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GLOBAL CURVE OF POSITIVE SOLUTIONS FOR $\varphi-$ LAPLACIAN DIRICHLET BVP WITH AT MOST ONE TURNING POINT

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Abstract. Under suitable conditions we prove that the set of positive solutions to the $\varphi-$Laplacian boundary value problem

$$-(\varphi(u'))' = \lambda f(u) \text{ in } (0,1);\ u(0) = u(1) = 0,$$

where $\lambda > 0$ is a real parameter, φ is an odd increasing homeomorphism of $\mathbb{R}$ and $f \in C([0,+\infty),[0,+\infty))$, consists on a curve $\|u\| \to \lambda(\|u\|)$.

1. Introduction

The study of existence of positive solutions to classes of semilinear boundary value problems (bvp for short), known as positone problems, has been undertaken by several authors over the last forty years (see for example [6], [13], [15], [27], [29], [32], and references therein). Such a study was initiated by Keller and Cohen [25].

Positive solutions for $\varphi-$Laplacian equations with Dirichlet boundary conditions were studied by Benmezai [7], Benmezai *et al.* [9], [10], de-Coster [12], Dang *et al.* [14], Garcia-Huidobro *et al.* [17], Huang [22], Kaper *et al.* [24], Manásevich *et al.* [30], Rynne [34] and Ubilla [35].

We investigate in this paper existence and exact number of positive solutions to the second order bvp

$$-\left(\varphi(u'(x))\right)' = \lambda f(u(x)),\ \ x \in (0,1), \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

$\lambda > 0$ is a real parameter, φ is an odd increasing homeomorphism of $\mathbb{R}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous where $\mathbb{R}^+ = [0,+\infty)$. In all this paper we assume that

$$f(u) > 0 \text{ for all } u > 0.$$

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By *a positive solution* to problem (1.1)-(1.2), we mean a pair $(\lambda, u) \in (0, +\infty) \times C^1([0,1])$ such that $u \geqslant 0$ in $(0,1)$, $u(x_0) > 0$ for some $x_0 \in (0,1)$, and (λ, u) satisfies (1.1)-(1.2).

Because of the autonomous character of our problem, the main tool of this paper will be the time mapping approach. This method have been used in many papers where several classes of problems related to second order differential equation are studied. For example this method have been used in [16] and [33] to prove existence of periodic solutions for some classes of second order differential eqautions. It has been also used in [1], [8] [15] and [29] to study existence of solutions for semi-linear second order BVPs and in [2], [3], [4] and [31] to study existence of solutions for second order BVPs involving the one dimensional $p-Laplacian$.

Roughly speaking, this method This consists to calculate the time $T(\lambda, \rho)$ required by a solution of the initial value problem (ivp for short)

$$\begin{cases} -(\varphi(u'))' = \lambda f(u), \\ u(1/2) = \rho, \ \ u'(1/2) = 0 \end{cases}$$

to reach the value 0, starting from an extremal value ρ. Clearly, positive solutions of $(1.1)-(1.2)$ are those of the above ivp satisfying $T(\lambda, \rho) = 1/2$.

In the same spirit as that of the papers [2], [11], [27]-[29] and [34], under regularity conditions on the functions φ and f, we obtain by means of the implicit function theorem that the set of positive solutions to (1.1)-(1.2) is reduced to a continuous curve $\lambda : (0, +\infty) \to (0, +\infty)$. Namely, for $\rho > 0$, the pair $(\lambda(\rho), u)$ is a positive solution to (1.1)-(1.2).

In all this paper, we understand by $\|\cdot\|$ the sup norm and for a continuously differentiable function u defined on a compact interval, $\|u\|_1 = \|u\| + \|u'\|$.

2. Preliminaries

We begin this section by introducing some notations. Let φ and f be as mentioned in the introduction.

- ψ denotes the inverse function of φ,
- for all $x \in \mathbb{R}$, $\Phi(x) = \int_0^x \varphi(s)ds$, $\Psi(x) = \int_0^x \psi(s)ds$, $W(x) = \Psi \circ \varphi(x) = x\varphi(x) - \Phi(x)$,
- Γ is the inverse function of the restriction of W to $\mathbb{R}^+$ and
- for all $x \geqslant 0$, $F(x) = \displaystyle\int_0^x f(s)\, ds$.

Let

$$A^+ = \left\{ u \in C^1([0,1]) : u > 0 \text{ in } (0,1) \text{ and } u \text{ is symmetrical about } \frac{1}{2} \right\}.$$

LEMMA 1. *If* (λ,u) *is a solution to* (1.1) *with* $u:(\alpha,\beta)\to\mathbb{R}$ *then there exists a real constant* C *such that*

$$W(u'(x))+\lambda F(u(x))=C, \quad \text{for all } x\in(\alpha,\beta). \tag{2.1}$$

Proof. Differentiating the function $x\to W(u'(x))+\lambda F(u(x))$ over $(0,1)$ we get

$$\psi(\varphi(u'(x)))(\varphi(u'(x)))'+f(u(x))u'(x)=[((\varphi(u'(x)))'+\lambda f(u(x)))]u'(x)=0.$$

REMARK 1. In fact Lemma 1 holds even f is not positive on $(0,+\infty)$.

Now, consider for $\lambda>0$ and $\rho>0$ the ivp

$$\begin{cases} -(\varphi(u'))'=\lambda f(u), \\ u(1/2)=\rho, \;\; u'(1/2)=0. \end{cases} \tag{2.2}$$

Setting $\upsilon=\varphi(u')$ the ivp (2.2) is reduced to the first order ivp

$$\begin{cases} u'=\psi(\upsilon), \\ \upsilon'=-\lambda f(u), \\ u(1/2)=\rho, \\ \upsilon(1/2)=0. \end{cases} \tag{2.3}$$

LEMMA 2. *For all* $\lambda,\rho>0$ *there exists a unique* $T(\lambda,\rho)>0$ *such that the ivp* (2.2) *admits a unique solution* u *defined on* $[1/2-T(\lambda,\rho),1/2+T(\lambda,\rho)]$. *Moreover we have:*

i) $u(1/2-T(\lambda,\rho))=u(1/2+T(\lambda,\rho))=0$ *and* $u(t)>0$ *for all* $t\in(1/2-T(\lambda,\rho), 1/2+T(\lambda,\rho))$,

ii) $u'(t)>0$ *for all* $t\in[1/2-T(\lambda,\rho),1/2)$, $u'(t)<0$ *for all* $t\in(1/2,1/2+T(\lambda,\rho)]$ *and* $\|u\|=u(1/2)=\rho$,

iii) u *is symmetrical about* $1/2$ *and*

iv) for all $t\in[1/2-T(\lambda,\rho),1/2+T(\lambda,\rho)]$, $u(t)\geqslant p(t)\rho$, *where*

$$p(t)=\max\left(1-\frac{1}{T(\lambda,\rho)}(1/2-t),1+\frac{1}{T(\lambda,\rho)}(1/2-t)\right).$$

