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EXISTENCE OF SOLUTIONS FOR A SINGULAR FOURTH–ORDER p–LAPLACIAN SYSTEM

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Abstract. This work is devoted to proving existence of solutions for a singular fourth-order p-Laplacian system. The nonlinearities depend on the solution and first derivatives and may exhibit singularities. Existence results are proved using the Leray-Schauder nonlinear alternative. Examples of applications illustrate each one of the obtained results.

1. Introduction

In this paper, we are concerned with the existence of solutions to the fourth-order nonlinear differential system associated with the p-Laplacian

$$\begin{aligned} &(\varphi_p(u''))'' = f_1(x,u,v,u',v'), \quad x \in (0,1),\\ &(\varphi_p(v''))'' = f_2(x,u,v,u',v'), \quad x \in (0,1),\\ &u(0) = u(1) = u''(0) = u''(1) = 0,\\ &v(0) = v(1) = v''(0) = v''(1) = 0, \end{aligned} \tag{1.1}$$

where for each $i = 1,2$, $f_i : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a Carathéodory function.

DEFINITION 1.1. $f : [0,1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is called a Carathéodory function if:

(i) the map $x \longmapsto f(x,u,v,y,z)$ is measurable for all $u,v,y,z \in \mathbb{R}$;

(ii) the map $(u,v,y,z) \longmapsto f(x,u,v,y,z)$ is continuous for almost every $x \in [0,1]$;

(iii) for every $r > 0$, there exists $h_r \in L^1([0,1],\mathbb{R}^+)$ such that $|f(x,u,v,y,z)| \leqslant h_r(x)$, for a.e. $x \in [0,1]$ and for all $u,v,y,z \in \mathbb{R}$, with $|u| \leqslant r$, $|v| \leqslant r$, $|y| \leqslant r$, and $|z| \leqslant r$.

Here $\varphi_p(s) = s|s|^{p-2}$ $(p > 1)$ refers to the p-Laplacian operator. By a solution to Problem (1.1), we understand a couple of functions $(u,v) \in C^2\big([0,1],\mathbb{R}^2\big)$ such that $(\varphi_p(u''),\varphi_p(v'')) \in C^2\big((0,1),\mathbb{R}^2\big)$ and (1.1) is satisfied for $x \in (0,1)$.

The typical problem for (1.1) stems from the modelization of the deformation of an elastic beam under an external force f and simply supported at both ends. The bending

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is described by the following linear boundary-value problem (see e.g., [2, 3, 12, 15, 17] and the references therein):

$$\begin{aligned} &u^{(4)}(x) = f(x), \quad x \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned}$$

The boundary conditions are motivated by the vanishing moments and shear forces at the ends of the attached beam (see [9]). Many nonlinear fourth-order boundary value problems, corresponding to $f = f(t,u)$, have been extensively studied in the recent literature (see for instance [1, 8, 14]). Several methods ranging from the fixed point index theory on cones of Banach spaces to monotone techniques have been used so far (see [4, 5] for the case of equations). However, the study of general nonlinear systems is less developed and only a few existence results are available in the literature. In [6], a p-Laplacian system is investigated and existence of positive solutions are proved. In fact, a recent vector version of the Krasnosel'skiĭ fixed point theorem [13] was used and new existence results were obtained. Our aim in this work is to complement some results obtained in [6] and [10, 11] by considering the regular and singular cases on one hand and the case when the nonlinearities depend on first derivatives on the other one.

In this section, we also give some preliminary results needed in this paper. The regular problem where f_i $(i = 1,2)$ depend on the solution and first derivatives is discussed in Section 2. In Section 3, more general growth conditions are assumed when the nonlinearities do not depend on the derivatives. Finally, in Section 4, we study the problem where $f_i(x,u,v)$ $(i = 1,2)$ may possess a singularity at the origin. Each existence result is illustrated by means of an example of application.

Let $C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with norm $\|u\|_0 = \sup\{|u(x)|,\ 0 \leqslant x \leqslant 1\}$ and denote by E the Banach space $C([0,1],\mathbb{R}^2) := C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ endowed with the sup-norm

$$\|(u,v)\|_0 = \|u\|_0 + \|v\|_0.$$

E_1 will refer to the Banach space $C^1([0,1],\mathbb{R}^2) := C^1([0,1],\mathbb{R}) \times C^1([0,1],\mathbb{R})$ endowed with the sup-norm

$$\|(u,v)\|_1 = \|u\|_1 + \|v\|_1,$$

where

$$\|u\|_1 = \max(\|u\|_0, \|u'\|_0).$$

$L^1(0,1)$ will denote the space of measurable functions which are Lebesgue integrable on $(0,1)$. The norm in this Banach space is denoted by

$$|u|_1 = \int_0^1 |u(t)|dt.$$

In order to transform Problem (1.1) into a fixed point problem, we need some auxiliary results that are collected in this section. The following lemma is immediate.

