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RIEMANN PROBLEM FOR ONE–DIMENSIONAL SYSTEM OF CONSERVATION LAWS OF MASS, MOMENTUM AND ENERGY IN ZERO–PRESSURE GAS DYNAMICS

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Abstract. This paper studies the one-dimensional Riemann problem for the system of conservation laws of mass, momentum and energy in zero-pressure gas dynamics. Using the characteristic analysis method, two kinds of solutions are obtained: vacuum and delta-shock solution. Under suitable generalized Rankine-Hugoniot relation and entropy condition, both existence and uniqueness of delta-shock solutions are established. These analytical results well match the results obtained through numerical simulations.

1. Introduction

Nonlinear hyperbolic conservation laws are a fundamental principle in building mathematical models for many natural process. As basic examples in fluid dynamics, Euler equations represent the conservation of mass, momentum and energy. However, the full Euler equations are so complicated that it is very difficult to do a complete investigation. Thus, to approach the full Euler equations, various mathematical simplification models are studied. Among them, a very important one is the zero-pressure gas dynamics, which can describe the motion of free particles which stick under collision and explain the formation of large-scale structures in the universe. The zero-pressure gas dynamics system consisting of conservation laws of mass and momentum has been investigated extensively and some excellent results have been obtained. For related results, see [2, 13, 19, 3, 6] and the papers cited therein. However, it is well known that for the media which can be considered as having no pressure, we must take into account energy transport. Therefore it is very necessary to consider the energy conservation law in zero-pressure gas dynamics.

These motivate us to consider the system of conservation laws of mass, momentum and energy in zero-pressure gas dynamics in one space dimension

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \\ \big(\rho u^2/2 + H\big)_t + \big((\rho u^2/2 + H)u\big)_x = 0, \end{cases} \tag{1.1}$$

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where ρ and u represent the density and velocity, respectively, $H = \rho\tau$ is the internal energy, τ is the internal energy per unit mass. The density ρ and internal energy H are nonnegative; the regions in the physical space where $\rho = 0$ and $H = 0$ are identified with vacuum regions of the flow. For convenience, we here consider H as an independent variable.

As we know, a flow is formed by two kinds of effects: the effect of inertia and the effect of pressure difference. When we neglect the effect of pressure difference in the Euler equation of nonisentropic gas dynamic

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho E)_t + \big((\rho E + p)u\big)_x = 0, \end{cases} \tag{1.2}$$

that is, the pressure p is a constant, system (1.2) is reduced to system (1.1), where $\rho E = \rho u^2/2 + \rho\tau$ is the total energy.

One main feature of system (1.1) is that ρ, H and $\partial u/\partial x$ blow up simultaneously in a finite time even starting from smooth initial data. Therefore, we have to understand (1.1) in the sense of measures and introduce delta shock waves in the piecewise smooth solutions of (1.1). In the delta-shock solutions, both the density and energy develop an extreme concentration. As for delta shock waves, there have been rich results. For instance, Bouchut [2], Li, Sheng and Zhang [13, 19], Zheng [23], Tan, Zhang and Zheng [20, 21], Keyfits and Kranzer [11], Hayes and Lefloch [10], Yang [22], Chen and Liu [3], Cheng and Yang [4, 5, 6], Danilov, Shelkovich and Panov [7, 8, 18, 17], Guo, Sheng and Zhang [9], etc.

Early, system (1.1) was studied in [12]. To construct a solution for arbitrary initial data, the discontinuities which are different from classical ones and carry mass, impulse and energy were needed. Recently, papers [16, 15] further considered system (1.1). In order to define delta-shock solutions, special integral identities were introduced. Using these integral identities, the Rankine-Hugoniot conditions for delta shock waves are obtained. The balance laws describing mass, momentum and energy transport from the area outside the delta shock wave front onto its front were derived. Next, in [1], the multidimensional case of system (1.1) was considered.