Proof. Let u be a maximal solution of (2.2) defined on some interval, say (α,β), where α and β can be infinite. The positiveness of f implies that u' is decreasing on (α,β) and u is positive and concave on (α,β). More precisely, $u'(t)>0$ for all $t\in(\alpha,1/2)$ and $u'(t)<0$ for all $t\in(1/2,\beta)$. Thus the limits $\lim_{t\to\beta}u(t)$ and $\lim_{t\to\beta}u'(t)$ exit and are finite. Applying Theorem I.3.2 in [23] on the ivp (2.3) we get $\left(\lim_{t\to\beta}u(t),\lim_{t\to\beta}u'(t)\right)\in\partial(\mathbb{R}^+\times\mathbb{R})=\{0\}\times\mathbb{R}$, that is $\lim_{t\to\beta}u(t)=0$. Similarly we have $\lim_{t\to\alpha}u(t)=0$.

Now let us prove that $-\infty < \alpha < \beta < +\infty$. By the contrary suppose that $\beta = +\infty$ (the case $\alpha > -\infty$ can be checked similarly) and set $\lim_{t\to+\infty} u'(t) = l \leqslant 0$. Then it follows from (2.1) that $l = -\Gamma(\lambda F(\rho)) < 0$. Thus, L'Hôpital's rule leads to the contradiction

$$0 = \lim_{x\to\to+\infty} \frac{u(x)}{x} = \lim_{x\to+\infty} u'(x) = l < 0.$$

Now let ϑ and ω be respectively the inverse functions of the restrictions of u to $(\alpha, 1/2)$ and $(1/2, \beta)$. We have

$$\vartheta'(u(t)) = \frac{1}{u'(t)}, \quad \text{for all } t \in (\alpha, 1/2)$$

and

$$\omega'(u(t)) = \frac{1}{u'(t)}, \quad \text{for all } t \in (1/2, \beta).$$

Then from (2.1) follows

$$\vartheta'(u(t)) = \frac{1}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } t \in (\alpha, 1/2),$$

and

$$\omega'(u(t)) = -\frac{1}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } t \in (1/2, \beta).$$

Integrating we get

$$\begin{aligned} x - \alpha &= \vartheta(u(x)) - \vartheta(0) = \int_0^{u(x)} \vartheta'(u(t))du(t) \\ &= \int_0^{u(x)} \frac{du(t)}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } x \in [\alpha, 1/2], \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \beta - x &= \omega(0) - \omega(u(x)) = \int_0^{u(x)} -\omega'(u(t))du(t) \\ &= \int_0^{u(x)} \frac{du(t)}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } x \in [1/2, \beta]. \end{aligned} \tag{2.5}$$

In particular for $x = 1/2$ we have

$$\frac{1}{2} - \alpha = \beta - \frac{1}{2} = \int_0^{\rho} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}$$

which implies

$$\alpha = \frac{1}{2} - T(\lambda, \rho) \text{ and } \beta = \frac{1}{2} + T(\lambda, \rho)$$

where

$$T(\lambda, \rho) = \int_0^{\rho} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}.$$

At this stage, i) and ii) are proved and let us prove iii). For any $x \in \left[\frac{1}{2} - T(\lambda,\rho), \frac{1}{2}\right]$ the symmetrical point to x relatively to $1/2$ is $y = 1 - x \in \left[\frac{1}{2}, \frac{1}{2} + T(\lambda,\rho)\right]$. Taking in consideration that $\alpha = \frac{1}{2} - T(\lambda,\rho)$ and $\beta = \frac{1}{2} + T(\lambda,\rho)$ we deduce respectively from (2.4) and (2.5) that

$$x - \left(\frac{1}{2} - T(\lambda,\rho)\right) = x + T(\lambda,\rho) - \frac{1}{2} = \int_0^{u(x)} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}$$

and

$$\begin{aligned} x + T(\lambda,\rho) - \frac{1}{2} &= \left(\frac{1}{2} + T(\lambda,\rho)\right) - (1-x) = \left(\frac{1}{2} + T(\lambda,\rho)\right) - y \\ &= \int_0^{u(y)} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))} = \int_0^{u(1-x)} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}. \end{aligned}$$

Since x is arbitrary, we deduce from the above that:

$$\text{for all } x \in \left[\tfrac{1}{2} - T(\lambda,\rho), \tfrac{1}{2}\right],\ u(x) = u(1-x).$$

At the end, iv) follows from the concavity of u and uniqueness of the solution to (2.2) is due to the fact that ϑ and ω depends only on ρ, λ f and φ.

REMARK 2. Consider the map $\Pi : (0,+\infty) \times (0,+\infty) \to (0,+\infty) \times C^1([0,1])$ where $\Pi(\lambda,\rho) = (\lambda,u)$,

$$u(x) = \begin{cases} \vartheta^{-1}(x), & \text{if } x \in [1/2 - T(\lambda,\rho), 1/2], \\ \vartheta^{-1}(1-x), & \text{if } x \in [1/2, 1/2 + T(\lambda,\rho)], \end{cases}$$

and for $\gamma \in [0,\rho]$

$$\vartheta(\gamma) = 1/2 - \int_\gamma^\rho \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}.$$

In fact Lemma 2 says that, for λ,ρ in $(0,+\infty)$, $\Pi(\lambda,\rho)$ satisfies (2.2).

REMARK 3. We understand from Lemma 2 and its proof that for any function $g \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $g(u) > 0$ for all $u > 0$ and all λ,ρ in $(0,+\infty)$,

$$T(\lambda,\rho) = \int_0^u \frac{ds}{\Gamma(\lambda(G(\rho) - G(s))} < \infty,$$

where $G(u) = \int_0^u g(t)dt$.

LEMMA 3. *For $\rho > 0$ fixed we have*

$$\lim_{\lambda \to 0} T(\lambda,\rho) = +\infty \text{ and } \lim_{\lambda \to +\infty} T(\lambda,\rho) = 0.$$

Proof. We have for $\rho > 0$ fixed,

$$\begin{aligned}\frac{\rho}{\Gamma(\lambda F(\rho))} &\leqslant T(\lambda,\rho) \\ &\leqslant \frac{1}{2}\frac{\rho}{\Gamma\left(\lambda\left(F(\rho)-F\left(\frac{\rho}{2}\right)\right)\right)} + \int_{1/2}^{1}\frac{\rho}{\Gamma(\lambda(F(\rho)-F(s\rho)))}ds,\end{aligned}$$

from which follows immediately that $\lim_{\lambda\to 0} T(\lambda,\rho) = +\infty$.

