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INFINITELY MANY SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS

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Abstract. This paper is devoted to the study of infinitely many solutions for a class of Kirchhoff type problems on a bounded domain. Based on the Fountain Theorem of Bartsch, we obtain the multiplicity results, which unify and sharply improve the recent results of He and Zou [X. He, W. Zou, Multiplicity of solutions for a class of Kirchhoff type problems, Acta Math. Appl. Sin. (Engl. Ser.) 26 (2010) 387-394].

1. Introduction and main results

Consider the following Kirchhoff type problem:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N=1,2,3)$, $a,b>0$ and the nonlinearity $f\in C(\Omega\times\mathbb{R},\mathbb{R})$ satisfies the subcritical conditions:
(f_1) there exists $a_1>0$ such that

$$|f(x,t)|\leqslant a_1(1+|t|^{p-1}) \qquad \text{for some } 4<p<2^*=\begin{cases} 6, & n=3, \\ +\infty, & n=1,2. \end{cases}$$

Let $X:=H_0^1(\Omega)$ be the usual Sobolev space endowed with the norm $\|u\|=\left(\int_\Omega|\nabla u|^2dx\right)^{1/2}$. Since Ω is a bounded domain, $X\hookrightarrow L^r(\Omega)$ continuously for $r\in[1,2^*]$, compactly for $r\in[1,2^*)$, and then there exists $\tau_r>0$ such that

$$|u|_r\leqslant\tau_r\|u\|, \qquad \forall u\in X, \tag{1.2}$$

where $|\cdot|_r$ denotes the usual L^r-norm. The condition (f_1) implies that the functional $\varphi: X\to\mathbb{R}$,

$$\varphi(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\int_\Omega F(x,u)dx \tag{1.3}$$

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is well defined and of C^1 class, and

$$\langle\varphi'(u),v\rangle=(a+b\|u\|^2)\int_\Omega \nabla u\cdot\nabla v dx-\int_\Omega f(x,u)v dx,\qquad \forall u,v\in X.$$

The weak solutions of problem (1.1) are precisely the critical points of φ.

The existence and multiplicity of solutions of problem (1.1) have been extensively studied by many researchers via variational methods, see [1-9] and references therein. Perera and Zhang [10] obtain nontrivial solutions of problem (1.1) with asymptotically 4-linear terms using Yang index. In [14], they revisit problem (1.1) and establish the existence of a positive, a negative, and a sign-changing solution of problem (1.1) by means of invariant sets of descent flow. Similar results can also be found in Mao and Zhang [9]. Sun and Tang [11] prove the existence of a mountain pass type positive solution of problem (1.1) under the conditions that $f(x,t)$ is asymptotically linear near zero and superlinear at infinity. Moreover, infinitely many nontrivial solutions are established in [11] via the fountain theorem of Bartsch and the symmetric mountain pass lemma due to Kajikiya [6].

Recently, under the Ambrosetti-Rabinowitz's 4-superlinear condition (see (f_2') below), and no Ambrosetti-Rabinowitz's 4-superlinear condition, He and Zou [4] obtain the following two theorems via the fountain theorem and the variant fountain theorem.

THEOREM A. (see [4, Theorem 3]) *Assume that* $f(x,t)$ *satisfies* (f_1) *and:*
(f_2') *there exist* $\mu>4$ *and* $L>0$ *such that*

$$0<\mu F(x,t)\leqslant tf(x,t),\qquad \forall x\in\Omega,\ |t|\geqslant L;$$

(f_3) $f(x,-t)=-f(x,t)$ *for all* $(x,t)\in\Omega\times\mathbb{R}$.
Then problem (1.1) *has infinitely many solutions* (u_k) *such that* $\varphi(u_k)\to+\infty$ *as* $k\to\infty$.

THEOREM B. (see [4, Theorem 4]) *The conclusion of Theorem A holds, if* $f(x,t)$ *satisfies* (f_1), (f_3) *and:*
(f_2'') $f(x,t)t\geqslant 0$ *for* $t>0$; $\lim_{|t|\to\infty}\frac{F(x,t)}{t^4}=+\infty$ *uniformly for* $x\in\Omega$; *and* $f(x,t)=o(|t|)$ *as* $|t|\to 0$ *uniformly in* $x\in\Omega$;
(f_4') $\frac{1}{4}f(x,t)t-F(x,t)\to+\infty$ *as* $|t|\to\infty$ *uniformly in* $x\in\Omega$;
(f_5') $f(x,t)/t^3$ *is an increasing function of* $t\geqslant 0$.