LEMMA 1.1. *For any $v \in C([0,1],\mathbb{R})$, the fourth-order boundary-value problem*

$$\begin{aligned} (\varphi_p(u''))''(x) &= v(x), \quad 0 < x < 1, \\ u(0) = u(1) &= u''(0) = u''(1) = 0, \end{aligned}$$

has the unique solution

$$u(x) = \int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)v(\tau)d\tau\Big)ds,$$

where φ_q stands for the inverse function $\varphi_q(s) = \varphi_p^{-1}(s)$ with conjugates p,q, i.e., $1/p+1/q=1$ and

$$G(x,s) = \begin{cases} x(1-s), & 0 \leqslant x \leqslant s \leqslant 1, \\ s(1-x), & 0 \leqslant s \leqslant x \leqslant 1, \end{cases} \tag{1.2}$$

is the Green's function of the second-order linear problem

$$\begin{aligned} -u''(x) &= 0, \quad 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned}$$

REMARK 1.1. It is clear that the Green's function G satisfies:

(i) $G(x,s) \leqslant G(s,s)$, for $0 \leqslant x,s \leqslant 1$

(ii) $G(x,s) \leqslant 1/4$, for $0 \leqslant x,s \leqslant 1$.

The following Lemma is easily proved by induction.

LEMMA 1.2. *Let $a_1,a_2,...,a_n$ be real numbers. Then*

$$|a_1+a_2...+a_n|^r \leqslant C_r(|a_1|^r+|a_2|^r+...+|a_n|^r), \quad \forall r > 0,$$

where

$$C_r = \begin{cases} 1, & 0 < r \leqslant 1, \\ n^{r-1}, & r > 1. \end{cases}$$

By Lemma 1.1, we deduce that (u,v) is a solution of Problem (1.1) if and only if

$$\begin{aligned} u(x) &= \int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_1\big(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau)\big)d\tau\Big)ds \\ &:= T_1(u,v)(x) \end{aligned}$$

and

$$v(x) = \int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_2\big(\tau,u(\tau),v(\tau),u'(\tau),v'(\tau)\big)d\tau\Big)ds$$

$$:= T_2(u,v)(x).$$

Hence, $(u,v) \in E_1$ is a solution of Problem (1.1) if and only if it is a fixed point of the operator

$$T = (T_1, T_2) : E_1 \longrightarrow E_1, \quad (u,v) \mapsto (T_1(u,v), T_2(u,v)).$$

The following result, known as the Leray-Schauder nonlinear alternative, will be needed in this paper.

THEOREM 1.1. (See [7, 16]) *Let X be a Banach space, $\Omega \subset X$ be a bounded and open set, $0 \in \Omega$, and $A : \overline{\Omega} \to X$ be a completely continuous operator. Then, either there exist $u \in \partial\Omega$ and $\lambda > 1$ such that $Au = \lambda u$ or A has a fixed point in $\overline{\Omega}$.*

2. The regular problem

In this section, we assume that the following conditions hold:

$(\mathscr{H}_1)$ there exist $a_i, b_i, c_i \in L^1([0,1], \mathbb{R}^+)$, such that for all $(x,u,v,y,z) \in [0,1] \times \mathbb{R}^4$ and for each $i = 1,2$,

$$|f_i(x,u,v,y,z)| \leqslant a_i(x) + b_i(x)\big(\varphi_p(|u|) + \varphi_p(|v|)\big) + c_i(x)\big(\varphi_p(|y|) + \varphi_p(|z|)\big),$$

$(\mathscr{H}_2)$

$$\varphi_q(2B) + \varphi_q(2C) < 1/4C_q,$$

where $B = \max\{|b_1|_1, |b_2|_1\}$, $C = \max\{|c_1|_1, |c_2|_1\}$, and

$$C_q = \begin{cases} 1, & 1 < q \leqslant 2, \\ 3^{q-2}, & q > 2. \end{cases}$$

The main result in this section is:

THEOREM 2.1. *Suppose that $(\mathscr{H}_1)$-$(\mathscr{H}_2)$ hold. Then Problem (1.1) has at least one solution $(u,v) \in E_1$.*

First, we prove

LEMMA 2.1. *Under the condition that $f_i\,(i = 1,2)$ are Carathéodory functions, for any bounded subset Ω of $C^1([0,1], \mathbb{R})$, the mapping $T : \overline{\Omega} \times \overline{\Omega} \to E_1$ is compact.*

Proof. Let $\Omega := \{u \in C^1([0,1], \mathbb{R}) : \|u\|_1 < R\}$ where R is some positive real constant. Since G and φ_q are continuous and f_1, f_2 are Carathéodory, the mappings T_1 and T_2 are continuous on $\overline{\Omega} \times \overline{\Omega}$.

Let $(u,v) \in \overline{\Omega} \times \overline{\Omega}$ with $\|(u,v)\|_1 = \|u\|_1 + \|v\|_1 \leqslant 2R$. Since $G(x,s) \leqslant 1$ for $0 \leqslant x,s \leqslant 1$, the following estimates hold true for $i = 1,2$:

$$|T_i(u,v)(x)| \leqslant \int_0^1 G(x,s)ds\, \varphi_q\Big(\int_0^1 G(s,\tau)h_R(\tau)d\tau\Big)$$

$$\leqslant \varphi_q(|h_R|_1) < \infty.$$

Hence, $\|T_i(u,v)\|_0 < \infty$, for $i = 1,2$. Similarly, we can check that $\|(T_i(u,v))'\|_0 < \infty$. So, $T_i(\overline{\Omega} \times \overline{\Omega})$ is uniformly bounded. Finally, if $x_1, x_2 \in (0,1)$, then