Let $\lambda_0 > 0$ and $\varkappa = \min_{x\in[\rho/2,\rho]} f(x)$. Then we have

$$\frac{\rho}{\Gamma(\lambda(F(\rho)-F(s\rho)))} \leqslant \frac{\rho}{\Gamma(\lambda_0\varkappa(\rho - s\rho))}$$

and from Remark 3 for $g \equiv \varkappa$, we have

$$\int_{1/2}^{1}\frac{\rho}{\Gamma(\lambda_0\varkappa(\rho-s\rho))}ds \leqslant \int_0^{\rho}\frac{d\xi}{\Gamma(\lambda_0\varkappa(\rho-\xi))} < \infty.$$

Thus, by the dominated covergence theorem, we deduce that $\lim_{\lambda\to+\infty} T(\lambda,\rho) = 0$.

LEMMA 4. *If (λ,u) is a positive solution to (1.1)-(1.2), then u satisfies (2.2) with $\rho = \|u\|$ and $u \in A^+$.*

Proof. The positiveness of f implies that u' is decreasing on $[0,1]$ and u is concave on $[0,1]$. More precisely, there exists a unique $\delta \in (0,1)$ such that $u'(t) > 0$ for all $t \in [0,\delta)$ and $u'(t) < 0$ for all $t \in (\delta,1]$. Thus arguing as in the proof of Lemma 2 we prove that $\delta = 1/2$ and u is a solution to (2.2) with $\rho = \|u\|$. Thus we deduce from iii) of Lemma 2 that u is symmetrical about $1/2$.

LEMMA 5. *Assume that*

$$\varphi \in C^1(\mathbb{R}\setminus\{0\}) \text{ and there exists } c > 0 \text{ such that } \frac{x^2\varphi'(x)}{W(x)} \geqslant c \text{ for all } x > 0. \tag{2.6}$$

Then T is differentiable with respect to λ and $\dfrac{\partial T}{\partial\lambda}(\lambda,\rho) < 0$ for all $\lambda,\rho > 0$.

Proof. We have

$$T(\lambda,\rho) = \int_0^1 g(s,\lambda,\rho)\,ds, \quad \text{where } g(s,\lambda,\rho) = \frac{\rho}{\Gamma(\lambda(F(\rho)-F(s\rho)))}.$$

Thus

$$\begin{aligned}\frac{\partial g}{\partial\lambda}(s,\lambda,\rho) &= \frac{-\rho}{\lambda}\cdot\frac{\lambda(F(\rho)-F(s\rho))(\Gamma)'(\lambda(F(\rho)-F(s\rho)))}{(\Gamma(\lambda(F(\rho)-F(s\rho))))^2} \\ &= \frac{-\rho}{\lambda}\cdot\frac{W(\Gamma(\lambda(F(\rho)-F(s\rho))))}{(\Gamma(\lambda(F(\rho)-F(s\rho))))^3\,\varphi'(\Gamma(\lambda(F(\rho)-F(s\rho))))}\end{aligned}$$

and

$$\left|\frac{\partial g}{\partial \lambda}(s,\lambda,\rho)\right| \leqslant \frac{1}{\lambda c} g(s,\lambda,\rho).$$

For an arbitrary interval $[\alpha,\beta] \subset (0,+\infty)$ we have

$$\int_0^1 \left|\frac{\partial g}{\partial \lambda}(s,\lambda,\rho)\right| ds \leqslant \int_0^1 \frac{1}{\alpha c} g(s,\alpha,\rho)\, ds = \frac{T(\alpha,\rho)}{\alpha c} < \infty.$$

So, $\dfrac{\partial T}{\partial \lambda}$ exists and $\dfrac{\partial T}{\partial \lambda}(\lambda,\rho) < 0$ for all $\lambda > 0$ and $\rho > 0$.

3. Global curve of positive solutions

We deduce from Lemma 2, if u is a solution of (2.2) with $T(\lambda,\rho) = 1/2$, then (λ,u) is a positive solution to (1.1)-(1.2). Reciprocally and from Lemma 4, if (λ,u) is a positive solution to (1.1)-(1.2), then u is a solution to (2.2) with $\rho = \|u\|$ and $T(\lambda,\|u\|) = 1/2$. Let $S \subset \mathbb{R}\times C^1([0,1])$ be the set of positive solutions to (1.1)-(1.2) and $D = \{(\lambda,\rho) \in (0,+\infty)\times(0,+\infty), T(\lambda,\rho) = 1/2\}$. The above means that the restriction of the map Π, defined in Remark 2, to D and S is one to one. Therefore, we identify the set S to the set D.

THEOREM 1. *Assume that*

$$f \in C^1\left(\mathbb{R}^+\right) \text{ and } \varphi,\psi \in C^1(\mathbb{R}). \tag{3.1}$$

Then the set of positive solutions to (1.1)-(1.2) *is reduced to a continuously differentiable curve* $\rho \to \lambda(\rho)$ *defined on* $(0,+\infty)$.

Proof. Note that u is a solution to (2.2) if and only if (u,u') is a solution to

$$\begin{cases} u' = \psi(v), \quad v' = -\lambda f(u), \\ u(1/2) = \rho, \quad v(1/2) = 0. \end{cases}$$

Thus, since $f \in C^1(\mathbb{R}^+)$ and $\psi \in C^1(\mathbb{R})$, u is continuously differentiable with respect to all its variables.

