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OSCILLATION CRITERIA FOR NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Some new oscillation criteria are established for second order neutral partial functional differential equations of the form

$$\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\right)\right]$$
$$=a(t)\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\Delta u(x,t-\rho_k(t))-q(x,t)u(x,t)$$
$$-\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\sigma_j)),(x,t)\in\Omega\times[0,\infty)\equiv G,$$

by integral average, where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean N-space $\mathbb{R}^N$.

1. Introduction

Recently, the oscillation problem for the partial functional differential equation has been studied by many authors. We refer the reader to [2,5,8] for parabolic equations and to [1,3,4,6,9] for hyperbolic equations.

In this paper, we study the oscillation of the solutions of neutral partial functional differential equation of the form

$$\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\right)\right]$$
$$=a(t)\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\Delta u(x,t-\rho_k(t))-q(x,t)u(x,t)$$
$$-\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\sigma_j)),(x,t)\in\Omega\times[0,\infty)\equiv G,$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, and

$$\Delta u(x,t)=\sum_{r=1}^{N}\frac{\partial^2 u(x,t)}{\partial x_r^2}.$$

In [7], Li and Cui have observed some oscillation properties of (1.1) under the following assumption

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{p(s)}=\infty,$$

in this paper, we will further the investigation and offer some new oscillation criteria of (1.1). Referencing the related work of [10], we will also study the oscillation and asymptotic behavior of (1.1) under the assumption

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{p(s)}<\infty.$$

We assume throughout this paper that the following conditions hold:

(A_1) $p(t)\in C^1([0,\infty);[0,\infty))$, $R(t)=\int_{t_0}^{t}\dfrac{ds}{p(s)}, t_0>0$;

(A_2) $\lambda_i\in C^2([0,\infty);[0,\infty))$, $0\leqslant\sum_{i=1}^{l}\lambda_i(t)\leqslant 1$, τ_i are nonnegative constants, $i\in I_l=\{1,2,\cdots,l\}$;

(A_3) $q,q_j\in C(\bar{G},(0,\infty))$, $q(t)=\min_{x\in\bar{\Omega}}q(x,t)$, $q_j(t)=\min_{x\in\bar{\Omega}}q_j(x,t)$, $j\in I_m=\{1,2,\cdots,m\}$;

(A_4) $a,a_k,\rho_k\in C([0,\infty);[0,\infty))$, $\lim_{t\to\infty}(t-\rho_k(t))=\infty$, σ_j are nonnegative constants, $j\in I_m, k\in I_s=\{1,2,\cdots,s\}$;

(A_5) $f_j\in C(\mathbb{R},\mathbb{R})$ are convex in $[0,\infty)$, $uf_j(u)>0$ and $\dfrac{f_j(u)}{u}\geqslant\alpha_j$ for $u\neq 0$, α_j are positive constants, $j\in I_m$.

We consider two kinds of boundary conditions,

$$\frac{\partial u(x,t)}{\partial\gamma}+g(x,t)u(x,t)=0,(x,t)\in\partial\Omega\times[0,\infty),\tag{1.1}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty)$, and

$$u(x,t)=0,(x,t)\in\partial\Omega\times[0,\infty).\tag{1.2}$$

As usual, a solution $u(x,t)$ of the problem (1.1), (1.2) (or (1.1), (1.3)) is called oscillatory in the domain $G=\Omega\times[0,\infty)$ if for any positive number μ there exists a point $(x_0,t_0)\in\Omega\times[\mu,\infty)$ such that $u(x_0,t_0)=0$ holds.

In the following two sections, by using a generalized Riccati transformation, we obtain some sufficient conditions for the oscillation of the problem (1.1), (1.2) as well as for (1.1), (1.3). We note that conditions for the oscillation of the solutions for $p(t)=1, \lambda_i(t)=0, f_j(u)=u$ have been obtained in the works of Cui et al.[4].

2. Oscillation of the problems (1.1) and (1.2)

First, we consider the case when $\lim_{t\to\infty} R(t) = \infty$.