$$\begin{aligned}
&\big|(T_i(u,v))(x_1) - (T_i(u,v))(x_2)\big| \\
&\quad = \left| \int_0^1 \big(G(x_1,s) - G(x_2,s)\big)\varphi_q\Big(\int_0^1 G(s,\tau) f_i\big(\tau, u(\tau), u'(\tau) v(\tau), v'(\tau)\big)\,d\tau\Big)\,ds \right| \\
&\quad \leqslant \varphi_q(|h_R|_1) \int_0^1 \big|G(x_1,s) - G(x_2,s)\big|\,ds
\end{aligned}$$

and the right-hand side tends to 0 as $|x_1 - x_2| \to 0$ since G is continuous. Also,

$$|(T_i(u,v))'(x_1) - (T_i(u,v))'(x_2)| \leqslant \varphi_q(|h_R|_1) \int_0^1 \left| \frac{\partial G}{\partial x}(x_1,s) - \frac{\partial G}{\partial x}(x_2,s) \right| ds$$

tends to 0 as $|x_1 - x_2| \to 0$ because $\partial G/\partial x$ is continuous. Consequently, $T_i(\overline{\Omega} \times \overline{\Omega})$ are equicontinuous for $i = 1,2$. The Arzelà-Ascoli theorem then implies that T_1 and T_2 are completely continuous. Hence, T is completely continuous. □

PROOF OF THEOREM 2.1 Let $A = \max(|a_1|_1, |a_2|_1)$. Using $(\mathscr{H}_2)$, let $M > 0$ be such that

$$M > \frac{\varphi_q(A)}{1/4C_q - \varphi_q(2B) - \varphi_q(2C)}$$

and consider the open ball

$$\Omega := \{u \in C^1([0,1], \mathbb{R}) : \|u\|_1 < M\}.$$

From Lemma 2.1, the mapping $T : \overline{\Omega} \times \overline{\Omega} \to E_1$ is completely continuous. We claim that $(u,v) \neq \lambda T(u,v)$, for any $(u,v) \in \partial(\Omega \times \Omega)$ and $\lambda \in (0,1)$. Indeed, let $(u,v) \in \partial(\Omega \times \Omega)$. By the relation

$$\partial(A \times B) = (\partial A \times \overline{B}) \bigcup (\overline{A} \times \partial B),$$

we have that

$$\begin{aligned}
(u,v) \in \partial(\Omega \times \Omega) \Leftrightarrow &\text{ either } (\|u\|_1 = M \text{ and } 0 \leqslant \|v\|_1 \leqslant M) \\
&\text{ or } (0 \leqslant \|u\|_1 \leqslant M \text{ and } \|v\|_1 = M).
\end{aligned}$$

Hence, $M \leqslant \|(u,v)\|_1 = \|u\|_1 + \|v\|_1 \leqslant 2M$. Moreover, by Assumption $(\mathscr{H}_1)$, we obtain, for each $i = 1,2$, that

$$\begin{aligned}
|T_i(u,v)(x)| \leqslant \int_0^1 G(x,s)\varphi_q\Bigg(\int_0^1 G(s,\tau) \times \Big(a_i(\tau) + b_i(\tau)\big(\varphi_p(|u(\tau)|) \\
+ \varphi_p(|v(\tau)|)\big) + c_i(\tau)\big(\varphi_p(|u'(\tau)|) + \varphi_p(|v'(\tau)|)\big)\Big)\,d\tau\Bigg)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \varphi_q\Big(|a_i|_1 + |b_i|_1\big(\varphi_p(\|u\|_0) + \varphi_p(\|v\|_0)\big) \\
&\qquad\qquad + |c_i|_1\big(\varphi_p(\|u'\|_0) + \varphi_p(\|v'\|_0)\big)\Big) \\
&\leqslant \varphi_q\Big(|a_i|_1 + |b_i|_1\big(\varphi_p(M) + \varphi_p(M)\big) + |c_i|_1\big(\varphi_p(M) + \varphi_p(M)\big)\Big) \\
&\leqslant \varphi_q\Big(|a_i|_1 + 2\varphi_p(M)\big(|b_i|_1 + |c_i|_1\big)\Big).
\end{aligned}
$$

Passing to the supremum over x and using Lemma 1.2, we infer that

$$
\begin{aligned}
\|T_i(u,v)\|_0 &\leqslant \varphi_q\Big(|a_i|_1 + 2\varphi_p(M)\big(|b_i|_1 + |c_i|_1\big)\Big) \\
&= \Big(|a_i|_1 + 2\varphi_p(M)(|b_i|_1 + |c_i|_1)\Big)^{q-1} \\
&\leqslant C_q\Big(\varphi_q(|a_i|_1) + M\varphi_q(2|b_i|_1) + M\varphi_q(2|c_i|_1)\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|T(u,v)\|_0 &= \|T_1(u,v)\|_0 + \|T_2(u,v)\|_0 \\
&\leqslant 2C_q(\varphi_q(A) + M\varphi_q(2B) + M\varphi_q(2C)).
\end{aligned}
$$

Similarly, we can show that

$$
\|(T(u,v))'\|_0 \leqslant 2C_q(\varphi_q(A) + M\varphi_q(2B) + M\varphi_q(2C)).
$$

Therefore,

$$
\begin{aligned}
\|T(u,v)\|_1 &= \|T(u,v)\|_0 + \|(T(u,v))'\|_0 \\
&\leqslant 4C_q(\varphi_q(A) + M\varphi_q(2B) + M\varphi_q(2C)) \\
&< M.
\end{aligned}
$$

