)=J_{\lambda,\mu}(tu,sv),\ \ t,s>0.$$

We will refer to such maps as fibering maps. It is clear that if (u,v) is a local minimizer of $J_{\lambda,\mu}$, then I has a local minimizer at $t=1$, $s=1$.

THEOREM 1. *Let $(u,v)\in W$, $u\neq 0$, $v\neq 0$ and $t,s>0$. Then $(tu,sv)\in S_{\lambda,\mu}$ if and only if $\frac{\partial I}{\partial t}=0$, $\frac{\partial I}{\partial s}=0$.*

Proof. The result is an immediate consequence of the fact that:

$$\begin{aligned}\frac{\partial I}{\partial t}&=J^{(1)}_{\lambda,\mu}(tu,sv)u=\frac{1}{t}J^{(1)}_{\lambda,\mu}(tu,sv)tu,\\ \frac{\partial I}{\partial s}&=J^{(2)}_{\lambda,\mu}(tu,sv)v=\frac{1}{t}J^{(2)}_{\lambda,\mu}(tu,sv)tv.\end{aligned}$$

Thus points in $S_{\lambda,\mu}$ correspond to stationary points of the map $I(t,s)$ and so it is natural to divide $S_{\lambda,\mu}$ into nine subsets. We have

$$\begin{aligned}\frac{\partial I}{\partial t}&=t^{p-1}\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k u\big|^p dx-\lambda t^{p-1}\int_{\mathbb{R}^N}f(x)\big||x|^{-(a+k)}u\big|^p dx\\ &\qquad -(\alpha+1)t^{\alpha}s^{\beta+1}G(u,v), \qquad (2.5)\\ \frac{\partial I}{\partial s}&=s^{q-1}\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k v\big|^q-\mu s^{q-1}\int_{\mathbb{R}^N}g(x)\big||x|^{-(a+k)}v\big|^q dx\\ &\qquad -(\beta+1)t^{\alpha+1}s^{\beta}G(u,v). \qquad (2.6)\end{aligned}$$

Moreover

$$\begin{aligned}\frac{\partial^2 I}{\partial t^2}&=(p-1)t^{p-2}\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k u\big|^p dx-(p-1)\lambda t^{p-2}\int_{\mathbb{R}^N}f(x)\big||x|^{-(a+k)}u\big|^p dx\\ &\qquad -\alpha(\alpha+1)t^{\alpha-1}s^{\beta+1}G(u,v),\\ \frac{\partial^2 I}{\partial s^2}&=(q-1)s^{q-2}\int_{\mathbb{R}^N}\big||x|^{-a}\nabla^k v\big|^q-(q-1)\mu s^{q-2}\int_{\mathbb{R}^N}g(x)\big||x|^{-(a+k)}v\big|^q dx\\ &\qquad -\beta(\beta+1)t^{\alpha+1}s^{\beta-1}G(u,v).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial^2 I}{\partial t^2}\Big|_{(1,1)}&=(p-1)\int_{\mathbb{R}^N}\big(\big||x|^{-a}\nabla^k u\big|^p dx-\lambda f(x)\big||x|^{-(a+k)}u\big|^p\big)dx\\ &\qquad -\alpha(\alpha+1)G(u,v),\end{aligned}$$

and

$$\frac{\partial^2 I}{\partial s^2}\Big|_{(1,1)} = (q-1)\int_{\mathbb{R}^N}\left(\left||x|^{-a}\nabla^k v\right|^q - \mu g(x)\left||x|^{-(a+k)}v\right|^q\right)dx \\ - \beta(\beta+1)G(u,v).$$

Hence, we define:

$$S^{++}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ \frac{\partial^2 I}{\partial t^2}\Big|_{(1,1)} > 0,\ \frac{\partial^2 I}{\partial s^2}\Big|_{(1,1)} > 0\Big\},$$
$$S^{--}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ \frac{\partial^2 I}{\partial t^2}\Big|_{(1,1)} < 0,\ \frac{\partial^2 I}{\partial s^2}\Big|_{(1,1)} < 0\Big\},$$
$$S^{00}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ \frac{\partial^2 I}{\partial t^2}\Big|_{(1,1)} = 0,\ \frac{\partial^2 I}{\partial s^2}\Big|_{(1,1)} = 0\Big\},$$
$$S^{+-}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ \frac{\partial^2 I}{\partial t^2}\Big|_{(1,1)} > 0,\ \frac{\partial^2 I}{\partial s^2}\Big|_{(1,1)} < 0\Big\}.$$

Similarly, we can define $S^{+0}_{\lambda,\mu}$, $S^{0+}_{\lambda,\mu}$, $S^{-0}_{\lambda,\mu}$, $S^{0-}_{\lambda,\mu}$ and $S^{-+}_{\lambda,\mu}$. Since $(u,v)\in S_{\lambda,\mu}$, (2.3) and (2.4) hold, which implies

$$S^{++}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}: \\ (\alpha+1)(p-1-\alpha)G(u,v) > 0, (\beta+1)(q-1-\beta)G(u,v) > 0\Big\}.$$

Since $p-1<\alpha$, $q-1<\beta$,

$$S^{++}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ G(u,v) < 0\Big\}.$$

Similarly,

$$S^{--}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ G(u,v) > 0\Big\},$$
$$S^{00}_{\lambda,\mu} = \Big\{(u,v)\in S_{\lambda,\mu}:\ G(u,v) = 0\Big\}.$$

Moreover, since $p-1<\alpha$, $q-1<\beta$, then $S^{+0}_{\lambda,\mu}$, $S^{0+}_{\lambda,\mu}$, $S^{-0}_{\lambda,\mu}$, $S^{0-}_{\lambda,\mu}$, $S^{-+}_{\lambda,\mu}$ and $S^{+-}_{\lambda,\mu}$ are empty. Thus, $S_{\lambda,\mu}$ is divided into three subsets $S^{++}_{\lambda,\mu}$, $S^{--}_{\lambda,\mu}$ and $S^{00}_{\lambda,\mu}$. We denote these simply as $S^{+}_{\lambda,\mu}$, $S^{-}_{\lambda,\mu}$ and $S^{0}_{\lambda,\mu}$ respectively. Then we have

LEMMA 1. *Let $(u,v)\in S_{\lambda,\mu}$. Then* $\frac{\partial I}{\partial t}\Big|_{(1,1)} = 0$, $\frac{\partial I}{\partial s}\Big|_{(1,1)} = 0$.

The following lemma shows that minimizers on $S_{\lambda,\mu}$ are usually critical points for $J_{\lambda,\mu}$.