In this work, with the aid of the classical Fountain Theorem of Bartsch, we can prove the same results under more general conditions, which unify and sharply improve Theorems A and B.

THEOREM 1.1. *Assume that* $f(x,t)$ *satisfies* (f_1), (f_3) *and:*
(f_2) $\lim\limits_{|t|\to\infty}\dfrac{F(x,t)}{t^4}=+\infty$ *uniformly for* $x\in\Omega$;
(f_4) *there exists* $L>0$ *such that*

$$tf(x,t)-4F(x,t)\geqslant 0,\qquad \forall x\in\Omega,\ |t|\geqslant L.$$

Then problem (1.1) *has infinitely many solutions* (u_k) *such that* $\varphi(u_k)\to+\infty$ *as* $k\to\infty$.

REMARK 1.1. Theorem 1.1 unifies and improves Theorems A and B. It follows from (f_2') that, for $x \in \Omega$, $|t| \geqslant L$ and $s \in [L/|t|, 1]$,

$$\frac{d}{ds}\left(\frac{F(x,st)}{s^{\mu}}\right) = \frac{f(x,st)st - \mu F(x,st)}{s^{\mu+1}} \geqslant 0,$$

which implies that

$$F(x,t) \geqslant \frac{|t|^{\mu}}{L^{\mu}} F(x, Lt/|t|) \geqslant \frac{|t|^{\mu}}{L^{\mu}} \inf_{x\in\Omega, |t|=L} F(x,t)$$

for all $x \in \Omega$ and $|t| \geqslant L$. Noticing that $\mu > 4$ and $\inf_{x\in\Omega,|t|=L} F(x,t) > 0$, the above inequality yields that

$$\frac{F(x,t)}{t^4} \geqslant \frac{|t|^{\mu-4}}{L^{\mu}} \inf_{x\in\Omega,|t|=L} F(x,t) \to +\infty \qquad \text{as } |t| \to \infty,$$

and then

$$tf(x,t) - 4F(x,t) \geqslant (\mu-4)F(x,t) \geqslant 0$$

for $|t|$ sufficiently large. Therefore, (f_2') implies (f_2) and (f_4), that is, Theorem 1.1 generalizes Theorem A. On the other hand, the conditions $f(x,t)t \geqslant 0$ for $t > 0$, $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \Omega$, and $f(x,t)/t^3$ is an increasing function of $t \geqslant 0$ in Theorem B are completely removed, and the uniformly coercivity condition (f_4') is replaced by the locally nonnegative condition (f_4). Hence Theorem 1.1 extends Theorem B. There are functionals $f(x,t)$ satisfying our Theorem 1.1 and not satisfying Theorems A and B. For example, let

$$f(x,t) = \begin{cases} 4t^3 \ln|t| + t^3, & |t| > 1, \\ -t^3|t| + 2t^3, & |t| \leqslant 1. \end{cases}$$

A simple computation shows that

$$F(x,t) = \begin{cases} t^4 \ln|t| + 3/10, & |t| > 1, \\ -\frac{1}{5}|t|^5 + \frac{1}{2}t^4, & |t| \leqslant 1, \end{cases}$$

and

$$tf(x,t) - 4F(x,t) = t^4 - \frac{6}{5}, \quad \forall x \in \Omega, \ |t| \geqslant 1.$$

Thus it is easy to check that f satisfies all the conditions of Theorem 1.1. But is does not satisfy the corresponding assumptions of Theorems A and B, because

$$\begin{aligned} tf(x,t) - \mu F(x,t) &= (4-\mu)t^4 \ln|t| + t^4 - \frac{3}{10}\mu \\ &= t^4\left[(4-\mu)\ln|t| + 1 - \frac{3\mu}{10t^4}\right] \\ &\to -\infty \qquad \text{as } |t| \to \infty \end{aligned}$$

for any $\mu > 4$, and $f(x,t)/t^3$ is nonincreasing in t for $t \in (0,1)$.