Differentiating with the respect to λ in the equality

$$u(1/2 + T(\lambda,\rho),\lambda,\rho) = 0,$$

we get

$$u'(1/2+T(\lambda,\rho),\lambda,\rho)\frac{\partial T}{\partial \lambda} + \frac{\partial u}{\partial \lambda}(1/2+T(\lambda,\rho),\lambda,\rho) = 0. \tag{3.2}$$

Let us prove that $\dfrac{\partial u}{\partial \lambda}(1/2+T(\lambda,\rho),\lambda,\rho) < 0$. Set $z(x,\lambda,\rho) = \dfrac{\partial u}{\partial \lambda}(x,\lambda,\rho)$. Then z satisfies

$$\begin{cases} -(\varphi'(u')z')' = f(u) + \lambda f'(u)z, \\ z(1/2) = 0, \quad z'(1/2) = 0. \end{cases} \tag{3.3}$$

Multiplying the differential equation in (3.3) by u', and integrating over $[1/2,x]$, we obtain

$$\varphi'(u'(x))z'(x)u'(x)+\lambda f(u(x))z(x)=F(\rho)-F(u(x)). \tag{3.4}$$

We deduce from (3.3) $(\varphi'(u')z')'<0$ and $z'<0$ in a right neighborhood of $1/2$, hence if $z(1/2+T(\lambda,\rho),\lambda,\rho)\geqslant 0$ then there exists some $x^*\in(1/2,T(\lambda,\rho))$ such that $z(x^*)=\min_{x\in[1/2,1/2+T(\lambda,\rho)]}z(x)<0$ and $z'(x^*)=0$. Inserting in (3.4) we arrive to the contradiction

$$0>\lambda f(u(x^*))z(x^*)=F(\rho)-F(u(x^*))>0.$$

Now, with

$$\frac{\partial u}{\partial\lambda}(1/2+T(\lambda,\rho),\lambda,\rho)<0 \quad\text{and}\quad u'(1/2+T(\lambda,\rho),\lambda,\rho)=-\Gamma(\lambda F(\rho))<0,$$

we deduce from (3.2) that

$$\frac{\partial T}{\partial\lambda}(\lambda,\rho)<0 \text{ for all } \lambda>0 \text{ and } \rho>0. \tag{3.5}$$

As in the proof of Theorem 2, for each $\rho>0$ there is a unique $\lambda=\lambda(\rho)$ solution to the equation $T(\lambda,\rho)=1/2$ and since the function $(\lambda,\rho)\to T(\lambda,\rho)$ is continuously differentiable on $(0,+\infty)\times(0,+\infty)$ and $\partial T/\partial\lambda<0$ the implicit function theorem leads to assertion of Theorem 1.

REMARK 4. We can see from the above proof that $z(x,\lambda,\rho)=\partial u/\partial\lambda(x,\lambda,\rho)<0$ for all $x\in(1/2,1/2+T(\lambda,\rho)]$.

It is easy to see that Theorem 1 does not cover the case $\varphi(x)=|x|^{p-2}x$ where $p\in(1,+\infty)$. The following result adapts to this case.

THEOREM 2. *Assume that* (2.6) *holds. Then the set of positive solutions to* (1.1)-(1.2) *is reduced to a continuous curve* $\rho\to\lambda(\rho)$ *defined on* $(0,+\infty)$.

Proof. We deduce from Lemma 3 and Lemma 5 that all $\rho>0$ there exists a unique $\lambda=\lambda(\rho)$ solution of the equation $T(\lambda,\rho)=1/2$. Moreover, since $\dfrac{\partial T}{\partial\lambda}(\lambda,\rho)<0$, we deduce from the implicit function theorem that

$$D=\{(\lambda(\rho),\rho),\ \rho>0\} \text{ and } \rho\to\lambda(\rho) \text{ is continuous.} \ \square$$

Now set

$$m_\sigma=\liminf_{\rho\to 0}\frac{\varphi(\sigma\rho)}{\varphi(\rho)},\qquad m^\sigma=\limsup_{\rho\to 0}\frac{\varphi(\sigma\rho)}{\varphi(\rho)},$$
$$M_\sigma=\liminf_{\rho\to+\infty}\frac{\varphi(\sigma\rho)}{\varphi(\rho)},\quad M^\sigma=\limsup_{\rho\to+\infty}\frac{\varphi(\sigma\rho)}{\varphi(\rho)}.$$

PROPOSITION 1. *Assume that (2.6) or (3.1) holds. We have:*

i) if

$$\begin{array}{l} \lim_{x\to 0} f(x)/\varphi(x) = 0 \text{ and } 0 < m_\sigma < \infty \text{ for all } \sigma > 1, \\ \text{or} \\ \limsup_{x\to 0} f(x)/\varphi(x) < \infty \text{ and } m_\sigma = \infty \text{ for all } \sigma > 1, \end{array} \tag{3.6}$$

then $\lim_{\rho\to 0} \lambda(\rho) = +\infty$.

ii) if

$$\lim_{x\to 0} f(x)/\varphi(x) = +\infty \text{ and } m^\sigma < \infty \text{ for all } \sigma > 1,\ m_\sigma > 0 \text{ for all } \sigma < 1, \tag{3.7}$$

then $\lim_{\rho\to 0} \lambda(\rho) = 0$.

iii) if

$$\begin{array}{l} \lim_{x\to +\infty} f(x)/\varphi(x) = 0 \text{ and } M_\sigma > 0 \text{ for all } \sigma > 1, \\ \text{or} \\ \limsup_{x\to +\infty} f(x)/\varphi(x) < \infty \text{ and } M_\sigma = \infty \text{ for all } \sigma > 1, \end{array} \tag{3.8}$$

then $\lim_{\rho\to +\infty} \lambda(\rho) = +\infty$.

iv) if

$$\lim_{x\to +\infty} f(x)/\varphi(x) = +\infty \text{ and } M^\sigma < \infty \text{ for all } \sigma > 1,\ M_\sigma > 0 \text{ for all } \sigma < 1, \tag{3.9}$$

then $\lim_{\rho\to +\infty} \lambda(\rho) = 0$.

Proof. Let $\rho > 0$ and u be the unique solution of

$$\begin{cases} -(\varphi(u'))' = \lambda(\rho) f(u), \\ u(1/2) = \rho, \quad u'(1/2) = 0. \end{cases}$$

Integrating twice, we get

$$\rho = \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{t} f(u(s))ds\right) dt. \tag{3.10}$$

i) If $\limsup_{x\to 0} f(x)/\varphi(x) = l$, then for arbitrary $\varepsilon > 0$, there exists $\delta > 0$, such that $f(x) \leqslant (l+\varepsilon)\varphi(x)$ for all $x \in [0,\delta]$. Thus, for $\rho \in (0,\delta)$ we have

$$\begin{aligned} \rho &= \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{t} f(u(s))ds\right) dt \\ &\leqslant \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{1} (l+\varepsilon)\varphi(u(s))ds\right) dt \end{aligned}$$

$$\leqslant \frac{1}{2}\psi\left(\frac{\lambda(\rho)}{2}(l+\varepsilon)\varphi(\rho)\right),$$

which implies

$$\lambda(\rho) \geqslant \frac{2\varphi(2\rho)}{\varphi(\rho)(l+\varepsilon)}.$$