In this paper, we continue to study solutions of system (1.1) by considering the Riemann problem with initial data

$$(\rho, u, H)(t = 0, x) = \begin{cases} (\rho_-, u_-, H_-), & x < 0, \\ (\rho_+, u_+, H_+), & x > 0. \end{cases} \tag{1.3}$$

By using the characteristic analysis method, we solve the Riemann problem (1.1) and (1.3) constructively. There are only two kinds of solutions: the one involving vacuum and the other containing delta shock wave. Then we focus our attention on delta shock waves. By a definition of measure solution to (1.1), we derive the generalized Rankine-Hugoniot relation, which consists of a system of ordinary differential equations and describes the relationship among the location, propagation speed, weights and assignment of u on their discontinuity relative to delta shock wave. To guarantee uniqueness, the delta shock waves must satisfy the entropy condition, which is overcompressive one. Thus the existence and uniqueness of solutions involving delta shock waves can be obtained by solving the generalized Rankine-Hugoniot relation under entropy condition.

The rest of this paper is arranged as follows. In Section 2, we present some preliminary knowledge about system (1.1) and construct the Riemann solutions by characteristic analysis method. Section 3 proposes the generalized Rankine-Hugoniot relation and entropy condition, and applies them to solving the Riemann problem. Finally, in Section 4, we simulate the Riemann solutions containing vacuum and delta shock wave.

2. Preliminaries and solutions obtained with characteristic method

2.1. Preliminaries

We consider the Riemann problem (1.3) for system (1.1). The system has a triple eigenvalue

$$\lambda = u \tag{2.1}$$

and two right eigenvectors

$$r_1 = (1,0,0)^T, \;\; r_2 = (0,0,1)^T \tag{2.2}$$

satisfying

$$\nabla\lambda \cdot r_i \equiv 0, \;\; i = 1,2. \tag{2.3}$$

Thus (1.1) is extremely nonstrictly hyperbolic and λ is linearly degenerate. The linear degeneracy also excludes the possibility of rarefaction wave and shock wave solutions. The characteristic equations can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho u_x, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = -H u_x, \tag{2.4}$$

which mean that characteristic lines are straight and u keeps constants along each of them.

As usual, we should seek the self-similar solution

$$(\rho,u,H)(t,x) = (\rho,u,H)(\xi), \quad \xi = x/t, \tag{2.5}$$

for which system (1.1) becomes

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2)_\xi = 0, \\ -\xi\left(\rho u^2/2 + H\right)_\xi + \left((\rho u^2/2 + H)u\right)_\xi = 0 \end{cases} \tag{2.6}$$

and initial condition (1.3) changes to the boundary condition

$$(\rho,u,H)(\pm\infty) = (\rho_\pm, u_\pm, H_\pm). \tag{2.7}$$

This is a two-point boundary value problem of first-order ordinary differential equations with the boundary values in the infinity.

For smooth solution, (2.6) is reduced to

$$\begin{pmatrix} u-\xi & \rho & 0 \\ 0 & u-\xi & 0 \\ 0 & H & u-\xi \end{pmatrix} \begin{pmatrix} \rho \\ u \\ H \end{pmatrix}_\xi = 0. \tag{2.8}$$

It provides either general solution (constant state)

$$(\rho,u,H)(\xi) = constant \qquad (\rho \neq 0, H \neq 0) \tag{2.9}$$

or singular solution (vacuum state, denoted by Vac)

$$\xi = u, \quad \rho = 0, \quad H = 0, \tag{2.10}$$

where $u(\xi)$ is arbitrary smooth functions. Thus the smooth solutions of system (1.1) only contain constants and vacuum solutions.

For a bounded discontinuity at $\xi = \omega$, the Rankine-Hugoniot relation can be written as

$$\begin{cases} -\omega[\rho] + [\rho u] = 0, \\ -\omega[\rho u] + [\rho u^2] = 0, \\ -\omega\big[\rho u^2/2 + H\big] + \big[(\rho u^2/2 + H)u\big] = 0, \end{cases} \tag{2.11}$$

where and after $[G] = G_l - G_r$ denotes the jump of G across the discontinuity. Solving (2.11), we obtain that

$$\xi = \omega = u_l(=\lambda_l) = u_r(=\lambda_r). \tag{2.12}$$

It is a contact discontinuity, denoted by J, which is just the characteristic line for both sides in (t,x)-plane. Two states (ρ_l,u_l,H_l) and (ρ_r,u_r,H_r) can be connected by a contact discontinuity J if and only if $u_l = u_r$, that is, these two states are located on the plane $u = u_l = u_r$ in the (ρ,u,H)–phase space. The contact discontinuity J in (t,x)-plane is characterized by $x/t = u_l = u_r$.