Furthermore, similar to the argument in Brown and Zhang [[5], Theorem 2.3] (or see Binding, Drábek, and Huang [3]), we can conclude the following result. We have

LEMMA 2. *Assume that* (u_0, v_0) *is a local minimizer for* $J_{\lambda,\theta}$ *on* $S_{\lambda,\theta}$ *and that* $(u_0, v_0) \notin S^0_{\lambda,\theta}$, *then* $J'_{\lambda,\theta}(u_0, v_0) = 0$ *in* W^{-1}.

It is easy to see that (2.5) and (2.6) are equivalent to

$$\begin{aligned}\frac{\partial I}{\partial t} &= t^{p-1}L(u) - (\alpha+1)t^{\alpha}s^{\beta+1}G(u,v),\\ \frac{\partial I}{\partial s} &= s^{q-1}R(v) - (\beta+1)t^{\alpha+1}s^{\beta}G(u,v).\end{aligned}$$

If $\frac{\partial I}{\partial t} = \frac{\partial I}{\partial s} = 0$, then

$$\begin{aligned}t^{p-1-\alpha} &= (\alpha+1)s^{\beta+1}\frac{G(u,v)}{L(u)},\\ s^{q-1-\beta} &= (\beta+1)t^{\alpha+1}\frac{G(u,v)}{R(v)}.\end{aligned}$$

Thus, if $L(u)$, $R(v)$ and $G(u,v)$ have the same signs, then $I(t,s)$ has exactly one turning point at

$$\begin{aligned}t &= \Big(\frac{(\alpha+1)^{\beta-q+1}|R(v)|^{\beta+1}}{(\beta+1)^{\beta+1}|G(u,v)|^{q}|L(u)|^{\beta-q+1}}\Big)^{\frac{1}{d}},\\ s &= \Big(\frac{(\beta+1)^{\alpha-p+1}|L(u)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}|G(u,v)|^{p}|R(v)|^{\alpha-p+1}}\Big)^{\frac{1}{d}},\end{aligned}\tag{2.7}$$

where $d = (\alpha+1)(\beta+1) - (\alpha-p+1)(\beta-q+1) > 0$. By calculation, t and s have the following property:

$$t(m_1u, m_2v) = \frac{1}{m_1}t(u,v), \quad s(m_1u, m_2v) = \frac{1}{m_2}s(u,v), \quad m_1, m_2 > 0.$$

Thus,

$$(t(m_1u, m_2v)m_1u, s(m_1u, m_2v)m_2v) = (t(u,v)u, s(u,v)v),$$

which play important role in our main results.

If $L(u)$, $R(v)$ and $G(u,v)$ have opposite signs, then $I(t,s)$ has no turning points. To get our results, we just verify that $L(u)$, $R(v)$ and $G(u,v)$ have the same signs.

We define

$$\begin{aligned}A^+ &= \{(u,v) \in W : \ ||u||_p = ||v||_q = 1, \ L(u) > 0, R(v) > 0\},\\ A^0 &= \{(u,v) \in W : \ ||u||_p = ||v||_q = 1, \ L(u) = 0, R(v) = 0\},\\ A^- &= \{(u,v) \in W : \ ||u||_p = ||v||_q = 1, \ L(u) < 0, R(v) < 0\},\end{aligned}$$

and

$$B^+ = \{(u,v) \in W : \ ||u||_p = ||v||_q = 1, \ G(u,v) > 0\},$$

$$B^0 = \{(u,v) \in W : \, ||u||_p = ||v||_q = 1, \, G(u,v) = 0\},$$
$$B^- = \{(u,v) \in W : \, ||u||_p = ||v||_q = 1, \, G(u,v) < 0\}.$$

Thus, if $(u,v) \in A^+ \cap B^+$, $I(t,s) > 0$ for t,s small and positive but $I(t,s) \to -\infty$ as $t \to \infty$ and $s \to \infty$; also $I(t,s)$ has a unique (maximum) stationary point at $(t(u,v), s(u,v))$ and $(t(u,v)u, s(u,v)v) \in S^-_{\lambda,\mu}$. Similarly, if $(u,v) \in A^- \cap B^-$, $I(t,s) < 0$ for t,s small and positive but $I(t,s) \to \infty$ as $t \to \infty$ and $s \to \infty$; also $I(t,s)$ has a unique (minimum) stationary point at $(t(u,v), s(u,v))$ and $(t(u,v)u, s(u,v)v) \in S^+_{\lambda,\mu}$.

Thus, if $(u,v) \in W$ and $u \neq 0$, $v \neq 0$, then:
(i) a multiple of u and a multiple of v lie in $S^-_{\lambda,\mu}$ if and only if $(\frac{u}{||u||_p}, \frac{v}{||v||_q}) \in A^+ \cap B^+$;
(ii) a multiple of u and a multiple of v lie in $S^+_{\lambda,\mu}$ if and only if $(\frac{u}{||u||_p}, \frac{v}{||v||_q}) \in A^- \cap B^-$;
(iii) when (u,v) is neither in $A^+ \cap B^+$ nor in $A^- \cap B^-$, no multiple (u,v) lies in $S_{\lambda,\mu}$.

3. Existence of at least one nontrivial critical point

Suppose that $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$. It is easy to deduce by contradiction with the first eigenvalue that there exists $\delta_0, \delta_1 > 0$ such that

$$L(u) \geqslant \delta_0 ||u||_p^p, \quad R(v) \geqslant \delta_1 ||v||_q^q, \quad (u,v) \in W.$$

Thus A^- and A^0 are empty and so $S^+_{\lambda,\mu}$ is empty and $S^0_{\lambda,\mu} = \{u = v = 0\}$. Moreover,

$$S^-_{\lambda,\mu} = \{(t(u,v)u, s(u,v)v), \, (u,v) \in B^+\}, \quad S_{\lambda,\mu} = S^-_{\lambda,\mu} \cup S^0_{\lambda,\mu}.$$

THEOREM 2. *Assume that $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$. Then (1.1) has at least one critical point.*

Proof. We investigate the behavior of $J_{\lambda,\mu}$ on $S^-_{\lambda,\mu}$. Clearly $J_{\lambda,\mu}(u,v) \geqslant 0$ if $(u,v) \in S^-_{\lambda,\mu}$ and so $J_{\lambda,\mu}(u,v)$ is bounded below by 0 on $S^-_{\lambda,\mu}$. We now show that

$$\inf_{(u,v) \in S^-_{\lambda,\mu}} J(u,v) > 0.$$

Suppose $(u,v) \in S^-_{\lambda,\mu}$. Let $\overline{u} = \frac{u}{||u||_p}$, $\overline{v} = \frac{v}{||v||_q}$, then $(\overline{u}, \overline{v}) \in A^+ \cap B^+$ and $u = t(\overline{u},\overline{v})\overline{u}, v = s(\overline{u},\overline{v})\overline{v}$, where t and s are determined by (2.7).