As a consequence,

$$
(u,v) \neq \lambda T(u,v), \quad \forall (u,v) \in \partial(\Omega\times\Omega) \text{ and } \forall \lambda \in (0,1).
$$

By Theorem 1.1, we deduce that the operator T has a fixed point (u,v) in $\overline{\Omega}\times\overline{\Omega}$, which is a solution of Problem (1.1). $\square$

2.1. Example

Let $f_i = f_i(x,u,v,u',v')\,(i=1,2)$ with

$$
f_1 = \frac{x}{2} + \frac{x}{8}\Big(\big(1+|u(x)|\big)^{\alpha_1} + \big(1+|v(x)|\big)^{\beta_1} + \big(1+|u'(x)|\big)^{\gamma_1} + (1+|v'(x)|)^{\delta_1}\Big),
$$

$$
f_2 = \frac{1-x}{4} + \frac{1-x}{16}\Big(\big(1+|u(x)|\big)^{\alpha_2} + \big(1+|v(x)|\big)^{\beta_2} + \big(1+|u'(x)|\big)^{\gamma_2} + \big(1+|v'(x)|\big)^{\delta_2}\Big)
$$

and where $0<\alpha_i,\beta_i,\gamma_i,\delta_i<1/2$ $(i=1,2)$. Clearly Assumption $(\mathscr{H}_1)$ in Theorem 2.1 is satisfied for:

$$\begin{aligned}
a_1(x) &= x/2+4x/8=x,\\
a_2(x) &= (1-x)/4+4(1-x)/16=(1-x)/2,\\
b_1(x) &= c_1(x)=x/8,\\
b_2(x) &= c_2(x)=(1-x)/16.
\end{aligned}$$

Hence, $A=1/2$ and $B=C=1/16$. Notice that

$$(1+|u|)^{\gamma}\leqslant(1+|u|)^{\frac{1}{2}}\leqslant 1+|u|^{1/2}\ \text{ for each } 0<\gamma<1.$$

Also, for $p=3/2$, we have $q=C_q=3$; hence condition $(\mathscr{H}_2)$ in Theorem 2.1 reads

$$(2B)^2+(2C)^2<\frac{1}{4.3}\Leftrightarrow 1/16<1/12.$$

Therefore, all assumptions in Theorem 2.1 are fulfilled which implies that the following problem has at least one solution:

$$\begin{aligned}
&\left(\operatorname{sgn}(u'')\sqrt{|u''|}\right)''=f_1(x,u,v,u',v'),\quad x\in(0,1),\\
&\left(\operatorname{sgn}(v'')\sqrt{|v''|}\right)''=f_2(x,u,v,u',v'),\quad x\in(0,1),\\
&u(0)=u(1)=u''(0)=u''(1)=0,\\
&v(0)=v(1)=v''(0)=v''(1)=0,
\end{aligned}$$

where $\operatorname{sgn}(s)\sqrt{|s|}=\varphi_{3/2}(s)$, $s\in\mathbb{R}$.

3. The case $f=f(x,u,v)$

In this section, we consider nonlinearities not depending on the first derivatives and prove an existence result for general growth conditions on f_i $(i=1,2)$. Consider the boundary value problem:

$$\begin{aligned}
&(\varphi_p(u''))''=f_1(x,u,v),\quad x\in(0,1),\\
&(\varphi_p(v''))''=f_2(x,u,v),\quad x\in(0,1),\\
&u(0)=u(1)=u''(0)=u''(1)=0,\\
&v(0)=v(1)=v''(0)=v''(1)=0,
\end{aligned}\tag{3.1}$$

where, for each $i=1,2$, $f_i:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are Carathéodory functions. By Lemma 1.1, (u,v) is a solution of Problem (3.1) if and only if

$$u(x)=\int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_1\big(\tau,u(\tau),v(\tau)\big)d\tau\Big)ds:=T_1(u,v)(x)$$

and

$$v(x)=\int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_2\big(\tau,u(\tau),v(\tau)\big)d\tau\Big)ds:=T_2(u,v)(x).$$

Hence, $(u,v)\in E$ is a solution of Problem (3.1) if and only if it is a fixed point of the operator

$$T=(T_1,T_2):E\longrightarrow E,\quad (u,v)\mapsto (T_1(u,v),T_2(u,v)).$$

Our second existence result is the following

THEOREM 3.1. *Suppose that:*

$(\mathscr{H}_3)$ *there exist functions* $h_1,h_2\in L^1([0,1],\mathbb{R}^+)$ *and nondecreasing positive functions on* $[0,+\infty)$, ψ_1,ψ_2,β_1, *and* β_2 *such that*

$$|f_1(x,u,v)|\leqslant h_1(x)\psi_1(|u|)\beta_1(|v|),\ \textit{for } x\in[0,1] \textit{ and } u,v\in\mathbb{R}$$

and

$$|f_2(x,u,v)|\leqslant h_2(x)\psi_2(|u|)\beta_2(|v|),\ \textit{for } x\in[0,1] \textit{ and } u,v\in\mathbb{R},$$

$(\mathscr{H}_4)$ *there exists* $M>0$ *such that*

$$4M>\varphi_q\Big(\frac{1}{4}|h_1|_1\psi_1(M)\beta_1(M)\Big)+\varphi_q\Big(\frac{1}{4}|h_2|_1\psi_2(M)\beta_2(M)\Big).$$

Then Problem (3.1) *has at least one solution* $(u,v)\in E$.