Letting $\rho \to 0,$ we get if $m_2 = +\infty$

$$\lim_{\rho\to 0}\lambda(\rho) = +\infty,$$

and if $m_2 < \infty$ and $l = 0$

$$\liminf_{\rho\to 0}\lambda(\rho) \geqslant \frac{2m_2}{\varepsilon}.$$

Since ε is arbitrary, this means that $\lim_{\rho\to 0}\lambda(\rho) = +\infty.$

ii) If $\lim_{x\to 0} f(x)/\varphi(x) = +\infty,$ then for arbitrary $K > 0,$ there exists $\delta > 0,$ such that $f(x) \geqslant K\varphi(x)$ for all $x \in [0,\delta]$. Thus, for $\rho \in (0,\delta)$ we have from (3.10) and iv) of Lemma 2

$$\begin{aligned}
\rho &\geqslant \int_{1/2}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{t}K\varphi(u(s))ds\right)dt \\
&\geqslant \int_{3/4}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{3/4}K\varphi(u(s))ds\right)dt \\
&\geqslant \int_{3/4}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{3/4}K\varphi(2\rho(1-s))ds\right)dt \\
&\geqslant \frac{1}{4}\psi\left(\frac{\lambda(\rho)}{4}K\varphi\left(\frac{\rho}{2}\right)\right),
\end{aligned}$$

which implies

$$\lambda(\rho) \leqslant \frac{4\varphi(4\rho)}{K\varphi(\rho/2)}.$$

Letting $\rho \to 0,$ we get

$$\limsup_{\rho\to 0}\lambda(\rho) \leqslant \frac{4m^4}{Km_{1/2}}.$$

Since K is arbitrary, this means that $\lim_{\rho\to 0}\lambda(\rho) = 0.$

iii) If $\limsup_{x\to+\infty} f(x)/\varphi(x) = l,$ then for arbitrary $\varepsilon > 0,$ there exists $C_\varepsilon > 0,$ such that $f(x) \leqslant (l+\varepsilon)\varphi(x) + C_\varepsilon$ for all $x \geqslant 0.$ Thus, for $\rho > 0$ we have

$$\begin{aligned}
\rho &= \int_{1/2}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{t}f(u(s))ds\right)dt \\
&\leqslant \int_{1/2}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{1}((l+\varepsilon)\varphi(u(s))+C_\varepsilon)ds\right)dt
\end{aligned}$$

$$\leqslant \frac{1}{2}\psi\left(\frac{\lambda(\rho)}{2}((l+\varepsilon)\varphi(\rho)+C_\varepsilon)\right)$$

which implies

$$\lambda(\rho) \geqslant \frac{2\varphi(2\rho)}{\varphi(\rho)} \frac{1}{(l+\varepsilon)+\frac{C_\varepsilon}{\varphi(\rho)}}.$$

As in i), we conclude that $\lim_{\rho\to+\infty}\lambda(\rho)=+\infty$.

iv) If $\lim_{x\to+\infty} f(x)/\varphi(x)=+\infty$, then for arbitrary $K>0$, there exists $B>0$, such that $f(x)\geqslant K\varphi(x)$ for all $x\geqslant B$. Thus, for $\rho\geqslant 2B$ as in ii) we have

$$\begin{aligned}
\rho &\geqslant \int_{3/4}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{3/4} f(u(s))ds\right)dt\\
&\geqslant \int_{3/4}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{3/4} K\varphi(u(s))ds\right)dt\\
&\geqslant \int_{3/4}^{1}\psi\left(\lambda(\rho)\int_{1/2}^{3/4} K\varphi(2\rho(1-s))ds\right)dt\\
&\geqslant \frac{1}{4}\psi\left(\frac{\lambda(\rho)}{4}K\varphi\left(\frac{\rho}{2}\right)\right),
\end{aligned}$$

which implies

$$\lambda(\rho) \leqslant \frac{4\varphi(4\rho)}{K\varphi(\rho/2)}.$$

Letting $\rho\to+\infty$, we get

$$\lim\sup_{\rho\to+\infty} \lambda(\rho) \leqslant \frac{4M^4}{KM_{1/2}}.$$

Since K is arbitrary, this means that $\lim_{\rho\to+\infty}\lambda(\rho)=0$.

REMARK 5. Conditions on m_σ, m^σ, M_σ and M^σ has been assumed in many papers where φ-Laplacian bvps are studied, see for example [9], [10], [14], [17], [18], [19], [20] and [21]. A typical example of a function φ satifying $0<m_\sigma, m^\sigma, M_\sigma, M^\sigma<\infty$ is $\varphi(x)=|x|^{p-2}x+|x|^{q-2}x$ with $1<p<q$.

PROPOSITION 2. *Assume that* (2.6) *or* (3.1) *hold. Then we have:*

i) if $f(0)>0$, *then* $\lim_{\rho\to 0}\lambda(\rho)=0$,

ii) if $\lim_{x\to 0}F(x)/W(x)=0$, *then* $\lim_{\rho\to 0}\lambda(\rho)=+\infty$ *and*

iii) if $\lim_{x\to+\infty}F(x)/W(x)=0$, *then* $\lim_{\rho\to+\infty}\lambda(\rho)=+\infty$.

Proof. Let $\rho > 0$ and u be the unique solution to (2.2) whith $\lambda = \lambda(\rho)$ and choose $\eta > 0$ small enough. Since $f(0) > 0$,

$$\delta_\eta = \min_{u \in [0,\eta]} f(u) > 0.$$

Thus, it follows from (3.10) that, if $\rho \in [0,\eta]$ then

$$\begin{aligned} \rho &\geqslant \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} f(u(s))ds\right)dt \\ &\geqslant \frac{1}{4}\psi\left(\frac{\lambda(\rho)\delta_\eta}{4}\right), \end{aligned}$$

leading to

$$\lim_{\rho\to 0}\lambda(\rho) \leqslant \lim_{\rho\to 0}\frac{4\varphi(4\rho)}{\delta_\eta} = 0,$$

which proves i). We have from (2.1),

$$\rho = \int_0^{1/2} u'(t)dt \leqslant \int_0^{1/2} u'(0)dt \leqslant \Gamma(\lambda(\rho)F(\rho)),$$

which implies

$$\lambda(\rho) \geqslant \frac{W(\rho)}{F(\rho)} = \left(\frac{F(\rho)}{W(\rho)}\right)^{-1},$$

leading to assertions ii) and iii) of the proposition.

REMARK 6. By L'Hôpital's rule, if $\varphi \in C^1(\mathbb{R} \setminus \{0\})$ and $\lim_{x\to 0} f(x)/x\varphi'(x) = 0$ (respectively $\lim_{x\to+\infty} f(x)/x\varphi'(x) = 0$) then $\lim_{x\to 0} F(x)/W(x) = 0$ (respectively $\lim_{x\to+\infty} F(x)/W(x) = 0$).