THEOREM 2.1. *If there exists a* $j_0 \in I_m$ *such that*

$$\int^{\infty} \left\{ R(t-\sigma_{j_0}) \Big[\alpha_{j_0} q_{j_0}(t) \Big(1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big) + q(t)\Big(1 - \sum_{i=1}^{l} \lambda_i(t)\Big)\Big] - \frac{1}{4p(t-\sigma_{j_0})R(t-\sigma_{j_0})} \right\} dt = \infty, \quad (2.1)$$

then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Proof. Assume to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.2) in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. Without loss of generality we assume that $u(x,t) > 0, u(x,t-\tau_i) > 0, u(x,t-\rho_k(t)) > 0$ and $u(x,t-\sigma_j) > 0$ in $\Omega \times [t_1,\infty), t_1 \geqslant t_0, i \in I_l, k \in I_s, j \in I_m$.

Integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx + \sum_{i=1}^{l}\lambda_i(t)\int_\Omega u(x,t-\tau_i)dx\right)\right] \\
&= a(t)\int_\Omega \Delta u(x,t)dx + \sum_{k=1}^{s} a_k(t)\int_\Omega \Delta u(x,t-\rho_k(t))dx - \int_\Omega q(x,t)u(x,t)dx \\
&\quad - \sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))dx, \ t \geqslant t_1. \qquad (2.2)
\end{aligned}$$

From Green's formula and boundary condition (1.2), it follows that

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial \gamma} dS = -\int_{\partial\Omega} g(x,t)u(x,t)dS \leqslant 0, \ t \geqslant t_1, \qquad (2.3)$$

and for $t \geqslant t_1$, $k \in I_s$,

$$\begin{aligned}
\int_\Omega \Delta u(x,t-\rho_k(t))dx &= \int_{\partial\Omega} \frac{\partial u(x,t-\rho_k(t))}{\partial\gamma} dS \\
&= -\int_{\partial\Omega} g(x,t-\rho_k(t))u(x,t-\rho_k(t))dS \\
&\leqslant 0, \qquad (2.4)
\end{aligned}$$

where dS is the surface element on $\partial\Omega$. Moreover, from $(A_3),(A_5)$ and Jensen's inequality, it follows that

$$\int_\Omega q(x,t)u(x,t)dx \geqslant q(t)\int_\Omega u(x,t)dx, \ t \geqslant t_1, \qquad (2.5)$$

and

$$\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))dx \geqslant q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))dx$$
$$\geqslant q_j(t)\int_\Omega dx\cdot f_j\left(\int_\Omega u(x,t-\sigma_j)dx\left(\int_\Omega dx\right)^{-1}\right),\ \ t\geqslant t_1. \quad (2.6)$$

Let

$$V(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx,\ t\geqslant t_1, \quad (2.7)$$

where $|\Omega|=\int_\Omega dx$.

In view of (2.3)-(2.7), (2.2) yields

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\Big(V(t)+\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)\Big)\right]+q(t)V(t)$$
$$+\sum_{j=1}^{m}q_j(t)f_j(V(t-\sigma_j))\leqslant 0,\ t\geqslant t_1. \quad (2.8)$$

Let $Z(t)=V(t)+\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)$, we have $Z(t)>0$ and $[p(t)Z'(t)]'\leqslant 0$ for $t\geqslant t_1$. Hence $p(t)Z'(t)$ is a decreasing function in the interval $[t_1,\infty)$. We can claim that $p(t)Z'(t)>0$ for $t\geqslant t_1$. In fact, if there exist a $T>t_1$ such that $p(T)Z'(T)<0$, this implies that

$$Z'(t)\leqslant\frac{p(T)Z'(T)}{p(t)}\ \ \text{for}\ \ t\geqslant T,$$

and

$$Z(t)-Z(T)\leqslant p(T)Z'(T)\int_T^t\frac{ds}{p(s)},\ t\geqslant T.$$

Therefore $\lim_{t\to\infty}Z(t)=-\infty$, which contradicts the fact that $Z(t)>0$.