Proof. We will apply again Theorem 1.1 to obtain the existence of a solution for Problem (3.1). Consider the open ball

$$\Omega:=\{u\in C([0,1],\mathbb{R}):\|u\|_0<M\},$$

where $M>0$ is as defined in condition $(\mathscr{H}_4)$. We claim that $(u,v)\neq\lambda T(u,v)$, for any $(u,v)\in\partial(\Omega\times\Omega)$ and $\lambda\in(0,1)$. Indeed, let $(u,v)\in\partial(\Omega\times\Omega)$, that is $M\leqslant\|(u,v)\|_0=\|u\|_0+\|v\|_0\leqslant 2M$. By assumptions $(\mathscr{H}_3)$ and the fact that $G(x,s)\leqslant 1/4$ for $0\leqslant x,s\leqslant 1$, we obtain

$$\begin{aligned}
\lambda|T_1(u,v)(x)| &\leqslant \int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)h_1(\tau)\psi_1(|u(\tau)|)\beta_1(|v(\tau)|)d\tau\Big)ds\\
&\leqslant \int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)h_1(\tau)\psi_1(\|u\|_0)\beta_1(\|v\|_0)d\tau\Big)ds\\
&\leqslant \frac{1}{4}\int_0^1 \varphi_q\Big(\int_0^1 \frac{1}{4}|h_1|_1\psi_1(\|u\|_0)\beta_1(\|v\|_0)d\tau\Big)ds\\
&=\frac{1}{4}\varphi_q\Big(\frac{1}{4}|h_1|_1\psi_1(\|u\|_0)\beta_1(\|v\|_0)\Big).
\end{aligned}$$

Passing to the supremum, we get

$$\lambda\|T_1(u,v)\|_0\leqslant\frac{1}{4}\varphi_q\Big(\frac{1}{4}|h_1|_1\psi_1(\|u\|_0)\beta_1(\|v\|_0)\Big).$$

Similarly, we have

$$\lambda \|T_2(u,v)\|_0 \leqslant \frac{1}{4}\varphi_q\Big(\frac{1}{4}|h_2|_1\psi_2(\|u\|_0)\beta_2(\|v\|_0)\Big).$$

Therefore, by Assumption $(\mathscr{H}_4)$

$$\begin{aligned}\lambda \|T(u,v)\|_0 &\leqslant \frac{1}{4}\varphi_q\Big(\frac{1}{4}|h_1|_1\psi_1(\|u\|_0)\beta_1(\|v\|_0)\Big)+\frac{1}{4}\varphi_q\Big(\frac{1}{4}|h_2|_1\psi_2(\|u\|_0)\beta_2(\|v\|_0)\Big)\\ &< M.\end{aligned}$$

As a consequence

$$(u,v)\neq \lambda T(u,v), \quad \forall (u,v)\in\partial(\Omega\times\Omega),\ \forall\lambda\in(0,1).$$

From Lemma 2.1, T is completely continuous. By Theorem 1.1, we conclude that the operator T has a fixed point (u,v) in $\overline{\Omega}\times\overline{\Omega}$, which is a solution of Problem (3.1). □

3.1. Example

$$\begin{aligned}&\Big(u''\sqrt{|u''|}\Big)''=\Big(\frac{1}{24}\Big)^{3/2}h_1(x)e^{3v/2}(1+u)^{1/3}\sqrt{1+v^2},\ x\in(0,1),\\ &\Big(v''\sqrt{|v''|}\Big)''=\Big(\frac{1}{24}\Big)^{3/2}h_2(x)e^{3v/4}(1+u)^{1/5}\sqrt{1+v^2},\ x\in(0,1),\\ &u(0)=u(1)=u''(0)=u''(1)=0,\\ &v(0)=v(1)=v''(0)=v''(1)=0,\end{aligned}\tag{3.2}$$

where $h_1,h_2\in L^1([0,1],\mathbb{R}^+)$ satisfy $|h_1|_1=|h_2|_1=4$. With $p=5/2$ and since

$$|1+u|^{1/3}\leqslant 1+|u|^{1/3},\ |1+u|^{1/5}\leqslant 1+|u|^{1/5} \text{ and } \sqrt{1+v^2}\leqslant 1+|v|,$$

we may take

$$\begin{aligned}&\psi_1(u)=(1/24)^{3/2}(1+u^{1/3}),\ \psi_2(u)=(1/24)^{3/2}(1+u^{1/5}),\\ &\beta_1(v)=\beta_2(v)=e^{3v/2}\sqrt{1+v}.\end{aligned}$$

Then Assumption $(\mathscr{H}_4)$ in Theorem 3.1 becomes

$$\exists M>0,\ 48M>e^M(1+M)^{2/3}\Big[\big(1+M^{1/3}\big)^{2/3}+\big(1+M^{1/5}\big)^{2/3}\Big].\tag{3.3}$$

A sufficient for (3.3) be satisfied is

$$e^M(1+M)^{2/3}\Big(2+M^{2/9}+M^{2/5}\Big)\leqslant 48M$$

which is true whenever $M=1$. Therefore, all assumptions in Theorem 3.1 are met and Problem (3.2) has at least one solution.