THEOREM 1.2. *The conclusion of Theorem 1.1 holds if we replace* (f_4) *with:*
(f_5) *there exists* $r>0$ *such that for all* $x\in\Omega$*,*

$$\frac{f(x,t)}{t^3} \text{ is increasing in } t \geqslant r.$$

REMARK 1.2. Theorem 1.2 also generalizes Theorem B, since the global monotonicity condition (f_5') is replaced by the local one (f_5) and we remove the conditions $f(x,t)\geqslant 0$ for $t>0$, $f(x,t)=o(|t|)$ as $|t|\to 0$ uniformly in x, and $\frac{1}{4}f(x,t)t-F(x,t)\to+\infty$ as $|t|\to\infty$ uniformly in x. Furthermore, Theorem 1.2 can be viewed as a useful complement of Sun and Tang [11, Theorem 3], in which a sequence of high energy solutions are obtained under (f_1), (f_2), (f_3) and the following condition introduced by Jeanjean [5]:
(f_5'') there exists $\theta\geqslant 1$ such that $\theta\mathscr{F}(x,t)\geqslant\mathscr{F}(x,st)$ for all $(x,t)\in\Omega\times\mathbb{R}$ and $s\in[0,1]$, where $\mathscr{F}(x,t):=f(x,t)t-4F(x,t)$.
Although (f_5'') is weaker than the assumption (f_5'), both (f_5') and (f_5'') are global conditions on $f(x,t)$, and therefore are not satisfactory. In Theorem 1.2 we consider the local condition (f_5) near infinity, which is a quite generic assumption.

2. Proofs of the theorems

We shall apply the Fountain Theorem (see [12, Theorem 3.6]) to find the critical points of φ. For the readers' convenience, we state it here.

Let X be a reflexive and separable Banach space, then there are $(e_n)_{n\in\mathbb{N}}\subset X$ and $(e_n^*)_{n\in\mathbb{N}}\subset X^*$ (the dual space of X) such that

$$X=\overline{\operatorname{span}\{e_n : n\in N\}},\ X^*=\overline{\operatorname{span}\{e_n^* : n\in\mathbb{N}\}},$$

and $\langle e_n,e_m\rangle=1$ if $n=m$, and $\langle e_n,e_m\rangle=0$ if $n\neq m$. Let $X_j=\operatorname{span}\{e_j\}$, then $X=\overline{\bigoplus_{j\geqslant 1}X_j}$. Now we define

$$Y_k=\bigoplus_{j=1}^{k}X_j \quad\text{and}\quad Z_k=\overline{\bigoplus_{j\geqslant k}X_j}. \tag{2.1}$$

Then we have the following Fountain Theorem.

PROPOSITION 2.1. (Fountain Theorem) *Assume that function* $\varphi\in C^1(X,R)$ *satisfies* $\varphi(-u)=\varphi(u)$*. For almost every* $k\in\mathbb{N}$*, there exist* $\rho_k>r_k>0$ *such that*
(i) $b_k:=\inf_{u\in Z_k,\|u\|=r_k}\varphi(u)\to+\infty$ *as* $k\to\infty$*;*
(ii) $a_k:=\max_{u\in Y_k,\|u\|=\rho_k}\varphi(u)\leqslant 0$*;*
(iii) φ *satisfies the Cerami condition (C), that is,* (u_n) *has a convergent subsequence in* X *whenever* $\{\varphi(u_n)\}$ *is bounded and* $\|\varphi'(u)\|(1+\|u_n\|)\to 0$ *as* $n\to\infty$*.*
Then φ *has a sequence of critical points* (u_k) *such that* $\varphi(u_k)\to+\infty$*.*

REMARK 2.1. In [12], the Fountain Theorem is established under the Palais-Smale (PS) condition. Since the Deformation Theorem is still valid under the Cerami condition, we see that like many critical point theorems, the Fountain Theorem holds true under the Cerami condition.