2.2. Structures of Riemann solutions

With the constants, vacuum and contact discontinuity, we solve Riemann problem (1.1) and (1.3) by two different cases.

Case 1. $u_- \leqslant u_+$. For this case, this is no overlap of characteristic lines and no characteristic line passes though the region $\Omega = \{(t,x)\big|u_- \leqslant x/t \leqslant u_+\}$ in (t,x)-plane. Therefore the vacuum should appear there. Thus we can construct the solution which consists of two contact discontinuities and a vacuum state besides two constants (see Fig.1(left)). The solution can be expressed as

$$(\rho,u,H)(\xi) = \begin{cases} (\rho_-,u_-,H_-), & -\infty < \xi < u_-, \\ (0,u(\xi),0), & u_- \leqslant \xi \leqslant u_+, \\ (\rho_+,u_+,H_+), & u_+ < \xi < +\infty, \end{cases} \tag{2.13}$$

where $u(\xi)$ satisfies that $u(u_-) = u_-$ and $u(u_+) = u_+$.

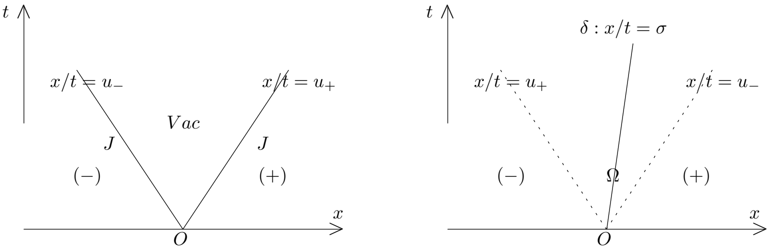

Fig.1. $u_- \leqslant u_+$ (left) and $u_- > u_+$ (right)

Case 2. $u_- > u_+$. For this case, the characteristics line from the x-axis will overlap in the domain $\Omega = \{(t,x) \big| u_+ \leqslant x/t \leqslant u_-\}$ in (t,x)-plane. So the singularity of solutions must develop in Ω. By a method similar to that in [22], one can prove that ρ, H and $\partial u/\partial x$ blow up simultaneously in a finite time even starting from smooth initial data. Therefore no solution exists in the bounded variation space.

Thus, motivated by [13, 19, 22, 6], for the case $u_- > u_+$, using delta shock wave, we can construct the solution which consists of a delta shock wave besides two constant states (see Fig.1(right)), where σ is the propagation speed of delta shock wave.

In the next section, we will study in more detail the existence and uniqueness of solutions involving delta shock waves.

3. Generalized Rankine-Hugoniot relations of delta shock waves

Due to the simultaneous occurrence of two blowup mechanism of solutions, it is natural to seek solutions in the space of Borel measures. Denote by $BM(\mathbb{R}^1)$ the space of bounded Borel measures on $\mathbb{R}^1$, then the definition of a measure solution of (1.1) in $BM(\mathbb{R}^1)$ can be given as follows.

DEFINITION 3.1. A triple distribution (ρ, u, H) constitutes a *measure solution* of (1.1) if it satisfies

(a) $\rho \in L^\infty\big([0,\infty), BM(\mathbb{R}^1)\big) \cap C\big([0,\infty), H^{-s}(\mathbb{R}^1)\big)$,

(b) $u \in L^\infty\big([0,\infty), L^\infty(\mathbb{R}^1)\big) \cap C\big([0,\infty), H^{-s}(\mathbb{R}^1)\big)$,

(c) $H \in L^\infty\big([0,\infty), BM(\mathbb{R}^1)\big) \cap C\big([0,\infty), H^{-s}(\mathbb{R}^1)\big)$, $s > 0$,

(d) u is measurable with respect to ρ and H at almost for all $t \geqslant 0$,

and

$$\begin{cases} \int_0^\infty \int_{\mathbb{R}^1} (\phi_t + u\phi_x) d\rho dt = 0, \quad \int_0^\infty \int_{\mathbb{R}^1} u(\phi_t + u\phi_x) d\rho dt = 0, \\ \int_0^\infty \int_{\mathbb{R}^1} (u^2/2)(\phi_t + u\phi_x) d\rho dt + \int_0^\infty \int_{\mathbb{R}^1} (\phi_t + u\phi_x) dH dt = 0, \end{cases} \tag{3.1}$$

hold in the sense of measures for all $\phi \in C_0^\infty\big([0,\infty) \times \mathbb{R}^1\big)$.