Now, we claim that, there exists a positive constant C_1 such that

$$G(\overline{u},\overline{v}) = \int_{\mathbb{R}^N} h(x) ||x|^{-b}\overline{u}|^{\alpha+1} ||x|^{-b}\overline{v}|^{\beta+1} dx \leqslant C_1 ||\overline{u}||_p^{\alpha+1} ||\overline{v}||_q^{\beta+1}.$$

Indeed, by condition (H2) we have

$$\frac{Np}{(\alpha+1)[N - p(a+k-b)]} - \frac{Nq}{Nq - (\beta+1)[N - q(a+k-b)]} > 0.$$

So, there exists ρ_0 such that

$$0<\rho_0<\frac{Np}{(\alpha+1)[N-p(a+k-b)]}-\frac{Nq}{Nq-(\beta+1)[N-q(a+k-b)]},$$

which implies

$$\frac{(\beta+1)(p^*-\rho_0(\alpha+1))}{p^*-(\rho_0+1)(\alpha+1)}<\frac{Nq}{N-q(a+k-b)}=q^*.$$

Then, by using the Hölder inequality and the Appendix A in [15], we get,

$$\begin{aligned}
G(\overline{u},\overline{v}) &\leqslant C\left(\int_{\mathbb{R}^N}[||x|^{-b}\overline{u}|^{\alpha+1}]^{\frac{p^*}{\alpha+1}-\rho_0}dx\right)^{\frac{\alpha+1}{p^*-\rho_0(\alpha+1)}}\\
&\qquad\times\left(\int_{\mathbb{R}^N}[||x|^{-b}\overline{v}|^{\beta+1}]^{\frac{p^*-\rho_0(\alpha+1)}{p^*-(\rho_0+1)(\alpha+1)}}dx\right)^{\frac{p^*-(\rho_0+1)(\alpha+1)}{p^*-\rho_0(\alpha+1)}}\\
&\leqslant C\left(\int_{\mathbb{R}^N}||x|^{-b}\overline{u}|^{p}dx\right)^{\frac{\alpha+1}{p}}\left(\int_{\mathbb{R}^N}||x|^{-b}\overline{v}|^{q}dx\right)^{\frac{\beta+1}{q}}\\
&\leqslant C_1||\overline{u}||_p^{\alpha+1}||\overline{v}||_q^{\beta+1}=C_1.
\end{aligned}$$

Hence

$$J_{\lambda,\mu}(u,v)=J_{\lambda,\mu}(t(\overline{u},\overline{v})\overline{u},s(\overline{u},\overline{v})\overline{v})=K\frac{(L(\overline{u}))^{\frac{(\alpha+1)q}{d}}(R(\overline{v}))^{\frac{(\beta+1)p}{d}}}{(G(\overline{u},\overline{v}))^{\frac{pq}{d}}},$$

where

$$K=\left(\frac{(\alpha+1)^{\frac{(\beta+1-q)p}{d}}}{p(\beta+1)^{\frac{p(\beta+1)}{d}}}+\frac{(\beta+1)^{\frac{(\alpha+1-p)q}{d}}}{q(\alpha+1)^{\frac{q(\alpha+1)}{d}}}-\frac{1}{(\alpha+1)^{\frac{q(\alpha+1)}{d}}(\beta+1)^{\frac{p(\beta+1)}{d}}}\right)\times\operatorname{sign}(G(\overline{u},\overline{v})).$$

Since $(\overline{u},\overline{v})\in A^+\cap B^+$, we have $K>0$ and thus

$$J_{\lambda,\mu}(u,v)\geqslant K\frac{(\delta_0)^{\frac{(\alpha+1)q}{d}}(\delta_1))^{\frac{(\beta+1)p}{d}}}{C_1^{\frac{pq}{d}}}.$$

Hence, $\inf_{(u,v)\in S^-_{\lambda,\mu}}J_{\lambda,\mu}(u,v)>0$.

We now show that there exists a minimizer on $S^-_{\lambda,\mu}$ which is a non-trivial critical point of $J_{\lambda,\mu}(u,v)$ in (1.1). Let $\{(u_n,v_n)\}\in S^-_{\lambda,\mu}$ be a minimizer sequence, i.e., $\lim_{n\to\infty}J_{\lambda,\mu}(u_n,v_n)=\inf_{(u,v)\in S^-_{\lambda,\mu}}J_{\lambda,\mu}(u,v)$. Since

$$J_{\lambda,\mu}(u_n,v_n)=\frac{1}{p}L(u_n)+\frac{1}{q}R(v_n)-G(u_n,v_n)$$

$$
\begin{aligned}
&= \Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big) G(u_n, v_n) \\
&= \Big(\frac{1}{p} + \frac{\beta+1}{q(\alpha+1)} - \frac{1}{\alpha+1}\Big) L(u_n) \\
&\geqslant \Big(\frac{1}{p} + \frac{\beta+1}{q(\alpha+1)} - \frac{1}{\alpha+1}\Big) \delta_0 ||u_n||_p^p,
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
J_{\lambda,\mu}(u_n, v_n) &= \Big(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\Big) R(v_n) \\
&\geqslant \Big(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\Big) \delta_1 ||v_n||_q^q,
\end{aligned}
$$

then (u_n, v_n) is bounded in W; we can pass to a subsequence if necessary and have that

$$
\begin{cases}
(u_n, v_n) \rightharpoonup (u_0, v_0), & \text{weakly in } W, \\
u_n \to u_0, v_n \to v_0, & \text{a.e. in } \mathbb{R}^N,
\end{cases}
$$

and

$$
G(u_n, v_n) \to G(u_0, v_0).
$$

Now,

$$
\begin{aligned}
0 < \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) &= \Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big) \lim_{n\to\infty} G(u_n, v_n) \\
&= \Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big) G(u_0, v_0)
\end{aligned}
$$

and so $u \neq 0$, $v \neq 0$. Since $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$, we have

$$
L(u_0) > 0, \quad R(v_0) > 0.
$$

Hence, a multiple of u_0 and a multiple of v_0 lie in $A^+ \cap B^+$. Now we prove that

$$
(u_n, v_n) \to (u_0, v_0), \qquad \text{strongly in } W.
$$

Supposing the contrary, by the lower semi-continuity of norm, then either the lower semi-continuity, $||u_0||_p < \lim_{n\to\infty} ||u_n||_p$, $||v_0||_q < \lim_{n\to\infty} ||v_n||_q$ and so

$$
L(u_0) - (\alpha+1)G(u_0, v_0) < \lim_{n\to\infty} (L(u_n) - (\alpha+1)G(u_n, v_n)) = 0.
$$