LEMMA 2.1. *Assume that* (f_1), (f_2) *and* (f_4) *hold; then the functional* φ *defined in* (1.3) *satisfies the Cerami condition (C).*

Proof. Let (u_n) be a Cerami sequence of φ. Since the embedding of $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ $(1 \leqslant r < 2^*)$ is compact, it suffices to show that (u_n) is bounded. If (u_n) is unbounded, up to a subsequence, we can assume that, for some $c_1 \in R$,

$$\varphi(u_n) \to c_1, \quad \|\varphi'(u_n)\|(1+\|u_n\|) \to 0 \quad \text{and} \quad \|u_n\| \to \infty \tag{2.2}$$

as $n \to \infty$. We consider $w_n = u_n/\|u_n\|$. Going if necessary to a subsequence, we may assume that

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{in } X, \\ &w_n \to w \qquad \text{in } L^r(\Omega) \ \ (1 \leqslant r < 2^*), \\ &w_n(x) \to w(x) \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{2.3}$$

We first consider the case $w = 0$. It follows from (f_1) that

$$|F(t,x)| \leqslant \int_0^1 |f(x,st)||t|ds \leqslant \int_0^1 a_1(|t| + s^{p-1}|t|^p)ds \leqslant a_1|t| + \frac{a_1}{p}|t|^p \tag{2.4}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Hence we have, for $x \in \Omega$ and $|t| \leqslant L$,

$$|tf(x,t) - 4F(x,t)| \leqslant 5a_1(|t| + |t|^p) \leqslant c_2|t|,$$

where $c_2 = 5a_1(1 + L^{p-1})$. This, together with (f_4), shows that

$$tf(x,t) - 4F(x,t) \geqslant -c_2|t|, \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Therefore,

$$\begin{aligned} \frac{1}{\|u_n\|^2}\left(\varphi(u_n) - \frac{1}{4}\langle \varphi'(u_n), u_n\rangle\right) &= \frac{a}{4} + \frac{1}{\|u_n\|^2}\int_\Omega \left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right)dx \\ &\geqslant \frac{a}{4} - \frac{c_2}{4\|u_n\|}\int_\Omega |w_n|dx, \end{aligned}$$

which implies that

$$0 \geqslant \frac{a}{4}$$

by (2.2) and (2.3). This is a contradiction.

For the second case $w \neq 0$, the set $\Omega_1 = \{x \in \Omega : w(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Omega_1$, we have $|u_n(x)| \to \infty$ as $n \to \infty$, so that, using (f_2),

$$\frac{F(x,u_n(x))}{|u_n(x)|^4}|w_n(x)|^4 \to +\infty \qquad \text{as } n \to \infty,$$

and then, via Fatou's Lemma (see [13]),

$$\int_{w\neq 0} \frac{F(x,u_n)}{|u_n|^4}|w_n|^4 dx \to +\infty \qquad \text{as } n \to \infty. \tag{2.5}$$

On the other hand, by (f_2), there exists $L_1 > 0$ such that

$$F(x,t) \geqslant 0, \qquad \forall x \in \Omega, \ \ |t| \geqslant L_1. \tag{2.6}$$

From (2.4), one has

$$|F(x,t)| \leqslant c_3|t|, \qquad \forall x \in \Omega, \ \ |t| \leqslant L_1,$$

where $c_3 = a_1 + a_1 L_1^{p-1}/p$. Combining this with (2.6), one has

$$F(x,t) \geqslant -c_3|t|, \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Hence we obtain, using (1.2),

$$\int_{w=0} \frac{F(x,u_n)}{\|u_n\|^4}dx \geqslant -\frac{c_3\int_{w=0}|u_n|dx}{\|u_n\|^4} \geqslant -\frac{c_3|u_n|_1}{\|u_n\|^4} \geqslant -\frac{c_3\tau_1\|u_n\|}{\|u_n\|^4},$$

which implies that

$$\liminf_{n\to\infty} \int_{w=0} \frac{F(x,u_n)}{\|u_n\|^4}dx \geqslant 0. \tag{2.7}$$

Note

$$\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 = \varphi(u_n) + \int_\Omega F(x,u_n)dx, \qquad \forall n.$$

Dividing both sides by $\|u_n\|^4$ and letting $n \to \infty$, we deduce via (2.7), (2.5) and the first limit of (2.2) that

$$\frac{b}{4} = \lim_{n\to\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|^4}dx = \lim_{n\to\infty} \left(\int_{w=0} + \int_{w\neq 0}\right) \frac{F(x,u_n)}{|u_n|^4}|w_n|^4 dx = +\infty.$$

This is impossible.

In any case, we deduce a contradiction. Hence (u_n) is bounded. □

Similar to the proof of [7, Lemma 2.3], we have the following lemma.