We deduce immediately from Theorem 2, Proposition 1, Proposition 2 and Remark 6 the following corollaries.

COROLLARY 1. *Assume that* (2.6) *or* (3.1) *holds. Then Problem* (1.1)-(1.2) *admits at least one positive solution for all* $\lambda > 0$, *in each of the following situations i)-vi):*

i) $\lim_{x\to 0} f(x)/x\varphi'(x) = 0$ *and* (3.9) *holds.*

ii) Hypothesies (3.6) *and* (3.9) *hold.*

iii) $f(0) = 0$, $\lim_{x\to+\infty} f(x)/x\varphi'(x) = 0$ *and* (3.7) *holds.*

iv) $f(0) = 0$, (3.7) *and* (3.8) *hold.*

v) $f(0) > 0$ *and* $\lim_{x\to+\infty} f(x)/x\varphi'(x) = 0$.

vi) $f(0) > 0$ *and* (3.8) *holds.*

COROLLARY 2. *Assume that* (2.6) *or* (3.1) *and the following condition*

$$\lim_{x\to 0}\frac{f(x)}{x\varphi'(x)} = \lim_{x\to +\infty}\frac{f(x)}{x\varphi'(x)} = 0$$

hold. Then there exists $\lambda^+ > 0$ *such that:*

i) Problem (1.1)-(1.2) *admits no positive solution if* $\lambda < \lambda^+$,

ii) Problem (1.1)-(1.2) *admits at least one positive solution if* $\lambda = \lambda^+$ *and*

iii) Problem (1.1)-(1.2) *admits at least two positive solutions if* $\lambda > \lambda^+$.

COROLLARY 3. *Assume that* (2.6), (3.6) *and* (3.8) *or* (3.1), (3.6) *and* (3.8) *hold. Then there exists* $\lambda^+ > 0$ *such that:*

i) Problem (1.1)-(1.2) *admits no positive solution if* $\lambda < \lambda^+$,

ii) Problem (1.1)-(1.2) *admits at least one positive solution if* $\lambda = \lambda^+$ *and*

iii) Problem (1.1)-(1.2) *admits at least two positive solutions if* $\lambda > \lambda^+$.

COROLLARY 4. *Assume that* (2.6), (3.7) *and* (3.9) *or* (3.1), (3.7) *and* (3.9) *hold. Then there exists* $\lambda^+ > 0$ *such that:*

i) Problem (1.1)-(1.2) *admits no positive solution if* $\lambda > \lambda^+$,

ii) Problem (1.1)-(1.2) *admits at least one positive solution if* $\lambda = \lambda^+$ *and*

iii) Problem (1.1)-(1.2) *admits at least two positive solutions if* $\lambda < \lambda^+$.

REMARK 7. We can prove, as in the proof of uniqueness in [7], that, if $\varphi(u)/u$ and $f(u)/u$ are respectively decreasing and increasing on $(0,+\infty)$ in the case i), or $\varphi(u)/u$ and $f(u)/u$ are respectively increasing and decreasing on $(0,+\infty)$ in the case ii), the positive solution obtained in Corollary 1 is unique.

Now, with more regularity on φ and f, we will prove that the curve $\rho \to \lambda(\rho)$ admits at most one critical point. To this aim we assume in the following that $f \in C^2(\mathbb{R}^+)$ and $\psi \in C^2(\mathbb{R})$. Note that in this case the unique solution $u(\cdot,\lambda,\rho)$ of (2.2) is twice continuously differentiable with respect to

$$(x,\lambda,\rho) \in [1/2 - T(\lambda,\rho), 1/2 + T(\lambda,\rho)] \times (0,+\infty) \times (0,+\infty)$$

and denote $v = \dfrac{\partial u}{\partial \rho}$ and $w = \dfrac{\partial^2 u}{\partial \rho^2}$. Then v and w satisfy respectively

$$\begin{cases} -(\varphi'(u')v')' = \lambda f'(u)v, \\ v(1/2) = 1, \ \ v'(1/2) = 0, \end{cases} \tag{3.11}$$

and

$$\begin{cases} -\left(\varphi''(u')(v')^2+\varphi'(u')w'\right)' = \lambda f''(u)v^2+\lambda f'(u)w, \\ w(1/2)=0, \;\; w'(1/2)=0. \end{cases} \tag{3.12}$$

LEMMA 6. *Assume that* $\psi \in C^1(\mathbb{R})$. *If* $u(\cdot,\lambda,\rho)$ *is such that* (λ,u) *is positive solution to problem* (1.1)-(1.2) *then* v *has at most one zero in* $[1/2,1]$.

Proof. First note that if $x_0 \in [1/2,1]$ is such that $v(x_0)=0$ then $v'(x_0)\neq 0$. Otherwise, if we set $\theta=\varphi'(u')v'$ then the pair (v,θ) is solution to the ivp

$$\begin{cases} v' = (\varphi'(u'))^{-1}\theta, \;\; \theta' = -\lambda f'(u)v, \\ v(x_0)=0, \;\; \theta(x_0)=0. \end{cases}$$

Note that $(\varphi'(u'))^{-1}=\psi'(\varphi(u'))$ and the right-hand side of the above system is locally Lipschitzian. This makes $v=0$, which contradicts to $v(1/2)=1$.

Note that v admits a finite number of zeros, indeed if $(x_n)_{n\geqslant 1}$ is a sequence of zeros of v and $x^*=\lim\limits_{n\to\infty}x_n$ then

$$v(x^*)=\lim_{n\to\infty}v(x_n)=0=v'(x^*)=\lim_{n\to\infty}\frac{v(x_n)-v(x^*)}{x-x_n}$$

and for the pair (v,θ) satisfies

$$\begin{cases} v' = (\varphi'(u'))^{-1}\theta, \;\; \theta' = -\lambda f'(u)v, \\ v(x^*)=0, \;\; \theta(x^*)=0. \end{cases}$$

So, we get for the same reasons $v=0$, which contradicts to $v(1/2)=1$.

Now multiplying (3.11) by u' and integrating over $[1/2,x]$, we get

$$\varphi'(u'(x))v'(x)u'(x)+\lambda f(u(x))v(x)=\lambda f(\rho). \tag{3.13}$$

Suppose that v admits more than one zero and let $x_1<x_2$ be the two first consecutive zeros of v. Then we have

$$v'(x_1)<0 \text{ and } v'(x_2)>0. \tag{3.14}$$

Substituting $x=x_2$ in (3.13) we get

$$\varphi'(u'(x_2))v'(x_2)u'(x_2)=\lambda f(\rho). \tag{3.15}$$

Since $\varphi'(u'(x_2))>0$, $u'(x_2)<0$ and $\lambda f(\rho)>0$, we deduce from (3.15) that $v'(x_2)<0$ which contradicts (3.14). This completes the proof of the lemma.