We will obtain a contradiction by considering the fibering map $I(t,s)$. We have

$$
\frac{\partial I}{\partial t}\Big|_{(1,1)} = L(u_0) - (\alpha+1)G(u_0, v_0) < 0.
$$

Then, there exists $(x_0,y_0) \neq (1,1)$, such that

$$\frac{\partial I}{\partial t}\Big|_{(x_0,y_0)} = 0, \quad \frac{\partial I}{\partial s}\Big|_{(x_0,y_0)} = 0 \quad \text{i.e., } (x_0u_0, y_0v_0) \in S^-_{\lambda,\mu}.$$

Now, $(x_0u_n, y_0v_n) \rightharpoonup (x_0u_0, y_0v_0)$ in W. Moreover, as $(u_n,v_n) \in S^-_{\lambda,\mu}$, the map $I(t,s)$ attains its maximum at $t = s = 1$. Hence,

$$J_{\lambda,\mu}(x_0u_0,y_0v_0) < \lim_{n\to\infty} J_{\lambda,\mu}(x_0u_n,y_0v_n) \leqslant \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$$

and this is a contradiction. Then $(u_n,v_n) \to (u_0,v_0)$ in W. It follows easily that

$$L(u_0) - (\alpha+1)G(u_0,v_0) = 0, \quad R(v_0) - (\beta+1)G(u_0,v_0) = 0$$

and so $(u_0,v_0) \in S^-_{\lambda,\mu}$. Also

$$J_{\lambda,\mu}(u_0,v_0) = \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$$

and so (u_0,v_0) is a minimizer on $S^-_{\lambda,\mu}$. Since

$$G(u_0,v_0) > 0, \ (u_0,v_0) \notin S^0_{\lambda,\mu}$$

and by Lemma 2, (u_0,v_0) is a critical point of $J_{\lambda,\mu}(u,v)$. Since

$$J_{\lambda,\mu}(|u|,|v|) = J_{\lambda,\mu}(u,v),$$

we may assume that (u_0,v_0) is positive critical point of (1.1).

The final result of this section examines the behavior of

$$\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu} \quad \text{as } \lambda \to \lambda_1(a)^-, \ \mu \to \mu_1(b)^-.$$

THEOREM 3. *Suppose that*

$$\int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|\phi|^{\alpha+1}|\psi|^{\beta+1}dx > 0.$$

Then

$$\lim_{\lambda\to\lambda_1(a)^-} \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = 0 \quad \textit{and} \quad \lim_{\mu\to\mu_1(b)^-} \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = 0.$$

Proof. Without loss of generality, we assume that $||\phi||_p = ||\psi||_q = 1$. It is clear that $(\phi,\psi) \in B^+$. Since $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$, we get $(\phi,\psi) \in A^+$ and so $(\phi,\psi) \in A^+ \cap B^+$. Hence,

$$(t(\phi,\psi)\phi, s(\phi,\psi)\psi) \in S^-_{\lambda,\mu}$$

and

$$\begin{aligned}
&J_{\lambda,\mu}(t(\phi,\psi)\phi,s(\phi,\psi)\psi)\\
&\quad = K\frac{(\int_{\mathbb{R}^N}(\lambda_1-\lambda)f(x)\big||x|^{-(a+k)}\phi\big|^p dx)^{\frac{(\alpha+1)q}{d}}}{(G(\phi,\psi))^{\frac{pq}{d}}}\\
&\qquad\qquad \times (\int_{\mathbb{R}^N}(\mu_1-\mu)g(x)\big||x|^{-(a+k)}\psi\big|^q dx)^{\frac{(\beta+1)p}{d}}\\
&\quad = K(\lambda_1-\lambda)^{\frac{(\alpha+1)q}{d}}(\mu_1-\mu)^{\frac{(\beta+1)p}{d}}\\
&\qquad\qquad \times \frac{(\int_{\mathbb{R}^N}f(x)\big||x|^{-(a+k)}\phi\big|^p dx)^{\frac{(\alpha+1)q}{d}}(\int_{\mathbb{R}^N}g(x)\big||x|^{-(a+k)}\psi\big|^q dx)^{\frac{(\beta+1)p}{d}}}{(G(\phi,\psi))^{\frac{pq}{d}}}\\
&\quad \to 0
\end{aligned}$$

as $\lambda\to\lambda_1(a)^-$, $\mu\to\mu_1(b)^-$.

Since $0<\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v)\leqslant J_{\lambda,\mu}(t(\phi,\psi)\phi,s(\phi,\psi)\psi)$, it follows that the conclusion is true.

4. Existence of a second critical point

In this section we show that conditions $\overline{A^-}\cap\overline{B^+}=\emptyset$ and $S^+_{\lambda,\mu}$ allow a second critical point of $J_{\lambda,\mu}\neq\emptyset$ to exist in $S^+_{\lambda,\mu}$. Our first objective will be to show that such a situation can indeed occur.

If $\lambda>\lambda_1(a)$, $\mu>\mu_1(b)$, then we have

$$\begin{aligned}
L(\phi)&=\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k\phi\big|^p-(\lambda_1(a)-\lambda)f(x)\big||x|^{-(a+k)}\phi\big|^p)dx<0,\\
R(\psi)&=\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k\psi\big|^q-(\mu_1(b)-\mu)g(x)\big||x|^{-(a+k)}\psi\big|^q)dx<0.
\end{aligned}$$

Thus, $(\phi,\psi)\in A^+$, Hence, if $G(\phi,\psi)<0$, then $(\phi,\psi)\in A^-\cap B^-$ and so $S^+_{\lambda,\mu}$ is nonempty. Thus, $S_{\lambda,\mu}$ may consists of two distinct components in this case which makes it possible to prove the existence of at least two critical points by showing that $J_{\lambda,\mu}(u,v)$ has a minimizer on each component.