LEMMA 2.2. *If (f_5) holds, then for any $x \in \Omega$, $\mathscr{F}(x,t)$ is increasing in $t \geqslant L$ and decreasing in $t \leqslant -L$, where $\mathscr{F}(x,t)$ is the same as in (f_5''). In particular, there exists a constant $c_4 > 0$ such that*

$$\mathscr{F}(x,s) \leqslant \mathscr{F}(x,t) + c_4 \tag{2.8}$$

for all $x \in \Omega$ and $|s| \leqslant |t|$.

LEMMA 2.3. *Assume that (f_1), (f_2) and (f_5) hold; then the functional φ defined in (1.3) satisfies the (C) condition.*

Proof. Like in the proof of Lemma 2.1, it suffices to consider the case $w = 0$ and $w \neq 0$.

If $w = 0$, inspired by [5], we choose a sequence $(s_n) \subset R$ such that

$$\varphi(s_n u_n) = \max_{s \in [0,1]} \varphi(s u_n).$$

For any $m > 0$, letting $v_n = \sqrt{2m} w_n$, one has

$$v_n \to 0 \quad \text{in } L^r(\Omega) \ \ (1 \leqslant r < 2^*) \ \text{ and } \ v_n(x) \to 0 \quad \text{a.e. } x \in \Omega \tag{2.9}$$

by (2.3). Since

$$|F(x,u_n)| \leqslant a_1 |v_n| + \frac{a_1}{p} |v_n|^p \in L^1(\Omega),$$

using Lebesgue dominated convergence theorem and the second limit of (2.9), we have

$$\lim_{n \to \infty} \int_\Omega F(x,v_n) dx = \int_\Omega F(x,0) dx = 0.$$

Now, for n sufficiently large, $\sqrt{2m}\|u_n\|^{-1} \in (0,1)$, we obtain

$$\begin{aligned} \varphi(s_n u_n) &\geqslant \varphi(v_n) \\ &\geqslant \frac{a}{2}\|v_n\|^2 - \int_\Omega F(x,v_n) dx \\ &\geqslant am - \int_\Omega F(x,v_n) dx, \end{aligned}$$

which implies that $\liminf_{n \to \infty} \varphi(s_n u_n) \geqslant am$. By the arbitrariness of m, we have

$$\lim_{n \to \infty} \varphi(s_n u_n) = +\infty. \tag{2.10}$$

Since $\varphi(0) = 0$ and $\varphi(u_n) \to c_1$ $(n \to \infty)$, we see that for n large enough, $s_n \in (0,1)$, and

$$a \int_\Omega |\nabla(s_n u_n)|^2 dx + b\left(\int_\Omega |\nabla(s_n u_n)|^2 dx\right)^2 - \int_\Omega f(x, s_n u_n) s_n u_n dx$$

$$= \langle \varphi'(s_n u_n), s_n u_n \rangle = s_n \left.\frac{d}{ds}\right|_{s=s_n} \varphi(s u_n) = 0.$$

Therefore, using (2.10) and (2.8),

$$\begin{aligned}
&\frac{a}{4}\int_\Omega |\nabla u_n|^2 dx + \int_\Omega \left(\frac{1}{4} f(x,u_n)u_n - F(x,u_n)\right) dx \\
&\quad \geqslant \frac{a}{4}\int_\Omega |\nabla (s_n u_n)|^2 dx + \int_\Omega \left(\frac{1}{4} f(x,s_n u_n)s_n u_n - F(x,s_n u_n)\right) dx - \frac{c_4}{4}|\Omega| \\
&\quad \geqslant \frac{a}{2}\int_\Omega |\nabla (s_n u_n)|^2 dx + \frac{b}{4}\left(\int_\Omega |\nabla (s_n u_n)|^2 dx\right)^2 - \int_\Omega F(x,s_n u_n) dx - \frac{c_4}{4}|\Omega| \\
&\quad = \varphi(s_n u_n) - \frac{c_4}{4}|\Omega| \\
&\quad \to +\infty.
\end{aligned}$$

However, (2.2) implies that

$$\begin{aligned}
\frac{a}{4}\int_\Omega |\nabla u_n|^2 dx + \int_\Omega \left(\frac{1}{4} f(x,u_n)u_n - F(x,u_n)\right) dx &= \varphi(u_n) - \frac{1}{4}\langle \varphi'(u_n), u_n \rangle \\
&\to c_1,
\end{aligned}$$

a contradiction.

If $w \neq 0$, the proof is identical to that of Lemma 2.1.