LEMMA 7. *Assume that* $f\in C^2(\mathbb{R}^+)$, $\varphi,\psi\in C^2(\mathbb{R})$ *and for all* $x>0$, $f''(x)>0$ *and* $\varphi''(x)\leqslant 0$. *Let* $u(\cdot,\lambda,\rho)$ *be such that* (λ,u) *is a positive solution to problem* (1.1)-(1.2). *If* $v(1)=0$, *then* $w(1)<0$.

Proof. Multiplying (3.11) by w and integrating on $(1/2,1)$ we get

$$-\varphi'\left(u'(1)\right)v'(1)w(1)+\int_{1/2}^{1}\varphi'\left(u'\right)v'w'=\lambda\int_{1/2}^{1}f'(u)vw. \tag{3.16}$$

Similarly, multiplying (3.12) by v and integrating on $(1/2,1)$ we get

$$\begin{aligned}-\varphi'(u'(1))(v'(1))^2v(1)-\varphi'(u'(1))v(1)w'(1)&\\ +\int_{1/2}^{1}(\varphi''(u')(v')^3+\varphi'(u')v'w')&\\ =\lambda\int_{1/2}^{1}f''(u)v^3+\lambda\int_{1/2}^{1}f'(u)vw.&\end{aligned} \tag{3.17}$$

Subtracting (3.17) and (3.16) and taking in consideration $v(1)=0$ we get

$$\varphi'\left(u'(1)\right)v'(1)w(1)=\lambda\int_{1/2}^{1}f''(u)v^3-\int_{1/2}^{1}\varphi''\left(u'\right)\left(v'\right)^3. \tag{3.18}$$

Note that since $v(1)=0$, Lemma 6 leads to $v>0$ in $[1/2,1)$ and $v'(1)<0$. Thus, the convexity of φ and the oddness of φ'' leads to $\varphi''(u')>0$ in $(1/2,1]$.

It remains to investigate the sign of v'. We deduce from (3.11)

$$-\varphi''\left(u'\right)u''v'-\varphi'\left(u'\right)v''=\lambda f'(u)v,\quad \text{in } (0,1). \tag{3.19}$$

As $v'(1/2)=0$, $v(1)=0$ and $v>0$ in $[1/2,1)$ we have:

- either $v'\leqslant 0$ in $[1/2,1]$,
- or there exists $x_0\in(1/2,1]$ such that $v'(x_0)>0$.

In fact the second situation does not occur, indeed if v' changes its sign then it will exist x_1 and x_2 belonging to $(1/2,1)$ such that $x_1<x_2$ and at both x_1 and x_2, v reachs respectively a local minimum and a local maximum. In this case substituting respectively $x=x_1$ and $x=x_2$ in (3.19) we get

$$\begin{aligned}-\varphi'(u'(x_1))v''(x_1)&=\lambda f'(u(x_1))v(x_1)\leqslant 0,\\ -\varphi'(u'(x_2))v''(x_2)&=\lambda f'(u(x_2))v(x_2)\geqslant 0,\end{aligned}$$

so

$$f'(u(x_1))\leqslant 0 \text{ and } f'(u(x_2))\geqslant 0.$$

But this is impossible because $u(x_1)>u(x_2)$ and f' is increasing. Hence $v'\leqslant 0$ in $[1/2,1]$.

Finally taking in consideration the fact that $v\geqslant 0$ and $v'\leqslant 0$ in $[1/2,1]$, $f''(u)>0$ and $\varphi''(u')>0$ in $[1/2,1]$, $\varphi'(u'(1))>0$ and $v'(1)<0$, we deduce from (3.18) that $w(1)<0$. This completes the proof.

LEMMA 8. *Assume that* $f\in C^2(\mathbb{R}^+)$, $\varphi,\psi\in C^2(\mathbb{R})$ *and for all* $x\in\mathbb{R}^+$, $f''(x)<0$ *and* $\varphi''(x)\geqslant 0$. *Let* $u(\cdot,\lambda,\rho)$ *be such that* (λ,u) *is a positive solution to problem* (1.1)-(1.2). *If* $v(1)=0$ *then* $w(1)>0$.

Proof. First we claim that $f'(x) \geqslant 0$ for all $x \geqslant 0$. Indeed, if there exists some $x_0 \geqslant 0$ such that $f'(x_0) < 0$ then $\lim_{x\to+\infty} f'(x) = 0$, otherwise, if $\lim_{x\to+\infty} f'(x) = l < 0$ then $\lim_{x\to+\infty} f(x)/x = l < 0$ which contradicts the positiveness of f. In this case there exists $\overline{x} > 0$ such that $f'(\overline{x}) = \min_{x\geqslant 0} f'(x)$ and $f''(\overline{x}) = 0$, which contradicts the positiveness of f''.

Now it is easy to see from (3.11) that if $v(1) = 0$ then $v > 0$ in $[1/2, 1)$ and $v' < 0$ in $(1/2, 1]$. Thus, as in the proof of Lemma 7 we deduce from (3.18) that $w(1) > 0$.

THEOREM 3. *Assume that* $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$ *and one of the following condition holds,*

$$f''(x) < 0 \text{ and } \varphi''(x) \geqslant 0 \text{ for all } x > 0, \tag{3.20}$$

$$f''(x) > 0 \text{ and } \varphi''(x) \leqslant 0 \text{ for all } x > 0. \tag{3.21}$$

Then $\rho \to \lambda(\rho)$ *admits on* $(0, +\infty)$ *at most one critical point.*

Proof. We obtain the desired by proving that if $\lambda'(\rho) = 0$ for some $\rho \in D$ then $\lambda''(\rho) < 0$ or $\lambda''(\rho) > 0$. We have for all $\rho > 0$

$$u(1, \lambda(\rho), \rho) = 0. \tag{3.22}$$

Differentiating in (3.22) with respect to ρ we get

$$\frac{\partial u}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda'(\rho) + v(1, \lambda(\rho), \rho) = 0, \tag{3.23}$$

and

$$\frac{\partial^2 u}{\partial \lambda^2}(1, \lambda(\rho), \rho)(\lambda'(\rho))^2 + \frac{\partial u}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda''(\rho) + \frac{\partial v}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda'(\rho) + w(1, \lambda(\rho), \rho) = 0. \tag{3.24}$$

Suppose that (3.21) holds (the other case can be checked similarly), then if for some $\rho_0 > 0$, $\lambda'(\rho_0) = 0$ then we deduce from (3.23) that $v(1, \lambda(\rho_0), \rho_0) = 0$ and it follows from Lemma 7, $w(1, \lambda(\rho_0), \rho_0) < 0$. Thus, we deduce from (3.24) and Remark 4 that $\lambda''(\rho_0) < 0$. This completes the proof.