DEFINITION 3.2. A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve L parameterized as $x = x(s), y = y(s) (c \leqslant s \leqslant d)$ is defined by

$$\langle w(s)\delta_L, \phi(x,y)\rangle = \int_c^d w(s)\phi\big(x(s),y(s)\big)\mathrm{d}s$$

for all $\phi \in C_0^\infty(\mathbb{R}^2)$.

DEFINITION 3.3. A triple distribution (ρ,u,H) is called a *delta shock wave* if it is represented in the form

$$(\rho,u,H)(t,x) = \begin{cases} (\rho_l,u_l,H_l)(t,x) & x < x(t), \\ \big(w(t)\delta(x-x(t)),u_\delta(t),h(t)\delta(x-x(t))\big), & x = x(t), \\ (\rho_r,u_r,H_r)(t,x), & x > x(t) \end{cases} \tag{3.2}$$

and satisfies Definition 3.1, where (ρ_l,u_l,H_l) and $(\rho_r,u_r,H_r)(t,x)$ are piecewise smooth bounded solutions of (1.1).

Setting $dx/dt = u_\delta(t)$ since the concentrations in ρ and H need to travel at the speed of delta shock wave (also [13, 22, 23], etc), then a delta shock wave must satisfy the relation

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\ \dfrac{dw(t)}{dt} = -[\rho]u_\delta(t) + [\rho u], \\ \dfrac{dw(t)u_\delta(t)}{dt} = -[\rho u]u_\delta(t) + [\rho u^2], \\ \dfrac{d\big(w(t)u_\delta^2(t)/2 + h(t)\big)}{dt} = -\big[\rho u^2/2 + H\big]u_\delta(t) + \Big[\big(\rho u^2/2 + H\big)u\Big]. \end{cases} \tag{3.3}$$

As a matter of fact, for any test function $\phi \in C_0^\infty\big([0,\infty)\times\mathbb{R}^1\big)$, from (3.1), using Green's formula and integrating by parts, we can calculate

$$\begin{aligned} 0 &= \int_0^\infty\int_{\mathbb{R}^1} \frac{u^2}{2}(\phi_t + u\phi_x)d\rho dt + \int_0^\infty\int_{\mathbb{R}^1}(\phi_t + u\phi_x)dHdt \\ &= \int_0^\infty\int_{-\infty}^{x(t)} \frac{\rho_l u_l^2}{2}(\phi_t + u_l\phi_x)dxdt + \int_0^\infty\int_{-\infty}^{x(t)} H_l(\phi_t + u_l\phi_x)dxdt \\ &\quad + \int_0^\infty\int_{x(t)}^\infty \frac{\rho_r u_r^2}{2}(\phi_t + u_r\phi_x)dxdt + \int_0^\infty\int_{x(t)}^\infty H_r\big(\phi_t + u_r\phi_x\big)dxdt \\ &\quad + \int_0^\infty \frac{w(t)u_\delta^2(t)}{2}\big(\phi(t,x(t)) + u_\delta(t)\phi_x(t,x(t))\big)dt \\ &\quad + \int_0^\infty h(t)\big(\phi_t(t,x(t)) + u_\delta(t)\phi_x(t,x(t))\big)dt \\ &= \int_0^\infty\int_{-\infty}^{x(t)} \Big(\big(\frac{\rho_l u_l^2}{2}\phi\big)_t + \big(\frac{\rho_l u_l^2}{2}u_l\phi\big)_x\Big)dxdt + \int_0^\infty\int_{-\infty}^{x(t)}\big((H_l\phi)_t + (H_l u_l\phi)_x\big)dxdt \end{aligned}$$