4. The singular problem

In this final section, we assume that the nonlinearities are positive, are allowed to be singular at the solution but do not depend on the first derivative:

$$\begin{aligned} &(\varphi_p(u''))'' = f_1(x,u,v), \quad x \in (0,1), \\ &(\varphi_p(v''))'' = f_2(x,u,v), \quad x \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \\ &v(0) = v(1) = v''(0) = v''(1) = 0, \end{aligned} \tag{4.1}$$

where, for each $i = 1,2$, the function $f_i : [0,1] \times (0,+\infty)^2 \longrightarrow \mathbb{R}^+$ is Carathéodory and may be singular at the solution, i.e.

$$\lim_{u+v \to 0\ (u,v \geqslant 0)} f_i(x,u,v) = +\infty.$$

We will assume that the following conditions hold:

$(\mathscr{H}_5)$ there exist $a_i, b_i \in L^1([0,1],\mathbb{R}^+)$ satisfying $\int_0^1 B(s)\varphi_p(1/s)ds < \infty$, such that for all $(x,u,v) \in [0,1] \times (0,+\infty) \times (0,+\infty)$ and for each $i = 1,2$,

$$0 \leqslant f_i(x,u,v) \leqslant a_i(x) + b_i(x)\varphi_p\Big(\frac{1}{u+v}\Big);$$

$(\mathscr{H}_6)$ there exists $M > 0$ such that

$$A + \varphi_p\Big(\frac{1}{M+1}\Big)\int_0^1 B(\tau)\varphi_p\big(1/p(\tau)\big)d\tau < \varphi_p\Big(\frac{M+1}{2}\Big).$$

Here

$$p(s) = \min_{s\in[0,1]}(s,1-s),\ B(s) = \max_{s\in[0,1]}\big(b_1(s),b_2(s)\big) \quad \text{and} \quad A = \max\big(|a_1|_1,|a_2|_1\big).$$

THEOREM 4.1. *Suppose that $(\mathscr{H}_5)$ and $(\mathscr{H}_6)$ hold. Then Problem (4.1) has at least one solution $(u,v) \in E$.*

To prove this theorem, we need the following lemma

LEMMA 4.1. *Let $w \in L^1([0,1])$, $w \geqslant 0$ a.e. and let u satisfy*

$$\begin{aligned} &(\varphi_p(u''))''(x) = w(x), \quad 0 < x < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{4.2}$$

Then

$$u(x) \geqslant p(x)\|u\|_0, \quad \forall x \in [0,1],$$

where

$$p(x) = \min(x,1-x),\ \text{for } x \in [0,1]. \tag{4.3}$$

Proof. Let $-\varphi_p(u'') = v$, then $(\varphi_p(u''))'' = -v''$ with $v(0) = v(1) = 0$. So, Problem (4.2) is equivalent to the linear problem

$$\begin{aligned} -v''(x) &= w(x), \quad 0 < x < 1, \\ v(0) &= v(1) = 0. \end{aligned} \tag{4.4}$$

Equivalently

$$v(x) = \int_0^1 G(x,s)w(s)ds,$$

where G is the function defined by (1.2). Since $v > 0$ on $(0,1)$, $\varphi_p(u'') = -v \leqslant 0$. Then $u'' \leqslant 0$, since φ_p is nondecreasing; thus u is concave on $(0,1)$, with $u(0) = u(1) = 0$. In addition, there exists some $0 < x_0 < 1$ such that $u'(x_0) = 0$. By Lemma 1.1, we have $u(x) > 0$, on $(0,1)$. Then, u is positive, concave and admits a unique maximum at x_0. Its graph is then above the lines joining $u(x_0)$ to the endpoints. It follows that:

$$u(x) \geqslant x\frac{u(x_0)}{x_0} \geqslant xu(x_0) = x\|u\|_0, \quad \forall x \in [0,x_0],$$

and

$$u(x) \geqslant (1-x)\frac{u(x_0)}{1-x_0} \geqslant (1-x)u(x_0) = (1-x)\|u\|_0, \quad \forall x \in [x_0,1].$$

The lemma is proved. $\square$

PROOF OF THEOREM 4.1 For $n \in \{1,2,\ldots\}$ and $(x,u,v) \in [0,1] \times (0,+\infty)^2$, define the operator $T^n = (T_1^n, T_2^n)$ $(i = 1,2)$ by

$$T_i^n(u,v)(x) := \int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_i\big(\tau,u_n(\tau),v_n(\tau)\big)d\tau\Big)ds,$$

where $u_n(.) = u(.) + 1/n$ and $v_n(.) = v(.) + 1/n$. Then $(u_n,v_n) \in E$ is a solution of the problem

$$\begin{aligned} &(\varphi_p(u_n''))'' = f_1(x,u_n,v_n), \quad x \in (0,1), \\ &(\varphi_p(v_n''))'' = f_2(x,u_n,v_n), \quad x \in (0,1), \\ &u_n(0) = u_n(1) = u_n''(0) = u_n''(1) = 0, \\ &v_n(0) = v_n(1) = v_n''(0) = v_n''(1) = 0 \end{aligned} \tag{4.5}$$

if and only if (u_n,v_n) is a fixed point of the operator T^n. Now consider the open ball

$$\Omega := \{u \in C([0,1],\mathbb{R}) : \|u\|_0 < M+1\},$$

where M is as defined in $(\mathscr{H}_6)$. Since the functions

$$f_i^n(x,u,v) = f_i(x,u+1/n,v+1/n)$$

are Carathéodory on $[0,1] \times (0,+\infty)^2$, Lemma 2.1 guarantees that the operator $T^n : \overline{\Omega} \times \overline{\Omega} \to E$ is completely continuous. The remainder of the proof is split into three steps:

Step 1. We claim that $(u,v) \neq \lambda T^n(u,v)$, for any $(u,v) \in \partial(\Omega \times \Omega)$ and $\lambda \in (0,1)$. Arguing by contradiction, let $(u,v) \in \partial(\Omega \times \Omega)$ be such that $(u,v) = \lambda T^n(u,v)$; that is,

$$M+1 \leqslant \|u\|_0 + \|v\|_0 = \|u+v\|_0 \leqslant 2+2M.$$

On one hand, we have by Lemma 4.1 and the positivity of f_i, that $u(s) \geqslant p(s)\|u\|_0$. So $u(s)+1/n > u(s) \geqslant p(s)\|u\|_0$. Also, $v(s)+1/n > v(s) \geqslant p(s)\|v\|_0$. Then,

$$p(s)(\|u\|_0 + \|v\|_0) \leqslant u(s)+v(s) \leqslant \|u\|_0 + \|v\|_0.$$

Moreover, for $s \in [0,1]$, we have

$$\frac{1}{\|u\|_0+\|v\|_0+2/n} \leqslant \frac{1}{u(s)+v(s)+2/n} \leqslant \frac{1}{p(s)(\|u\|_0+\|v\|_0)} \leqslant \frac{1}{(M+1)p(s)}.$$

On the other hand, by Assumption $(\mathscr{H}_5)$ and since $G(x,s) \leqslant 1$ for $0 \leqslant x,s \leqslant 1$, we have for each $i = 1,2$:

$$\begin{aligned}
|T_i^n(u,v)(x)| &\leqslant \varphi_q\left(\int_0^1 a_i(\tau)+b_i(\tau)\varphi_p\left(\frac{1}{u(\tau)+v(\tau)+2/n}\right)d\tau\right)\\
&\leqslant \varphi_q\left(\int_0^1 a_i(\tau)+b_i(\tau)\varphi_p\left(\frac{1}{(M+1)p(\tau)}\right)d\tau\right)\\
&\leqslant \varphi_q\left(|a_i|_1+\varphi_p\left(\frac{1}{M+1}\right)\int_0^1 b_i(\tau)\varphi_p\left(\frac{1}{p(\tau)}\right)d\tau\right).
\end{aligned}$$

Passing to the supremum over x, we find that

$$\|T_i^n(u,v)\|_0 \leqslant \varphi_q\left(|a_i|_1+\varphi_p\left(\frac{1}{M+1}\right)\int_0^1 b_i(\tau)\varphi_p\left(\frac{1}{p(\tau)}\right)d\tau\right).$$

Consequently,

$$\begin{aligned}
\|T^n(u,v)\|_0 &= \|T_1^n(u,v)\|_0 + \|T_2^n(u,v)\|_0\\
&\leqslant 2\varphi_q\left(A+\varphi_p\left(\frac{1}{M+1}\right)\int_0^1 B(\tau)\varphi_p\left(\frac{1}{p(\tau)}\right)d\tau\right).
\end{aligned}$$

Since $\varphi_q = \varphi_p^{-1}$, assumption $(\mathscr{H}_6)$ implies that

$$\|T^n(u,v)\|_0 < M+1, \quad \forall (u,v) \in \partial(\Omega \times \Omega)$$

while $\|(u,v)\|_0 = \|u\|_0 + \|v\|_0 \geqslant M+1$. Therefore,

$$(u,v) \neq \lambda T^n(u,v), \quad \forall (u,v) \in \partial(\Omega \times \Omega),\ \forall \lambda \in (0,1).$$

Theorem 1.1 yields that the operator T^n has a fixed point denoted by (u_n, v_n) in $\overline{\Omega} \times \overline{\Omega}$, which is a solution of Problem (4.5).

Step 2. Equicontinuity. Consider the sequences $\{(u_n(.)\}$ and $\{v_n(.)\}$. Since $\|u_n\|_0 \leqslant M+1$ and $\|v_n\|_0 \leqslant M+1$, it follows that they are uniformly bounded on $[0,1]$. We show that they are equicontinuous on $[0,1]$. First, for $(u_n,v_n) \in \overline{\Omega} \times \overline{\Omega}$, we have for each $s \in [0,1]$,

$$\frac{1}{\|u_n\|_0+\|v_n\|_0+2/n} \leqslant \frac{1}{u_n+v_n+2/n} \leqslant \frac{1}{p(s)(\|u_n\|_0+\|v_n\|_0)} \leqslant \frac{1}{(M+1)p(s)}.$$

Let $x_1,x_2 \in (0,1)$; then we have

$$\begin{aligned}
|u_n(x_1)-u_n(x_2)| &= \int_0^1 |G(x_1,s)-G(x_2,s)| \\
&\qquad \times \varphi_q\left(\int_0^1 G(s,\tau) f_1\left(\tau, u_n(\tau)+\frac{1}{n}, v_n(\tau)+\frac{1}{n}\right) d\tau\right) ds \\
&\leqslant \int_0^1 |G(x_1,s)-G(x_2,s)| \\
&\qquad \times \varphi_q\left(\int_0^1 a_i(\tau)+b_i(\tau)\varphi_p\left(\frac{1}{u_n(\tau)+v(\tau)+2/n}\right) d\tau\right) ds \\
&\leqslant \int_0^1 |G(x_1,s)-G(x_2,s)| \\
&\qquad \times \varphi_q\left(A+\int_0^1 B(\tau)\varphi_p\left(\frac{1}{(M+1)p(\tau)}\right) d\tau\right) ds.
\end{aligned}$$

By continuity of G, the right-hand side tends to 0, as $|x_1-x_2| \to 0$. Hence $\{u_n(x)\}$ and $\{v_n(x)\}$ are equicontinuous on $[0,1]$.