LEMMA 3. *Assume that conditions (1.3), (1.4),(1.5) hold and $G(\phi,\psi)<0$. Then there exists $\delta>0$, $\sigma>0$ such that $\overline{A^-}\cap\overline{B^+}=\emptyset$ whenever $\lambda_1(a)\leqslant\lambda<\lambda_1(a)+\delta$, $\mu_1(a)\leqslant\mu<\mu_1(b)+\sigma$.*

Proof. If there is no such δ, then there exists sequences $\{(\lambda_n,\mu_n)\}$ and $\{(u_n,v_n)\}$ such that $||u_n||_p=1$, $||v_n||_q=1$, $\lambda_n\to\lambda_1(a)^+$, $\mu_n\to\mu_1(b)^+$ and

$$L(u_n)=\int_{\mathbb{R}^N}(\big||x|^{-a}\nabla^k u_n\big|^p-\lambda f(x)\big||x|^{-(a+k)}u_n\big|^p)dx\leqslant 0, \tag{4.1}$$

$$R(v_n) = \int_{\mathbb{R}^N} (\left||x|^{-a}\nabla^k v_n\right|^q - \mu g(x)\left||x|^{-(a+k)} v_n\right|^q) dx \leqslant 0, \tag{4.2}$$

$$G(u_n, v_n) = \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|u_n|^{\alpha+1}|v_n|^{\beta+1} dx \geqslant 0. \tag{4.3}$$

Since $\{(u_n, v_n)\}$ is bounded, we may assume that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_0, v_0), & \text{weakly in } W, \\ u_n \to u_0, v_n \to v_0, & \text{a.e. in } \mathbb{R}^N, \end{cases}$$

and

$$G(u_n, v_n) \to G(u_0, v_0).$$

Supposing the contrary, by the lower semi-continuity of norm, then either the lower semi-continuity, $||u_0||_p < \lim_{n\to\infty} ||u_n||_p$, $||v_0||_q < \lim_{n\to\infty} ||v_n||_q$ and so by (4.1),

$$0 \leqslant L(u_0) < \lim_{n\to\infty} L(u_n) \leqslant 0.$$

which is impossible. Hence, $(u_n, v_n) \to (u_0, v_0)$ in W holds and so $||u_0||_p = ||v_0||_q = 1$. From (4.1)- (4.3), it follows that

$$L(u_0) \leqslant 0, \quad R(v_0) \leqslant 0, \quad G(u_0, v_0) \geqslant 0.$$

The first two inequalities imply that $u_0 = k_1\phi$, $v_0 = k_2\psi$. But from the last inequality and the condition $G(\phi, \psi) < 0$, we deduce that $k_1 = 0$ or $k_2 = 0$ or $k_1 = k_2 = 0$. It is impossible for $||u_0||_p = ||v_0||_q = 1$. so there are $\delta > 0$, $\sigma > 0$ as require.

The condition $\overline{A^-} \cap \overline{B^+} = \emptyset$ guarantees that the Nehari manifold has several desirable properties, which we state in the next proposition.

PROPOSITION 1. *Suppose that* $\overline{A^-} \cap \overline{B^+} = \emptyset$. *Then*
(i) $S^0_{\lambda,\mu} = \{(0,0)\}$*;*
(ii) for any $(0,0) \notin \overline{S^-_{\lambda,\mu}}$*, and* $S^-_{\lambda,\mu}$ *is closed;*
(iii) $\overline{S^+_{\lambda,\mu}} \cap \overline{S^-_{\lambda,\mu}} = \emptyset$*;*
(iv) $S^+_{\lambda,\mu}$ *is bounded.*

Proof. (i) If $(u, v) \in S^0_{\lambda,\mu} \setminus \{(0,0)\}$ but $u \neq 0$, $v \neq 0$. Then

$$\left(\frac{u}{||u||_p}, \frac{v}{||v||_q}\right) \in A^0 \cap B^0 \subset \overline{A^-} \cap \overline{B^+} = \emptyset$$

and this is a contradiction.

(ii) Assume that $(0,0) \in \overline{S^-_{\lambda,\mu}}$. Then there exists $\{(u_n, v_n)\} \subset S^-_{\lambda,\mu}$ such that $(u_n, v_n) \to (0,0)$ in W. Thus we have

$$0 < L(u_n) = (\beta+1)G(u_n, v_n) \to 0,$$

$$0 < R(v_n) = (\alpha+1)G(u_n, v_n) \to 0.$$

We may assume that $\overline{u}_n = \frac{u_n}{||u_n||_p}$ and $\overline{v}_n = \frac{v_n}{||v_n||_q}$ are such that

$$\begin{cases} \overline{u}_n \rightharpoonup u_0, & \text{weakly in } Y_p, \\ \overline{v}_n \rightharpoonup v_0, & \text{weakly in } Y_q, \\ \overline{u}_n \to u_0, \quad \overline{v}_n \to v_0, & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

Clearly, as $n \to \infty$,

$$0 < L(\overline{u}_n) = (\alpha+1)||u_n||_p^{\alpha+1-p} \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|\overline{u}_n|^{\alpha+1}|v_n|^{\beta+1}dx \to 0, \tag{4.4}$$
$$0 < R(\overline{v}_n) = (\beta+1)||v_n||_q^{\beta+1-q} \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|u_n|^{\alpha+1}|\overline{v}_n|^{\beta+1}dx \to 0.$$

Since $||\overline{u}_n||_p = ||\overline{v}_n||_q = 1$, we deduce that

$$\begin{aligned} 0 = \lim_{n\to\infty} L(\overline{u}_n) &= 1 - \lambda \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)\big||x|^{-(a+k)}\overline{u}_n\big|^p dx \\ &= 1 - \lambda \int_{\mathbb{R}^N} f(x)\big||x|^{-(a+k)}u_0\big|^p dx, \end{aligned}$$

and

$$\begin{aligned} 0 = \lim_{n\to\infty} R(\overline{v}_n) &= 1 - \mu \lim_{n\to\infty} \int_{\mathbb{R}^N} g(x)\big||x|^{-(a+k)}\overline{v}_n\big|^q dx \\ &= 1 - \mu \int_{\mathbb{R}^N} g(x)\big||x|^{-(a+k)}v_0\big|^q dx, \end{aligned}$$

and so $u_0 \neq 0$, $v_0 \neq 0$. Moreover,

$$\begin{aligned} L(u_0) &\leqslant \lim_{n\to\infty} L(u_n) = 0, \\ R(v_0) &\leqslant \lim_{n\to\infty} R(v_n) = 0, \end{aligned}$$

which implies

$$\int_{\mathbb{R}^N} \Big(\Big||x|^{-a}\nabla^k \frac{u_0}{||u_0||_p}\Big|^p - \lambda f(x)\Big||x|^{-(a+k)}\frac{u_0}{||u_0||_p}\Big|^p\Big)dx \leqslant 0,$$
$$\int_{\mathbb{R}^N} \Big(\Big||x|^{-a}\nabla^k \frac{v_0}{||v_0||_q}\Big|^q - \mu g(x)\Big||x|^{-(a+k)}\frac{v_0}{||v_0||_q}\Big|^q\Big)dx \leqslant 0.$$