We deduce from Theorems 2, 1 and Proposition 2 the following corollaries.

COROLLARY 5. *Assume that* $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$, (3.21), (3.7) *and* (3.9) *hold. Then there exists* $\lambda^+ > 0$ *such that:*

i) Problem (1.1)-(1.2) *admits no positive solution if* $\lambda > \lambda^+$,

ii) Problem (1.1)-(1.2) *admits exactly one positive solution if* $\lambda = \lambda^+$ *and*

iii) Problem (1.1)-(1.2) *admits exactly two positive solutions if* $\lambda < \lambda^+$.

COROLLARY 6. *Assume that* $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$, (3.8) *and* (3.20) *hold. Then there exists* $\lambda^+ \geqslant 0$ *such that:*

i) Problem (1.1)-(1.2) *admits no positive solution if* $\lambda \leqslant \lambda^+$ *and*

ii) Problem (1.1)-(1.2) *admits exactly one positive solution if* $\lambda > \lambda^+$.

COROLLARY 7. *Assume that* $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$ *and* (3.20), *hold. If* $\lim_{x\to+\infty}(f(x)/x\varphi'(x)) = 0$ *then there exists* $\lambda^+ \geqslant 0$ *such that*

i) Problem (1.1)-(1.2) *admits no positive solution if* $\lambda \leqslant \lambda^+$ *and*

ii) Problem (1.1)-(1.2) *admits exactly one positive solution if* $\lambda > \lambda^+$.

REMARK 8. It is clear that in corollaries 6 and 7, $\lambda^+ = \lim_{\rho\to 0}\lambda(\rho)$ and because of the inequality $\lambda(\rho) \geqslant \frac{W(\rho)}{F(\rho)}$ it can happens that $\lambda^+ > 0$.

REMARK 9. In corollaries 6 and 7 we can not assume that $\lim_{x\to 0}(f(x)/x\varphi'(x)) = 0$ or $\lim_{x\to 0}(f(x)/\varphi(x)) = 0$ in order to obtain multiplicity results. This obstruction is caused by the fact that $\varphi'(0) > 0$ and $(f(x)/x)$ and $(x/\varphi(x))$ are decreasing functions on $(0,+\infty)$.

REMARK 10. In the case where $\varphi(x) = x$, many exact multiplicity results have been obtained under a requirement on the convexity of the nonlinear term. See Theorem 3.2 in [29], Theorem 2 in [5], Theorem 1 and Theorem 2 in [11]. Moreover, note that for $\lambda = 1$ Problem (1.1)-(1.2) admits at most one positive solution. This result has been obtained by Korman and Li in [26].

4. Examples

EXAMPLE 1. Consider the bvp (1.1)-(1.2) with

$$\varphi(u) = \begin{cases} e^u - 1, & \text{for } u \geqslant 0, \\ 1 - e^{-u}, & \text{for } u \leqslant 0, \end{cases}$$

and $f(u) = e^u$.

We have $\varphi, \psi, f \in C^1(\mathbb{R})$ and

$$f(0) > 0 \text{ and } \lim_{x\to+\infty} \frac{f(x)}{x\varphi'(x)} = 0.$$

Thus, we deduce from iii) of Corollary 1 that for all $\lambda > 0$, the bvp (1.1)-(1.2) admits at least one positive solution.

Note that in this example we have

$$\lim_{u\to+\infty}(f(u)/\varphi(u)) = 1 \text{ and}$$

$$\lim_{\rho\to+\infty}(\varphi(\sigma\rho)/\varphi(\rho)) = +\infty \text{ for all } \sigma > 1.$$

EXAMPLE 2. Consider the bvp (1.1)-(1.2) with

$$\varphi(u) = \begin{cases} e^u - 1, & \text{for } u \geqslant 0, \\ 1 - e^{-u}, & \text{for } u \leqslant 0, \end{cases}$$

and $f(u) = u^2$.

See that

$$\lim_{u \to 0} \frac{f(u)}{u\varphi'(u)} = \lim_{u \to +\infty} \frac{f(u)}{u\varphi'(u)} = 0.$$

Thus, we deduce from Corollary 2 that there exists $\lambda^+ > 0$ such that:

i) Problem (1.1)-(1.2) admits no positive solution if $\lambda < \lambda^+$,

ii) Problem (1.1)-(1.2) admits at least one positive solution if $\lambda = \lambda^+$ and

iii) Problem (1.1)-(1.2) admits at least two positive solutions if $\lambda > \lambda^+$.

EXAMPLE 3. Consider the bvp (1.1)-(1.2) with

$$\varphi(u) = \sinh(u) \text{ and } f(u) = \sqrt{1+u}.$$

Since $\lim_{\rho \to 0} \lambda(\rho) = 0$, we have from Corollary 7 that for all $\lambda > 0$ Problem (1.1)-(1.2) admits exactly one positive solution.

EXAMPLE 4. Consider the bvp (1.1)-(1.2) with

$$\varphi(u) = u + \frac{u}{\sqrt{1+u^2}} \text{ and } f(u) = 1 + u^2.$$

We have from Corollary 5 that there exists $\lambda^+ > 0$ such that:

i) Problem (1.1)-(1.2) admits no positive solution if $\lambda > \lambda^+$,

ii) Problem (1.1)-(1.2) admits exactly one positive solution if $\lambda = \lambda^+$ and

iii) Problem (1.1)-(1.2) admits exactly two positive solutions if $\lambda < \lambda^+$.

EXAMPLE 5. Consider the bvp (1.1)-(1.2) with $\varphi(u) = |u|^{p-2}u + |u|^{q-2}u$ where $1 < p < q$ and $f(u) = e^u$.

We have from Corollary 4 that there exists $\lambda^+ > 0$ such that:

i) Problem (1.1)-(1.2) admits no positive solution if $\lambda > \lambda^+$,

ii) Problem (1.1)-(1.2) admits at least one positive solution if $\lambda = \lambda^+$ and

iii) Problem (1.1)-(1.2) admits at least two positive solutions if $\lambda < \lambda^+$.

The case $p = q$ has been considered in [2] where exactness result is obtained.

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