Step 3. A sequential argument. The Arzelà-Ascoli theorem together with Step 2 imply that $\{u_n(.)\}$ and $\{v_n(.)\}$ are relatively compact on $C([0,1],\mathbb{R})$. Hence there exists a subsequence, still denoted $\{u_n(.)\}$, and a function $u_0(.)$ in $C([0,1],\mathbb{R})$, and there exists a subsequence still denoted $\{v_n(.)\}$ and a function $v_0(.)$ in $C([0,1],\mathbb{R})$ such that

$$\lim_{n\to+\infty} \sup_{x\in[0,1]} |u_n(x)-u_0(x)| = 0$$

and

$$\lim_{n\to+\infty} \sup_{x\in[0,1]} |v_n(x)-v_0(x)| = 0.$$

Now

$$u_n(x) = \int_0^1 G(x,s)\varphi_q\left(\int_0^1 G(s,\tau) f_1\left(\tau, u(\tau)+\frac{1}{n}, v(\tau)+\frac{1}{n}\right) d\tau\right) ds$$

and

$$v_n(x) = \int_0^1 G(x,s)\varphi_q\left(\int_0^1 G(s,\tau) f_2\left(\tau, u(\tau)+\frac{1}{n}, v(\tau)+\frac{1}{n}\right) d\tau\right) ds,$$

where for each $i=1,2$:

$$0 \leqslant f_i\left(\tau, u(\tau)+\frac{1}{n}, v(\tau)+\frac{1}{n}\right) \leqslant a_i(\tau)+b_i(\tau)\varphi_p\left(\frac{1}{u(\tau)+v(\tau)+2/n}\right)$$

$$\leqslant a_i(\tau)+b_i(\tau)\varphi_p\Big(\frac{1}{p(\tau)(M+1)}\Big).$$

Letting $n\to+\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$u_0(x)=\int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_1\big(\tau,u_0(\tau),v_0(\tau)\big)d\tau\Big)ds$$

and

$$v_0(x)=\int_0^1 G(x,s)\varphi_q\Big(\int_0^1 G(s,\tau)f_2\big(\tau,u_0(\tau),v_0(\tau)\big)d\tau\Big)ds.$$

Differentiating, we get

$$\big(\varphi_p(u_0''(x))\big)''=f_1(x,u_0(x),v_0(x)),\quad 0<x<1,$$

and

$$\big(\varphi_p(v_0''(x))\big)''=f_2(x,u_0(x),v_0(x)),\quad 0<x<1,$$

and $u_0(0)=u_0(1)=u_0''(0)=u_0''(1)=0$, $v_0(0)=v_0(1)=v_0''(0)=v_0''(1)=0$. Therefore, $(u_0(x),v_0(x))$ is a couple of solutions to Problem (4.1), ending the proof of the theorem. □

4.1. Example

Consider the $5/2$-Laplacian fourth-order boundary value problem:

$$\begin{aligned}
&\Big(u''\sqrt{|u''|}\Big)''=a_1(x)+\frac{b_1(x)}{\sqrt{u+v}},\quad x\in(0,1),\\
&\Big(v''\sqrt{|v''|}\Big)''=a_2(x)+b_2(x)\Big(\frac{1}{u+\sqrt{u+v}}+\frac{1}{v+\sqrt{u+v}}\Big),\quad x\in(0,1),\\
&u(0)=u(1)=u''(0)=u''(1)=0,\\
&v(0)=v(1)=v''(0)=v''(1)=0,
\end{aligned}\tag{4.6}$$

where $a_1,a_2\in L^1([0,1],(0,+\infty))$ satisfy $|a_1|_1=|a_2|_1=1$ and $b_1(x)=2b_2(x)=p(x)^{3/2}$. Since $\int_0^1 B(s)\varphi_p(1/p(s))ds=1<\infty$, and for all $u,v\in(0,+\infty)$,

$$\frac{1}{u+\sqrt{u}}<\frac{1}{\sqrt{u}},\ \frac{1}{u+\sqrt{u+v}}<\frac{1}{\sqrt{u+v}}\ \text{and}\ \frac{1}{v+\sqrt{u+v}}<\frac{1}{\sqrt{u+v}},$$

assumption $(\mathscr{H}_5)$ in Theorem 4.1 is fulfilled. Also, assumption $(\mathscr{H}_6)$ in Theorem 4.1 reads

$$\exists M>0,\quad 1+\frac{1}{(M+1)^{3/2}}<\Big(\frac{M+1}{2}\Big)^{3/2},$$

which is satisfied for $M=3$. Therefore, all assumptions in Theorem 4.1 are met, which implies that Problem (4.6) has at least one solution $(u,v)\in C([0,1],\mathbb{R}^2)$.

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