From (2.8), for the j_0 in (2.1) we obtain

$$[p(t)Z'(t)]'+q(t)V(t)+\alpha_{j_0}q_{j_0}(t)V(t-\sigma_{j_0})\leqslant 0,\ t\geqslant t_1. \quad (2.9)$$

or

$$[p(t)Z'(t)]'+q(t)\left[Z(t)-\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)\right]$$
$$+\alpha_{j_0}q_{j_0}(t)\left[Z(t-\sigma_{j_0})-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})V(t-\tau_i-\sigma_{j_0})\right]\leqslant 0,\ t\geqslant t_1.$$

Since $Z(t)\geqslant V(t)$, $Z(t)$ is increasing, it follows that

$$[p(t)Z'(t)]' + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big] Z(t)$$
$$+ \alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] Z(t-\sigma_{j_0}) \leqslant 0,\ t \geqslant t_1.$$

That is

$$[p(t)Z'(t)]' + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big] Z(t-\sigma_{j_0})$$
$$+ \alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] Z(t-\sigma_{j_0}) \leqslant 0,\ t \geqslant t_1.$$

Define

$$w(t) = R(t-\sigma_{j_0}) \frac{p(t)Z'(t)}{Z(t-\sigma_{j_0})}, \tag{2.10}$$

then $w(t) > 0$, and

$$\begin{aligned} w'(t) &= R'(t-\sigma_{j_0}) \frac{p(t)Z'(t)}{Z(t-\sigma_{j_0})} \\ &\quad + R(t-\sigma_{j_0}) \frac{[p(t)Z'(t)]'Z(t-\sigma_{j_0}) - p(t)Z'(t)Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})} \\ &\leqslant \frac{w(t)}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})} - R(t-\sigma_{j_0})\Big\{\alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] \\ &\quad + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big]\Big\} - \frac{R(t-\sigma_{j_0})p(t)Z'(t)Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})}. \end{aligned}$$

Using the fact that $p(t)Z'(t)$ is decreasing, we have

$$Z'(t-\sigma_{j_0}) \geqslant \frac{p(t)Z'(t)}{p(t-\sigma_{j_0})},\ t \geqslant t_1.$$

Thus,

$$\begin{aligned} w'(t) &\leqslant \frac{w(t)}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})} - R(t-\sigma_{j_0})\Big\{\alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] \\ &\quad + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big]\Big\} - \frac{R(t-\sigma_{j_0})[p(t)Z'(t)]^2}{p(t-\sigma_{j_0})Z^2(t-\sigma_{j_0})} \\ &= \frac{w(t)}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})} - R(t-\sigma_{j_0})\Big\{\alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] \\ &\quad + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big]\Big\} - \frac{w^2(t)}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1}{4} - \left(w(t) - \frac{1}{2}\right)^2}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})} - R(t-\sigma_{j_0})\Big\{\alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] \\
&\qquad + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big]\Big\} \\
&\leqslant \frac{1}{4p(t-\sigma_{j_0})R(t-\sigma_{j_0})} - R(t-\sigma_{j_0})\Big\{\alpha_{j_0} q_{j_0}(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big] \\
&\qquad + q(t)\Big[1 - \sum_{i=1}^{l} \lambda_i(t)\Big]\Big\}.
\end{aligned}
$$

Integrating the above inequality from some T_0 to t $(T_0 \geqslant t_1)$, we have

$$
\begin{aligned}
w(t) \leqslant w(T_0) - \int_{T_0}^{t} \Big\{ R(s-\sigma_{j_0})\Big[\alpha_{j_0} q_{j_0}(s)\Big(1 - \sum_{i=1}^{l} \lambda_i(s-\sigma_{j_0})\Big) \\
+ q(s)\Big(1 - \sum_{i=1}^{l} \lambda_i(s)\Big)\Big] - \frac{1}{4p(s-\sigma_{j_0})R(s-\sigma_{j_0})}\Big\} ds. \quad (2.11)
\end{aligned}
$$

Letting $t \to \infty$ in (2.11), from (2.1) we get contradiction. This completes the proof of Theorem 2.1.