$$
\begin{aligned}
&-\int_0^{\infty}\int_{-\infty}^{x(t)}\Big(\big(\frac{\rho_l u_l^2}{2}\big)_t+\big(\frac{\rho_l u_l^2}{2}u_l\big)_x\Big)\phi dxdt\\
&-\int_0^{\infty}\int_{-\infty}^{x(t)}\big((H_l)_t+(H_l u_l)_t\big)\phi dxdt\\
&+\int_0^{\infty}\int_{x(t)}^{\infty}\Big(\big(\frac{\rho_r u_r^2}{2}\phi\big)_t+\big(\frac{\rho_r u_r^2}{2}u_l\phi\big)_x\Big)dxdt\\
&+\int_0^{\infty}\int_{x(t)}^{\infty}\big((H_r\phi)_t+(H_r u_r\phi)_x\big)dxdt\\
&-\int_0^{\infty}\int_{x(t)}^{\infty}\Big(\big(\frac{\rho_r u_r^2}{2}\big)_t+\big(\frac{\rho_r u_r^2}{2}u_r\big)_x\Big)\phi dxdt-\int_0^{\infty}\int_{x(t)}^{\infty}\big((H_r)_t+(H_r u_r)_t\big)\phi dxdt\\
&+\int_0^{\infty}\frac{w(t)u_\delta^2(t)}{2}\frac{d\phi(t,x(t))}{dt}dt+\int_0^{\infty}h(t)\frac{d\phi(t,x(t))}{dt}dt\\
=&\int_0^{\infty}\Big\{-\big[\rho u^2/2+H\big]u_\delta(t)+\Big[\big(\rho u^2/2+H\big)u\Big]\\
&\qquad\qquad-\frac{d\big(w(t)u_\delta^2(t)/2+h(t)\big)}{dt}\Big\}\phi(t,x(t))dt,
\end{aligned}
$$

which implies the fourth identity in (3.3). In the same way as above, we can check that the second and third identities hold. Thus the proof is complete.

Relations (3.3) is called the generalized Rankine-Hugoniot relation. It reflects the exact relationship among the limit states on two sides, location, propagation speed, weights and the assignments of u on delta shock waves.

In addition, to guarantee uniqueness, the delta shock wave should satisfy

$$u_r < u_\delta(t) < u_l. \tag{3.4}$$

Condition (3.4) is called the entropy condition, which is an overcompressive one and means that all characteristics on both sides of delta shock wave are incoming.

Now we in particular apply the generalized Rankine-Hugoniot relation to solving Riemann problem (1.1) and (1.3) for the case $u_- > u_+$. At this moment, the Riemann problem is reduced to solving (3.3) with initial data

$$t=0:\ \ x(0)=0,\quad w(0)=0,\quad h(0)=0 \tag{3.5}$$

under entropy condition (3.4) which is

$$u_+ < u_\delta(t) < u_-. \tag{3.6}$$

From (3.3) and (3.5), it follows that

$$
\begin{cases}
w(t)=-[\rho]x(t)+[\rho u]t,\\
w(t)u_\delta(t)=-[\rho u]x(t)+[\rho u^2]t,\\
\frac{w(t)u_\delta^2(t)}{2}+h(t)=-[\frac{\rho u^2}{2}+H]x(t)+\big[(\frac{\rho u^2}{2}+H)u\big]t.
\end{cases}
\tag{3.7}
$$

Multiplying the first equation in (3.7) by $u_\delta(t)$ and then subtracting it from the second one, we obtain

$$[\rho]x(t)u_\delta(t)-[\rho u]u_\delta(t)t-[\rho u]x(t)+[\rho u^2]t=0, \tag{3.8}$$

that is,

$$\frac{d\big(\frac{[\rho]}{2}x^2(t)-[\rho u]x(t)t+\frac{[\rho u^2]}{2}t^2\big)}{dt}=0, \tag{3.9}$$

which provides

$$\frac{[\rho]}{2}x^2(t)-[\rho u]x(t)t+\frac{[\rho u^2]}{2}t^2=0. \tag{3.10}$$

From (3.10), one can find $u_\delta(t):=u_\delta$ is a constant and $x(t)=u_\delta t$. Then Eq.(3.10) can be rewritten into

$$[\rho]u_\delta^2-2[\rho u]u_\delta+[\rho u^2]=0, \tag{3.11}$$

which is just a quadratic equation with respect to u_δ.