Thus (u_n) is bounded. $\square$

PROOF OF THEOREM 1.1 For the Hilbert space $X = H_0^1(\Omega)$, denoted by $0 < \lambda_1 < \lambda_2 < \cdots$ the distinct Dirichlet eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, and by $e_1, e_2, e_3, \cdots$ the eigenfunctions corresponding to the eigenvalues. Then define Y_k and Z_k as in (2.1), where $X_j = \text{span}(e_j)$. According to Lemma 2.1 and the oddness of f, we know that φ satisfies the (C) condition and $\varphi(-u) = \varphi(u)$. It remains to verify the conditions (i) and (ii) of Proposition 2.1.

Verification of (i). For $1 \leqslant r < 2^*$, taking

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} |u|_r,$$

one has $\beta_k \to 0$ as $k \to \infty$ (see [12, Lemma 3.8]). Set

$$r_k := \left(\frac{bp}{16 a_1 \beta_k^p}\right)^{1/(p-4)}.$$

Since $p > 4$, we get

$$r_k \to +\infty \quad \text{as } k \to \infty.$$

So choosing k large enough such that $r_k \geqslant \left(\frac{16 a_1 \tau_1}{b}\right)^{1/3}$, we obtain, for $u \in Z_k$ with $\|u\| = r_k$,

$$\varphi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u) dx$$

$$\begin{aligned}
&\geqslant \frac{b}{4}\|u\|^4 - a_1|u|_1 - \frac{a_1}{p}|u|_p^p \\
&\geqslant \frac{b}{4}\|u\|^4 - a_1\tau_1\|u\| - \frac{a_1}{p}\beta_k^p\|u\|^p \\
&\geqslant \frac{br_k^4}{8}
\end{aligned}$$

by (2.4) and (1.2), which yields that

$$\inf_{u\in Z_k,\|u\|=r_k}\varphi(u) \geqslant \frac{br_k^4}{8} \to +\infty \qquad \text{as } k\to\infty.$$

Verification of (ii). Since on the finite-dimensional space Y_k all norms are equivalent, there exists $C_k > 0$ such that

$$C_k|u|_4 \geqslant \|u\|, \qquad \forall u\in Y_k. \tag{2.11}$$

From (f_2) we deduce that, there exists $\delta_k > 0$ such that

$$F(x,t) \geqslant C_k^4 bt^4, \qquad \forall x\in\Omega,\ |t|\geqslant \delta_k.$$

By (2.4), one has

$$|F(x,t)| \leqslant \left(a_1 + \frac{a_1}{p}\delta_k^{p-1}\right)|t|, \qquad \forall x\in\Omega,\ |t|\leqslant\delta_k,$$

which implies that

$$F(x,t)\geqslant C_k^4 bt^4 - c_5|t|, \qquad \forall (x,t)\in\Omega\times\mathbb{R},$$

where $c_5 = C_k^4 b\delta_k^3 + a_1 + a_1\delta_k^{p-1}/p$. Combining this with (2.11) and (1.2), we obtain

$$\begin{aligned}
\varphi(u) &\leqslant \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_k^4 b|u|_4^4 + c_5|u|_1 \\
&\leqslant \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - b\|u\|^4 + c_5\tau_1\|u\| \\
&\leqslant \frac{a}{2}\|u\|^2 + c_5\tau_1\|u\| - \frac{3b}{4}\|u\|^4
\end{aligned} \tag{2.12}$$

for all $u\in Y_k$. So choosing

$$\rho_k > \max\left\{\left(\frac{2a}{b}\right)^{1/2}, \left(\frac{4c_5\tau_1}{b}\right)^{1/3}, r_k\right\},$$

inequality (2.12) implies that

$$\max_{u\in Y_k,\|u\|=\rho_k}\varphi(u) \leqslant -\frac{b\rho_k^4}{4} < 0.$$

Consequently, by Proposition 2.1, φ possesses a sequence of critical points (u_k) such that $\varphi(u_k) \to +\infty$ as $k \to \infty$. $\square$

PROOF OF THEOREM 1.2 By virtue of Lemma 2.3 and assumption (f_3), we see that φ satisfies the (C) condition and is even in u. Like in the proof of Theorem 1.1, assumptions (f_1) and (f_2) indicate that φ satisfies the conditions (i) and (ii) of Proposition 2.1. Hence Theorem 1.2 holds. $\square$

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