COROLLARY 2.1. *If the inequality* (2.8) *has no eventually positive solutions, then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

COROLLARY 2.2. *Assume that for* $t_1 \geqslant t_0$,

$$
\begin{aligned}
\liminf_{t\to\infty} \frac{1}{\ln R(t-\sigma_{j_0})} \int_{t_1}^{t} R(s-\sigma_{j_0})\Big[\alpha_{j_0} q_{j_0}(s)\Big(1 - \sum_{i=1}^{l} \lambda_i(s-\sigma_{j_0})\Big) \\
+ q(s)\Big(1 - \sum_{i=1}^{l} \lambda_i(s)\Big)\Big] ds > \frac{1}{4}, \quad (2.12)
\end{aligned}
$$

then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Proof. It is not hard to verify that (2.12) yields the existence $\varepsilon > 0$ such that for all large t,

$$
\begin{aligned}
\frac{1}{\ln R(t-\sigma_{j_0})} \int_{t_1}^{t} R(s-\sigma_{j_0})\Big[\alpha_{j_0} q_{j_0}(s)\Big(1 - \sum_{i=1}^{l} \lambda_i(s-\sigma_{j_0})\Big) \\
+ q(s)\Big(1 - \sum_{i=1}^{l} \lambda_i(s)\Big)\Big] ds \geqslant \frac{1}{4} + \varepsilon,
\end{aligned}
$$

which follows that

$$\int_{t_1}^{t} R(s-\sigma_{j_0})\left[\alpha_{j_0} q_{j_0}(s)\left(1-\sum_{i=1}^{l}\lambda_i(s-\sigma_{j_0})\right)\right.$$
$$\left.+q(s)\left(1-\sum_{i=1}^{l}\lambda_i(s)\right)\right]ds \geqslant \frac{1}{4}\ln R(t-\sigma_{j_0})+\varepsilon\ln R(t-\sigma_{j_0}),$$

so we have

$$\int_{t_1}^{t}\left\{R(s-\sigma_{j_0})\left[\alpha_{j_0} q_{j_0}(s)\left(1-\sum_{i=1}^{l}\lambda_i(s-\sigma_{j_0})\right)+q(s)\left(1-\sum_{i=1}^{l}\lambda_i(s)\right)\right]\right.$$
$$\left.-\frac{1}{4p(s-\sigma_{j_0})R(s-\sigma_{j_0})}\right\}ds \geqslant \frac{1}{4}\ln R(t_1-\sigma_{j_0})+\varepsilon\ln R(t-\sigma_{j_0}). \quad (2.13)$$

Now, it is obvious that (2.13) implies (2.1) and this completes the proof of corollary 2.

COROLLARY 2.3. *Assume that*

$$\liminf_{t\to\infty} R^2(t-\sigma_{j_0})\left[\alpha_{j_0} q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right)\right.$$
$$\left.+q(t)\left(1-\sum_{i=1}^{l}\lambda_i(t)\right)\right]p(t-\sigma_{j_0}) > \frac{1}{4}, \quad (2.14)$$

then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillatory in* G.

Proof. It is easy to see that (2.14) yields the existence of an $\varepsilon>0$ such that for all large t,

$$R^2(t-\sigma_{j_0})\left[\alpha_{j_0} q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right)\right.$$
$$\left.+q(t)\left(1-\sum_{i=1}^{l}\lambda_i(t)\right)\right]p(t-\sigma_{j_0}) \geqslant \frac{1}{4}+\varepsilon,$$

that is

$$R(t-\sigma_{j_0})\left[\alpha_{j_0} q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right)+q(t)\left(1-\sum_{i=1}^{l}\lambda_i(t)\right)\right]$$
$$\geqslant \frac{1}{4p(t-\sigma_{j_0})R(t-\sigma_{j_0})}+\frac{\varepsilon}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})},$$

so we have

$$R(t-\sigma_{j_0})\left[\alpha_{j_0} q_{j_0}(t)\left(1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\right)+q(t)\left(1-\sum_{i=1}^{l}\lambda_i(t)\right)\right]$$

$$-\frac{1}{4p(t-\sigma_{j_0})R(t-\sigma_{j_0})} \geqslant \frac{\varepsilon}{p(t-\sigma_{j_0})R(t-\sigma_{j_0})}. \tag{2.15}$$

It is obvious that (2.1) holds and corollary 3 is evident by Theorem 1.