When $[\rho]=\rho_- -\rho_+\neq 0$, by virtue of the discriminant

$$\Delta=4[\rho u]^2-4[\rho][\rho u^2]=4\rho_-\rho_+(u_- -u_+)^2\geqslant 0,$$

we find two solutions

$$u_\delta=\frac{\sqrt{\rho_-}u_- +\sqrt{\rho_+}u_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} \tag{3.12}$$

and

$$u_\delta=\frac{\sqrt{\rho_-}u_- -\sqrt{\rho_+}u_+}{\sqrt{\rho_-}-\sqrt{\rho_+}}. \tag{3.13}$$

Next, with the help of the entropy condition (3.6), we will choose the admissible solution from (3.12) and (3.13). For solution (3.12), we have

$$u_\delta-u_- =-\frac{\sqrt{\rho_+}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(u_- -u_+)<0 \tag{3.14}$$

and

$$u_\delta-u_+ =\frac{\sqrt{\rho_-}}{\sqrt{\rho_-}+\sqrt{\rho_+}}(u_- -u_+)>0, \tag{3.15}$$

which imply the entropy condition (3.6) is valid. For solution (3.13), when $[\rho]=\rho_- -\rho_+>0$,

$$u_\delta-u_- =\frac{\sqrt{\rho_+}}{\sqrt{\rho_-}-\sqrt{\rho_+}}(u_- -u_+)>0, \tag{3.16}$$

and when $[\rho]=\rho_- -\rho_+<0$,

$$u_\delta-u_+ =\frac{\sqrt{\rho_-}}{\sqrt{\rho_-}-\sqrt{\rho_+}}(u_- -u_+)<0. \tag{3.17}$$

These show that the solution (3.13) does not satisfy the entropy condition (3.6). Thus from (3.7), we can calculate

$$\begin{cases} x(t) = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}t, \\ w(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)t, \\ u_\delta = \dfrac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ h(t) = \frac{\rho_-\rho_+(u_- - u_+)^2 + 2(\sqrt{\rho_-}+\sqrt{\rho_+})(H_-\sqrt{\rho_+}+H_+\sqrt{\rho_-})}{2(\sqrt{\rho_-}+\sqrt{\rho_+})^2}(u_- - u_+)t. \end{cases} \tag{3.18}$$

When $[\rho] = \rho_- - \rho_+ = 0$, the situation is very simple. One can easily calculate the solution

$$\begin{cases} x(t) = \dfrac{u_- + u_+}{2}t, \\ w(t) = \rho_-(u_- - u_+)t \\ u_\delta = \frac{u_- + u_+}{2}, \\ h(t) = \frac{\rho_-(u_- - u_+)^2 + 4(H_- + H_+)}{8}(u_- - u_+)t, \end{cases} \tag{3.19}$$

which obviously satisfies (3.6).

THEOREM 3.1. *Let $u_- > u_+$. Then Riemann problem (1.1) and (1.3) admits one and only one entropy measure solution of the form*

$$(\rho, u, H)(t,x) = \begin{cases} (\rho_-, u_-, H_-), & x < u_\delta t, \\ \big(w(t)\delta(x - u_\delta t), u_\delta, h(t)\delta(x - u_\delta t)\big), & x = u_\delta t, \\ (\rho_+, u_+, H_+), & x > u_\delta t, \end{cases} \tag{3.20}$$

where $w(t)$, u_δ and $h(t)$ are shown by (3.18) when $[\rho] \neq 0$ or (3.19) when $[\rho] = 0$.

4. Numerical simulations for Riemann solutions

In this section, we simulate the obtained Riemann solutions by employing the Nessyahu-Tadmor scheme [14] with 300×300 cells and CFL = 0.475. For the case $u_- \leqslant u_+$, we take $\rho_- = 1.0, u_- = 0.2, H_- = 2.48$ and $\rho_+ = 2.0, u_+ = 2.0, H_+ = 4.0$. The numerical results are presented in Fig.2. For the case $u_- > u_+$, we take $\rho_- = 1.0, u_- = 0.2, H_- = 2.48$ and $\rho_+ = 2.0, u_+ = 0.1, H_+ = 7.99$. The numerical results are shown in Fig.3.