We conclude that $\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in \overline{A^-}$. On the other hand, by Fatou's lemma,

$$\begin{aligned} 0 \leqslant \lim_{n\to\infty} G(u_n, v_n) = \lim_{n\to\infty} &\int_{h(x)<0} h(x)|x|^{-b(\alpha+\beta+2)}|u_n|^{\alpha+1}|v_n|^{\beta+1}dx \\ &+ \lim_{n\to\infty} \int_{h(x)\geqslant 0} h(x)|x|^{-b(\alpha+\beta+2)}|u_n|^{\alpha+1}|v_n|^{\beta+1}dx \end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{h(x)<0} h(x)|x|^{-b(\alpha+\beta+2)}|u_0|^{\alpha+1}|v_0|^{\beta+1}dx \\
&\qquad + \int_{h(x)\geqslant 0} h(x)|x|^{-b(\alpha+\beta+2)}|u_0|^{\alpha+1}|v_0|^{\beta+1}dx \\
&= \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|u_0|^{\alpha+1}|v_0|^{\beta+1}dx. \qquad (4.5)
\end{aligned}$$

Consequently

$$\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in \overline{B^+}.$$

Therefore, we have the contradiction

$$\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in \overline{A^-} \cap \overline{B^+}.$$

It follows that $(0,0) \notin \overline{S^-_{\lambda,\mu}}$. By the assertion (i),

$$\overline{S^-_{\lambda,\mu}} \subset S^-_{\lambda,\mu} \cup S^0_{\lambda,\mu} = \subset S^-_{\lambda,\mu} \cup \{(0,0)\}.$$

Since $(0,0) \notin \overline{S^-_{\lambda,\mu}}$, which implies that $\overline{S^-_{\lambda,\mu}} = S^-_{\lambda,\mu}$, i.e., $S^-_{\lambda,\mu}$ is closed.

(iii) By assertions (i) and (ii), we have

$$\begin{aligned}
\overline{S^-_{\lambda,\mu}} \cap \overline{S^+_{\lambda,\mu}} = S^-_{\lambda,\mu} \cap \overline{S^+_{\lambda,\mu}} &\subseteq S^-_{\lambda,\mu} \cap (S^+_{\lambda,\mu} \cup S^0_{\lambda,\mu}) \\
&= (S^-_{\lambda,\mu} \cap S^+_{\lambda,\mu}) \cup (S^-_{\lambda,\mu} \cap S^0_{\lambda,\mu}) \\
&= \emptyset,
\end{aligned}$$

and so $\overline{S^+_{\lambda,\mu}} \cap \overline{S^-_{\lambda,\mu}} = \emptyset$.

(iv) If $S^+_{\lambda,\mu}$ is unbounded, then we can fined a sequence $\{(u_n,v_n)\} \subset S^+_{\lambda,\mu}$ such that $||(u_n,v_n)|| \to \infty$. There will be three cases that occur:
(a) u_n is not bounded in Y_p and v_n is bounded in Y_q;
(b) u_n is bounded in Y_p and v_n is not bounded in Y_q;
(c) u_n is not bounded in Y_p and v_n is not bounded in Y_q.

Assume that case (a) occurs. We may assume that $\overline{u}_n = \frac{u_n}{||u_n||_p}$. Since,

$$L(\overline{u}_n) = (\alpha+1)||u_n||_p^{\alpha+1-p} \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|\overline{u}_n|^{\alpha+1}|v_n|^{\beta+1}dx < 0.$$

As the left hand side is uniformly bounded but the term $||u_n||_p^{\alpha+1-p} \to \infty$ and

$$\lim_{n\to\infty} L(\overline{u}_n) = 1 - \lambda \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x)\big||x|^{-(a+k)}\overline{u}_n\big|^p dx = 1 - \lambda \int_{\mathbb{R}^N} f(x)\big||x|^{-(a+k)}u_0\big|^p dx,$$

is finite, it must be true that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)|x|^{-b(\alpha+\beta+2)}|\overline{u}_n|^{\alpha+1}|v_n|^{\beta+1}dx = 0.$$

We now show that if

$$||u_0||_p = \lim_{n\to\infty} ||\overline{u}_n||_p = 1, \quad ||v_0||_q = \lim_{n\to\infty} ||\overline{v}_n||_q,$$

it follows from (4.6) that $\left(u_0, \frac{v_0}{||v_0||_q}\right) \in B^0$, which implies that $\left(u_0, \frac{v_0}{||v_0||_q}\right) \in \overline{B^+}$. Since

$$L(u_0) = \lim_{n\to\infty} L(\overline{u}_n) \leqslant 0,$$
$$R(v_0) = \lim_{n\to\infty} R(v_n) \leqslant 0,$$

we see that $\left(u_0, \frac{v_0}{||v_0||_q}\right) \in \overline{A^-} \cap \overline{B^+}$. This is impossible. If the case (b) or case (c) occurs, we also get a contradiction. Thus, $S^+_{\lambda,\mu}$ is bounded.

LEMMA 4. *Assume that conditions* (1.3), (1.4), (1.5) *hold and that* $G(\phi,\psi) < 0$, $(u,v) \in S^-_{\lambda,\mu}$. *Then there exists* $\delta_1 > 0$, $\delta_2 > 0$ *such that*

$$L(u_n) \geqslant \delta_1 ||u_n||_p^p \quad \textit{and} \quad R(v_n) \geqslant \delta_2 ||v_n||_q^q,$$

whenever $\lambda_1(a) \leqslant \lambda < \lambda_1(a) + \delta$, $\mu_1(a) \leqslant \mu < \mu_1(b) + \sigma$.

Proof. The proof is similar to the proof of Theorem 1(iii) in [3].

The following results are the main points of this section.

THEOREM 4. *Under the conditions* (1.3), (1.4), (1.5) *and* $\lambda,\mu \in R$, *if* $G(\phi,\psi) < 0$, *then*
(i) every minimizer sequence for $J_{\lambda,\mu}(u,v)$ *in* $S^-_{\lambda,\mu}$ *is bounded;*
(ii) $\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) > 0$*;*
(iii) there exists a minimizer of $J_{\lambda,\mu}(u,v)$ *on* $S^-_{\lambda,\mu}$.