Now, let's consider the case when

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{p(s)} < \infty. \tag{2.16}$$

THEOREM 2.2. *Assume that* (2.1) *and* (2.16) *hold, suppose that there exists a continuously differentiable* $\rho(t)$ *such that* $\rho(t)>0$, $\rho'(t)\geqslant 0$, *we also suppose that* $\lambda_i'(t)\geqslant 0$ *for* $t\geqslant t_0$, $i\in I_l$, $\lim_{t\to\infty}\sum_{i=1}^{l}\lambda_i(t)=A$. *If for some* $j_0\in I_m$,

$$\int^{\infty}\frac{1}{\rho(t)p(t)}\left(\int^{t}\rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds\right)dt=\infty, \tag{2.17}$$

then every solution $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillates or*

$$\lim_{t\to\infty}\int_{\Omega}u(x,t)dx=0.$$

Proof. Suppose that $u(x,t)$ is a nonoscillatory solution of the problem (1.1), (1.2). Without loss of generality we assume that $u(x,t)$ is an eventually positive solution of the problem (1.1), (1.2). Then $Z(t)=V(t)+\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)>0$, from $[p(t)Z'(t)]'\leqslant 0$, it is easy to conclude that there exist two possible cases of the sign of $Z'(t)$:

Case (1): Suppose $Z'(t)>0$ for sufficiently large t, then we are back to the case of Theorem 1. Thus the proof of Theorem 1 goes through, and we get contradiction by (2.1).

Case (2): Suppose $Z'(t)\leqslant 0$ for sufficiently large t. From the following conditions

$$\lambda_i'(t)\geqslant 0, Z'(t)=V'(t)+\sum_{i=1}^{l}\lambda_i'(t)V(t-\tau_i)+\sum_{i=1}^{l}\lambda_i(t)V'(t-\tau_i),$$

we have $V'(t)\leqslant 0$, and hence there exists $\lim_{t\to\infty}Z(t)=a\geqslant 0$. Now we claim that $a=0$. Otherwise, $\lim_{t\to\infty}Z(t)=a>0$, so $\lim_{t\to\infty}V(t)=a/(1+A)>0$, there exists a constant $M>0$ such that $V(t)\geqslant M, V(t-\sigma_{j_0})\geqslant M$ for the j_0 in (2.17) and all $t\geqslant t_1\geqslant t_0$. From (2.9) we get

$$\begin{aligned}[p(t)Z'(t)]' &\leqslant -q(t)V(t)-\alpha_{j_0}q_{j_0}(t)V(t-\sigma_{j_0})\\ &\leqslant -Mq(t)-\alpha_{j_0}Mq_{j_0}(t)=-M(q(t)+\alpha_{j_0}q_{j_0}(t)),\ t\geqslant t_1.\end{aligned} \tag{2.18}$$

Define $Q(t)=\rho(t)p(t)Z'(t)$, then $Q(t)\leqslant 0$, from (2.18) we get

$$\begin{aligned}Q'(t) &= \rho(t)[p(t)Z'(t)]'+\rho'(t)p(t)Z'(t)\leqslant \rho(t)[p(t)Z'(t)]'\\ &\leqslant -M\rho(t)(q(t)+\alpha_{j_0}q_{j_0}(t)).\end{aligned}$$

Integrating it from t_1 to t, we get

$$Q(t) \leqslant Q(t_1) - M\int_{t_1}^{t} \rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds \leqslant -M\int_{t_1}^{t} \rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds,$$

that is

$$\rho(t)p(t)Z'(t) \leqslant -M\int_{t_1}^{t} \rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds,$$

so that

$$Z'(t) \leqslant -\frac{M}{\rho(t)p(t)}\int_{t_1}^{t} \rho(s)(q(s)+\alpha_{j_0}q_{j_0}(s))ds.$$

Integrating the above inequality from t_1 to t, we obtain

$$Z(t) \leqslant Z(t_1) - M\int_{t_1}^{t} \frac{1}{\rho(s)p(s)}\left(\int_{t_1}^{s} \rho(\xi)(q(\xi)+\alpha_{j_0}q_{j_0}(\xi))d\xi\right)ds.$$

We can easily obtain a contradiction. So that $\lim_{t\to\infty} Z(t) = 0$, then $\lim_{t\to\infty} V(t) = 0$. This completes the proof of Theorem 2.