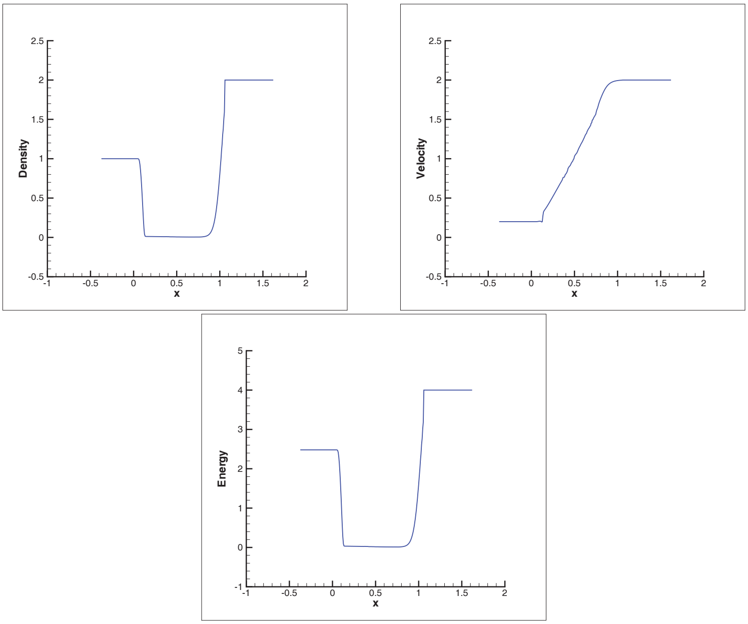

Fig.2. Numerical results for the case $u_- \leqslant u_+$ at $t = 0.5$

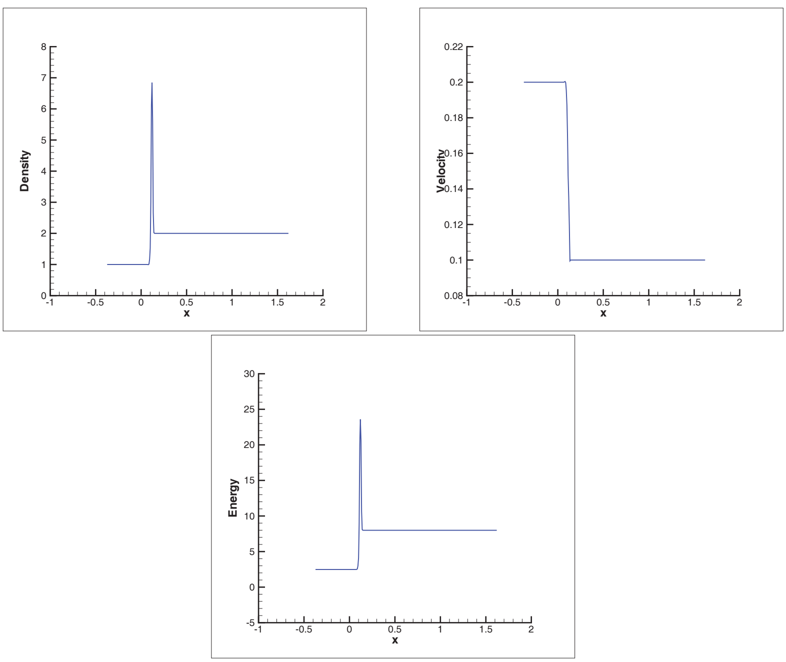

Fig.3. Numerical results for the case $u_- > u_+$ at $t = 0.8$

It can be clearly observed that the vacuum develops in Fig.2 while a delta shock wave in Fig.3. All the numerical results are in complete agreement with the theoretical analysis.

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REFERENCES

[1] S. ALBEVERIO, O. S. ROZANOVA, V. M. SHELKOVICH, *Transport and concentration processes in the multidimensional zero-pressure gas dynamics model with the energy conservation law*, preprint: arXiv:1101.5815v1 [math-ph] 30 Jan 2011.