Proof. (i) suppose that $\{(u_n,v_n)\} \subset S^-_{\lambda,\mu}$ is a minimizing sequence such that $\{(u_n,v_n)\}$ is unbounded. Without loss of generality, we may assume that u_n is unbounded in Y_p and v_n is bounded in Y_q. There exist $c_1, c_2 \geqslant 0$ such that

$$L(u_n) = (\alpha+1)G(u_n,v_n) \to c_1, \tag{4.6}$$
$$R(v_n) = (\beta+1)G(u_n,v_n) \to c_2. \tag{4.7}$$

Let $\overline{u}_n = \frac{u_n}{||u_n||_p}$. Diving (4.6) by $||u_n||_p^p$, we obtain

$$L(\overline{u}_n) = (\alpha+1)||u_n||_p^{\alpha+1-p} G(\overline{u}_n, v_n) \to 0.$$

But by Lemma 4, $L(\overline{u}_n) \geqslant \delta_1 ||\overline{u}_n||_p^p = \delta_1$. This is impossible.

(ii) Obviously, $\overline{A^-} \cap \overline{B^+} = \emptyset$ from Lemma 3. Since

$$J_{\lambda,\mu}(u,v) = \Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big)G(u,v) \geqslant 0, \quad \text{on } S^-_{\lambda,\mu},$$

so we have $\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) \geqslant 0$. Now, we show that $\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) > 0$. In fact, if $\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) = 0$. A minimizing sequence $\{(u_n,v_n)\} \subset S^-_{\lambda,\mu}$ satisfies

$$L(u_n) = (\alpha+1)G(u_n,v_n) \to 0,$$
$$R(v_n) = (\beta+1)G(u_n,v_n) \to 0.$$

Since (i) asserts that $\{(u_n,v_n)\}$ is bounded, we may assume that

$$\begin{cases} (u_n,v_n) \rightharpoonup (u_0,v_0), & \text{weakly in } W, \\ u_n \to u_0, v_n \to v_0, & \text{a.e. in } \mathbb{R}^N, \end{cases}$$

We claim that $(u_0,v_0) \notin S^0_{\lambda,\mu}$.

In fact, if $(u_0,v_0) = (0,0)$, i.e., $(u_n,v_n) \to (u_0,v_0)$ in W, then $(0,0) \in S^-_{\lambda,\mu}$ since $S^-_{\lambda,\mu}$ is closed, which is impossible.

If $u_0 \neq 0$, $v_0 = 0$. By the lower semi-continuity,

$$L(u_0) \leqslant \lim_{n\to\infty} L(u_n) = 0. \tag{4.8}$$

By like in the proof of Lemma 3, we can obtain that there exist $\delta,\sigma > 0$ such that $L(u_0) \geqslant 0$, $R(v_0) \geqslant 0$ when $\lambda_1(a) \leqslant \lambda < \lambda_1(a)+\delta$, $\mu_1(a) \leqslant \mu < \mu_1(b)+\sigma$ and $(u_n,v_n) \in S^-_{\lambda,\mu}$, $(u_n,v_n) \rightharpoonup (u_0,v_0)$. Therefore, $||u_0||_p = ||u_n||_p = 1$ and $(u_n,v_n) \to (u_0,v_0) \in S^-_{\lambda,\mu}$. This is a contradiction.

Similarly, we can get the contradiction when $u_0 = 0$, $v_0 \neq 0$. Then we have $u_0 \neq 0$, $v_0 \neq 0$.

Since $G(u_0,v_0) = 0$ from 4.8, we have $\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in B^0$. Therefore,

$$\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in \overline{B^+}.$$

By lower semi-continuity,

$$L(u_0) \leqslant \lim_{n\to\infty} L(u_n) = 0,$$
$$R(v_0) \leqslant \lim_{n\to\infty} R(v_n) = 0,$$

which implies

$$\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in \overline{A^-},$$

thus

$$\left(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\right) \in \overline{A^-} \cap \overline{B^+}.$$

This is impossible. Therefore, $\inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v) > 0$.

(iii) Obviously, $\overline{A^-} \cap \overline{B^+} = \emptyset$. Once again due to (i), we may assume that the minimizing sequence $\{(u_n, v_n)\} \subset S^-_{\lambda,\mu}$ satisfies

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_0, v_0), & \text{weakly in } W, \\ u_n \to u_0, v_n \to v_0, & \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{4.9}$$

and

$$G(u_n, v_n) \to G(u_0, v_0).$$

By like in (4.5) and Fatou's Lemma,

$$\begin{aligned} \Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big) G(u_0, v_0) &\geqslant \lim_{n\to\infty} \Big(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\Big) G(u_n, v_n) \\ &\to \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v) > 0. \end{aligned}$$

Now, by assumption $\overline{A^-} \cap \overline{B^+} = \emptyset$ and so $B^+ \subset A^+$. Hence,

$$\Big(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\Big) \in A^+ \cap B^+.$$

This shows that $(t(u_0, v_0)u_0, s(u_0, v_0)v_0) \in S^-_{\lambda,\mu}$, where t, s are given in (2.7). If $u_n \not\to u_0$ in Y_p and $v_n \not\to v_0$ in Y_q, then by the lower semi-continuity

$$||u_0||_p < \lim_{n\to\infty} ||u_n||_p \quad \text{and} \quad ||v_0||_q < \lim_{n\to\infty} ||v_n||_q.$$

Then

$$\begin{aligned} L(u_0) &< \lim_{n\to\infty} L(u_n) = (\alpha+1) \lim_{n\to\infty} G(u_n, v_n)) = (\alpha+1) G(u_0, v_0), \\ R(v_0) &< \lim_{n\to\infty} R(v_n) = (\beta+1) \lim_{n\to\infty} G(u_n, v_n)) = (\beta+1) G(u_0, v_0). \end{aligned}$$

Thus $(t(u_0, v_0), s(u_0, v_0)) \neq (1, 1)$. Since

$$(t(u_0, v_0)u_n, s(u_0, v_0)v_n) \rightharpoonup (t(u_0, v_0)u_0, s(u_0, v_0)v_0),$$

and the map $(t, s) \to J_{\lambda,\mu}(tu_n, sv_n)$ attains its maximum value at $t = s = 1$, we have that

$$\begin{aligned} J_{\lambda,\mu}(t(u_0, v_0)u_0, s(u_0, v_0)v_0) &< \lim_{n\to\infty} J_{\lambda,\mu}(t(u_0, v_0)u_n, s(u_0, v_0)v_n) \\ &\leqslant \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) \\ &= \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v). \end{aligned}$$

This is a contradiction. Then we have $||u_0||_p = \lim_{n\to\infty} ||u_n||_p$, $||v_0||_q = \lim_{n\to\infty} ||v_n||_q$. So, we can obtain

$$L(u_0) = (\alpha+1) G(u_0, v_0),$$

$$R(v_0) = (\beta+1)G(u_0,v_0).$$

Therefore $(u_0,v_0) \in S_{\lambda,\mu}$. Since $G(u_0,v_0) > 0$, which implies $(u_0,v_0) \in S^-_{\lambda,\mu}$. Also

$$J_{\lambda,\mu}(u_0,v_0) = \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = \inf_{(u,v)\in S^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).$$

This shows that (u_0,v_0) is a minimizer of $J_{\lambda,\mu}(u,v)$ on $S^-_{\lambda,\mu}$.