COROLLARY 2.4. *If* (2.12), (2.16) *and* (2.17) *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)dx = 0$.

COROLLARY 2.5. *If* (2.14), (2.16) *and* (2.17) *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.2) *is oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)dx = 0$.

3. Oscillation of the problems (1.1) and (1.3)

The following fact will be used. The smallest eigenvalue β_0 of the Dirichlet problem

$$\begin{cases} \Delta w(x) + \beta w(x) = 0 & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

THEOREM 3.1. *If all conditions of Theorem 2.1 hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates in* G.

Proof. To the contrary, if there is a nonoscillatory solution $u(x,t)$ of the problem (1.1), (1.3) in $\Omega\times[t_0,\infty)$ for some $t_0>0$, without loss of generality, we assume that $u(x,t)>0$, $u(x,t-\tau_i)>0$, $u(x,t-\rho_k(t))>0$ and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty)$, $t_1\geqslant t_0$, $i\in I_l$, $k\in I_s$, $j\in I_m$.

Multiplying both side of (1.1) by $\varphi(x)>0$ and integrating it with respect to x over the domain Ω, for $t\geqslant t_1$ we have

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)\varphi(x)dx + \sum_{i=1}^{l}\lambda_i(t)\int_\Omega u(x,t-\tau_i)\varphi(x)dx\right)\right]$$

$$= a(t)\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_{k=1}^{s} a_k(t)\int_\Omega \Delta u(x,t-\rho_k(t))\varphi(x)dx$$
$$-\int_\Omega q(x,t)u(x,t)\varphi(x)dx - \sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\varphi(x)dx.$$

From Green's formula and boundary condition (1.3), it follows that

$$\int_\Omega \Delta u(x,t)\varphi(x)dx = \int_\Omega u(x,t)\Delta\varphi(x)dx = -\beta_0\int_\Omega u(x,t)\varphi(x)dx \leqslant 0,\ t\geqslant t_1,$$

and for $t\geqslant t_1$, $k\in I_s$,

$$\int_\Omega \Delta u(x,t-\rho_k(t))\varphi(x)dx = \int_\Omega u(x,t-\rho_k(t))\Delta\varphi(x)dx$$
$$= -\beta_0\int_\Omega u(x,t-\rho_k(t))\varphi(x)dx \leqslant 0. \tag{3.2}$$

From $(A_3),(A_5)$ and Jensen's inequality, it follows that

$$\int_\Omega q(x,t)u(x,t)\varphi(x)dx \geqslant q(t)\int_\Omega u(x,t)\varphi(x)dx,\ t\geqslant t_1, \tag{3.3}$$

and

$$\int_\Omega q_j(x,t)f_j(u(x,t-\sigma_j))\varphi(x)dx \geqslant q_j(t)\int_\Omega f_j(u(x,t-\sigma_j))\varphi(x)dx$$
$$\geqslant q_j(t)\int_\Omega \varphi(x)dx\cdot f_j\left(\int_\Omega u(x,t-\sigma_j)\varphi(x)dx\left(\int_\Omega\varphi(x)dx\right)^{-1}\right),\ t\geqslant t_1. \tag{3.4}$$

Set

$$V(t) = \int_\Omega u(x,t)\varphi(x)dx\left(\int_\Omega\varphi(x)dx\right)^{-1},\ t\geqslant t_1.$$

Combining (3.2)-(3.7) we obtain

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V(t)+\sum_{i=1}^{l}\lambda_i(t)V(t-\tau_i)\right)\right] + q(t)V(t)$$
$$+\sum_{j=1}^{m} q_j(t)f_j(V(t-\sigma_j)) \leqslant 0,\ t\geqslant t_1. \tag{3.5}$$

Since the remainder of the proof is similar to that of Theorem 1, we omit it.