[2] F. BOUCHUT, *On zero pressure gas dynamics*, Advances in kinetic theory and computing, Series on Advances in Mathematics for Applied Sciences, World Scientific, Singapore, **22** (1994), 171–190.

[3] G. Q. CHEN, H. L. LIU, *Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the isentropic Euler equations*, SIAM J. Math. Anal., **34** (2003), 925–938.

[4] H. J. CHENG, H. C. YANG, *Riemann problem for the relativistic Chaplygin Euler equations*, J. Math. Anal. Appl., **381** (2011), 17–26.

[5] H. J. CHENG, H. C. YANG, *Delta shock waves in chromatography equations*, J. Math. Anal. Appl., **380** (2011), 475–485.

[6] H. J. CHENG, H. C. YANG, *Delta shock waves as limits of vanishing viscosity for 2-D steady pressureless isentropic flow*, Acta Appl. Math, **113** (2011), 323–348.

[7] V. G. DANILOV, V. M. SHELKOVICH, *Dynamics of propagation and interaction of δ-shock waves in conservation laws systems*, J. Differ. Equations, **221** (2005), 333–381.

[8] V. G. DANILOV, V. M. SHELKOVICH, *Delta-shock wave type solution of hyperbolic systems of conservation laws*, Q. Appl. Math., **63**, 3 (2005), 401–427.

[9] L. H. GUO, W. C. SHENG, T. ZHANG, *The two-dimensional Riemann problem for isentropic Chaplygin gas dynamic system*, Commun. Pure Appl. Anal., **9** (2010), 431–458.

[10] B. T. HAYES, P. G. LEFLOCH, *Measure solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity, **9** (1996), 1547–1563.

[11] B. L. KEYFITZ, H. C. KRANZER, *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Differ. Equations, **118** (1995), 420–451.

[12] A. KRAIKO, *Discontinuity surfaces in medium without self-pressure*, Prikladnaia Matematika i Mekhanika, **43** (1979), 539–549. (In Russian)

[13] J. Q. LI, T. ZHANG, *Generalized Rankine-Hugoniot relations of delta-shocks in solutions of transportation equations*, Advance in Nonlinear PDE and Related Areas, World Scientific, Singapore, 1998, 219–232.

[14] H. NESSYAHU, E. TADMOR, *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comput. Phys., **87** (1990), 408–463.

[15] B. NILSSON, O. S. ROZANOVA, V. M. SHELKOVICH, *Mass, momentum and energy conservation laws in zero-pressure gas dynamics and delta-shocks: II*, Appl. Anal., **90** (2011), 831–842.

[16] B. NILSSON, V. M. SHELKOVICH, *Mass, momentum and energy conservation laws in zero-pressure gas dynamics and delta-shocks*, Appl. Anal., **89** (2010), 1–13.

[17] E. Y. PANOV, V. M. SHELKOVICH, *δ'-shock waves as a new type of solutions to systems of conservation laws*, J. Differ. Equations, **228** (2006), 49–86.

[18] V. M. SHELKOVICH, *The Riemann problem admitting δ, δ'-shocks, and vacuum states (the vanishing viscosity approach)*, J. Differ. Equations, **231** (2006), 459–500.

[19] W. C. SHENG, T. ZHANG *The Riemann problem for transportation equation in gas dynamics*, Mem. Amer. Math. Soc., **137** (654), (1999), 1–77.

[20] D. C. TAN, T. ZHANG, *Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws(I) Four-J cases*, J. Differ. Equations, **111** (1994), 203–254.

[21] D. C. TAN, T. ZHANG, Y. X. ZHENG, *Delta shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, J. Differ. Equations, **112** (1994), 1–32.
[22] H. C. YANG, *Riemann problems for a class of coupled hyperbolic systems of conservation laws*, J. Differ. Equations, **159** (1999), 447–484.
[23] Y. X. ZHENG, *Systems of conservation laws with incomplete sets of eigenvectors everywhere*, Advances in nonlinear partial differential equations and related areas (Beijing, 1997): A Volume Dedicated to Professor Xiaqi Ding's 70th Birthday, World Scientific, River Edge, NJ, 1998, 399–426.

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