THEOREM 5. *Suppose that $S^+_{\lambda,\mu} \neq \emptyset$ and $\overline{A^-}\cap\overline{B^+} = \emptyset$, then there exists a $(u_0,v_0) \in S^+_{\lambda,\mu}$ such that $J_{\lambda,\mu}(u_0,v_0) = \inf_{(u,v)\in S^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v)$.*

Proof. By definition $b := \inf_{(u,v)\in S^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v) < 0$. Then by Proposition 1(iv), $S^+_{\lambda,\mu}$ is bounded, and so b is finite.

We may assume that the minimizing sequence $\{(u_n,v_n)\} \subset S^+_{\lambda,\mu}$ satisfies

$$\begin{cases} (u_n,v_n) \rightharpoonup (u_0,v_0), & \text{weakly in } W, \\ u_n \to u_0, v_n \to v_0, & \text{a.e. in } \mathbb{R}^N, \end{cases}$$

and

$$G(u_n,v_n) \to G(u_0,v_0).$$

Since

$$L(u_0) < \lim_{n\to\infty} L(u_n) = \Big(\frac{\alpha+1}{p(\beta+1)} + \frac{1}{q} - \frac{1}{\beta+1}\Big)^{-1} b < 0,$$
$$R(v_0) < \lim_{n\to\infty} R(v_n) = \Big(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\Big)^{-1} b < 0.$$

and $\overline{A^-}\cap\overline{B^+} = \emptyset$, $G(u_0,v_0) < 0$. So

$$u_0 \neq 0,\ v_0 \neq 0,\ \Big(\frac{u_0}{||u_0||_p}, \frac{v_0}{||v_0||_q}\Big) \in A^- \cap B^-.$$

This shows that $(t(u_0,v_0)u_0, s(u_0,v_0)v_0) \in S^+_{\lambda,\mu}$, where t, s are given in (2.7).

If $u_n \nrightarrow u_0$ in Y_p and $v_n \nrightarrow v_0$ in Y_q, then by the lower semi-continuity

$$||u_0||_p < \lim_{n\to\infty} ||u_n||_p \quad \text{and} \quad ||v_0||_q < \lim_{n\to\infty} ||v_n||_q.$$

Then

$$L(u_0) < \lim_{n\to\infty} L(u_n) = (\alpha+1) \lim_{n\to\infty} G(u_n,v_n)) = (\alpha+1)G(u_0,v_0),$$
$$R(v_0) < \lim_{n\to\infty} R(v_n) = (\beta+1) \lim_{n\to\infty} G(u_n,v_n)) = (\beta+1)G(u_0,v_0).$$

Thus $(t(u_0,v_0),s(u_0,v_0)) \neq (1,1)$. Since

$$(t(u_0,v_0)u_n, s(u_0,v_0)v_n) \rightharpoonup (t(u_0,v_0)u_0, s(u_0,v_0)v_0),$$

and the map $(t,s) \to J_{\lambda,\mu}(tu_n, sv_n)$ attains its maximum value at $t = s = 1$, we have that

$$\begin{aligned} J_{\lambda,\mu}(t(u_0,v_0)u_0, s(u_0,v_0)v_0) &< \lim_{n\to\infty} J_{\lambda,\mu}(t(u_0,v_0)u_n, s(u_0,v_0)v_n) \\ &\leqslant \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = b. \end{aligned}$$

This is a contradiction. Then we have $||u_0||_p = \lim_{n\to\infty} ||u_n||_p$, $||v_0||_q = \lim_{n\to\infty} ||v_n||_q$. So, we can obtain

$$\begin{aligned} L(u_0) &= (\alpha+1)G(u_0,v_0), \\ R(v_0) &= (\beta+1)G(u_0,v_0). \end{aligned}$$

Therefore $(u_0,v_0) \in S_{\lambda,\mu}$. Since $G(u_0,v_0) > 0$, which implies $(u_0,v_0) \in S^+_{\lambda,\mu}$. Also

$$J_{\lambda,\mu}(u_0,v_0) = \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = b.$$

This shows that (u_0,v_0) is a minimizer of $J_{\lambda,\mu}(u,v)$ on $S^+_{\lambda,\mu}$.

THEOREM 6. *Suppose that $G(\phi,\psi) < 0$, then there exists $\delta > 0$, $\sigma > 0$ such that (1.1) has at least two critical points whenever $\lambda_1(a) \leqslant \lambda < \lambda_1(a)+\delta$, $\mu_1(a) \leqslant \mu < \mu_1(b)+\sigma$.*

Proof. When $\lambda > \lambda_1(a)$, $\mu > \mu_1(a)$, we easily get $(\phi,\psi) \in A^- \cap B^-$. By Lemma 3, Theorems 4 and 5, we know that there exists $\delta > 0$, $\sigma > 0$ such that when $\lambda_1(a) \leqslant \lambda < \lambda_1(a)+\delta$, $\mu_1(a) \leqslant \mu < \mu_1(b)+\sigma$, $J_{\lambda,\mu}(u,v)$ has a minimizer in each of $S^-_{\lambda,\mu}$ and $S^+_{\lambda,\mu}$. As $J_{\lambda,\mu}(|u|,|v|) = J_{\lambda,\mu}(u,v)$, we may assume that these minimizers of $J_{\lambda,\mu}(u,v)$ are positive. From Theorem 1(iii), we get that $S^-_{\lambda,\mu}$ and $S^+_{\lambda,\mu}$ are separated and $S^0_{\lambda,\mu} = \{(0,0)\}$. It follows that the minimizers of $J_{\lambda,\mu}(u,v)$ are its local minimizers in $S_{\lambda,\mu}$ which do not lie in $S^0_{\lambda,\mu}$, and so are critical points of (1.1) by Theorem 1.

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(Received November 21, 2011)
(Revised March 14, 2012)

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