COROLLARY 3.1. *If the inequality* (3.7) *has no eventually positive solutions, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillatory in* G.

The following conclusions can be proved analogously.

COROLLARY 3.2. *Let the conditions of Corollary 2.2 hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates in* G.

COROLLARY 3.3. *Let the conditions of Corollary 2.3 hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates.*

THEOREM 3.2. *Let the conditions of Theorem 2.2 hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)\varphi(x)dx = 0$ *in* G, *where* $\varphi(x)$ *is as in* (3.1).

COROLLARY 3.4. *If* (2.12), (2.16) *and* (2.17) *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)\varphi(x)dx = 0$, *where* $\varphi(x)$ *is as in* (3.1).

COROLLARY 3.5. *If* (2.14), (2.16) *and* (2.17) *hold, then every solution* $u(x,t)$ *of the problem* (1.1), (1.3) *is oscillates or* $\lim_{t\to\infty}\int_\Omega u(x,t)\varphi(x)dx = 0$, *where* $\varphi(x)$ *is as in* (3.1).

4. Examples

EXAMPLE 4.1. Consider the equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{t+\pi}\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{t+\pi}u(x,t-2\pi)\right)\right]\\
&\quad=\left(\frac{1}{t+\pi}+\frac{1}{(t+\pi)^2}-\frac{3}{(t+\pi)^4}\right)\Delta u(x,t)\\
&\qquad+\left(\frac{3}{(t+\pi)^3}+\frac{1}{(t+\pi)^2}\right)\Delta u(x,t-\frac{3\pi}{2})+\left(\frac{1}{2t^3\ln t}+\frac{1}{t^3}\right)\Delta u(x,t-\pi)\\
&\qquad-\left(\frac{1}{2t^3\ln t}+\frac{2}{t^3}\right)u(x,t)-\frac{1}{t^3}u(x,t-\pi),\ (x,t)\in(0,\pi)\times[0,\infty)
\end{aligned}\tag{4.1}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\ t\geqslant 0. \tag{4.2}$$

Here,

$$\begin{aligned}
&N=1,\ p(t)=\frac{1}{t+\pi},\ l=1,\ \lambda_1(t)=\frac{1}{t+\pi},\ \tau_1=2\pi,\\
&a(t)=\frac{1}{t+\pi}+\frac{1}{(t+\pi)^2}-\frac{3}{(t+\pi)^4},\ s=2,\\
&a_1(t)=\frac{3}{(t+\pi)^3}+\frac{1}{(t+\pi)^2},\ \rho_1(t)=\frac{3\pi}{2},\\
&a_2(t)=\frac{1}{2t^3\ln t}+\frac{1}{t^3},\ \rho_2(t)=\pi,
\end{aligned}$$

$$q(x,t) = \frac{1}{2t^3 \ln t} + \frac{2}{t^3},\ m = 1,$$
$$q_1(x,t) = \frac{1}{t^3},\ \sigma_1 = \pi,\ f_1(u) = u,$$

it is easy to see that

$$\begin{aligned}
q_{j_0}(t) &= q_1(t) = \frac{1}{t^3},\ \alpha_{j_0} = 1,\\
\lambda_1(t-\sigma_{j_0}) &= \lambda_1(t-\pi) = \frac{1}{t},\\
q(t) &= q(x,t) = \frac{1}{2t^3 \ln t} + \frac{2}{t^3},\\
p(t-\sigma_{j_0}) &= \frac{1}{t},\\
R(t) &= \int_{t_0}^{t} \frac{ds}{p(s)} = \frac{t^2}{2} + \pi t,
\end{aligned}$$

then we have

$$\begin{aligned}
\liminf_{t\to\infty} R^2(t-\sigma_{j_0}) \Bigg[\alpha_{j_0} q_{j_0}(t) \Big(1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\Big) \\
+ \Big(1 - \sum_{i=1}^{l} \lambda_i(t)\Big)\Bigg] p(t-\sigma_{j_0}) = \frac{3}{4} > \frac{1}{4},
\end{aligned}$$

which shows that all conditions of Corollary 8 are verified. Thus every solutions of problem (4.1), (4.2) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u(x,t) = \sin(x)\cos(t)$ is such a solution.

EXAMPLE 4.2. Consider the equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[(t+\pi)^2 \frac{\partial}{\partial t}\left(u(x,t) + \left(1 - \frac{1}{(t+\pi)^4}\right) u(x,t-2\pi)\right)\right]\\
&\quad = \left(1 - \frac{1}{t^2} + 3t + \frac{4}{(t+\pi)^2}\right)\Delta u(x,t)\\
&\qquad + \left(\frac{12}{(t+\pi)^4} + \frac{2}{(t+\pi)^3} + 4(t+\pi)\right)\Delta u(x,t-\frac{3\pi}{2})\\
&\qquad + \left(2t + \frac{1}{(t+\pi)^2}\right)\Delta u(x,t-\pi) - \left(\frac{\pi^2 t^2}{3} - \frac{1}{t^2} + \frac{2}{t^3} + 1\right) u(x,t)\\
&\qquad - \left(\frac{\pi^2 t^2}{3} + \frac{1}{t^3} + t + \frac{3}{(t+\pi)^2}\right) u(x,t-\pi),\quad (x,t)\in(0,\pi)\times[0,\infty),
\end{aligned} \tag{4.3}$$

with the boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, t \geqslant 0. \tag{4.4}$$

Here,

$$N=1,\ p(t)=(t+\pi)^2,\ l=1,\ \lambda_1(t)=1-\frac{1}{(t+\pi)^4},\ \tau_1=2\pi,$$

and

$$\begin{aligned}
a(t)&=1-\frac{1}{t^2}+3t+\frac{4}{(t+\pi)^2},\ s=2,\\
a_1(t)&=\frac{12}{(t+\pi)^4}+\frac{2}{(t+\pi)^3}+4(t+\pi),\\
a_2(t)&=2t+\frac{1}{(t+\pi)^2},\\
q(x,t)&=\frac{\pi^2t^2}{3}-\frac{1}{t^2}+\frac{2}{t^3}+1,\quad \text{and}\quad f_1(u)=u.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&q_{j_0}(t)=q_1(t)=q_1(x,t)=\frac{\pi^2t^2}{3}+\frac{1}{t^3}+t+\frac{3}{(t+\pi)^2},\ \alpha_{j_0}=1,\\
&\lambda_1(t-\sigma_{j_0})=\lambda_1(t-\pi)=\frac{1}{t},\\
&q(t)=q(x,t)=\frac{\pi^2t^2}{3}-\frac{1}{t^2}+\frac{2}{t^3}+1,\\
&p(t-\sigma_{j_0})=p(t-\pi)=(t+\pi)^2,\\
&R(t)=\int_0^t\frac{ds}{p(s)}=\int_0^t\frac{ds}{(s+\pi)^2}=\frac{1}{\pi}-\frac{1}{t+\pi},
\end{aligned}$$

let $\rho(s)=1$, then

$$\begin{aligned}
\liminf_{t\to\infty}R^2(t-\sigma_{j_0})\Big[\alpha_{j_0}q_{j_0}(t)\Big(1-\sum_{i=1}^{l}\lambda_i(t-\sigma_{j_0})\Big)&\\
+q(t)\Big(1-\sum_{i=1}^{l}\lambda_i(t)\Big)\Big]p(t-\sigma_{j_0})&=\frac{2}{3}>\frac{1}{4},
\end{aligned}$$

and

$$\int^{\infty}\left[\frac{1}{p(t)}\int_1^t(q(s)+q_1(s))ds\right]dt=\infty,$$

which shows that all conditions of Corollary 5 are verified. Thus every solution of problem (4.3), (4.4) oscillates or $\lim_{t\to\infty}\int_\Omega u(x,t)dx=0$ in $(0,\pi)\times[0,\infty)$. However, the main results of [7] fail to the problem (4.3), (4.4) because $\lim_{t\to\infty}R(t)<\infty$.

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