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SYSTEMS OF ELLIPTIC EQUATIONS INVOLVING MULTIPLE INVERSE–SQUARE POTENTIALS AND CRITICAL EXPONENTS

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Abstract. In this paper, a system of elliptic equations is investigated, which involves multiple critical Sobolev exponents and singular points. The best Sobelev constant related to the system is studied, which is verified to be independent of the location of singular points. By a variant of the concentration compactness principle and the mountain-pass argument, the existence of positive solutions to the system is proved. At last, the existence of sign-changing solutions to the system is also established on the basis of the mountain-pass-type positive solutions.

1. Introduction

In this paper, we study the following system of elliptic equations:

$$\begin{cases} -\Delta u - \sum_{i=1}^{k} \dfrac{\lambda_i u}{|x-a_i|^2} = \dfrac{\sigma\alpha}{2^*}|u|^{\alpha-2}|v|^{\beta}u + |u|^{2^*-2}u + \sigma_1 u + \sigma_2 v, \\ -\Delta v - \sum_{i=1}^{k} \dfrac{\mu_i v}{|x-b_i|^2} = \dfrac{\sigma\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v + |v|^{2^*-2}v + \sigma_2 u + \sigma_3 v, \\ u, v \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N\,(N \geqslant 3)$ is a bounded domain with the smooth boundary $\partial\Omega$ such that

$$a_i,\ b_i \in \Omega,\ \lambda_i, \mu_i < \bar{\lambda} := \left(\frac{N-2}{2}\right)^2, 1 \leqslant i \leqslant k,$$

$$\sigma \geqslant 0,\ \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R},\ \alpha, \beta > 1,\ \alpha + \beta = 2^* := \frac{2N}{N-2},$$

$H_0^1(\Omega) =: H$ denotes the completion of $C_0^\infty(\Omega)$ with respect to

$$\left(\int_\Omega |\nabla \cdot|^2 \,\mathrm{d}x\right)^{1/2},$$

$\bar{\lambda}$ is the best Hardy constant and 2^* is the critical Sobolev exponent.

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Define the functional of (1.1) on the product space $H_0^1(\Omega)\times H_0^1(\Omega)=:H\times H$ by

$$J(u,v):=\frac{1}{2}\int_\Omega\Big(|\nabla u|^2+|\nabla v|^2-\sum_{i=1}^k\Big(\frac{\lambda_i u^2}{|x-a_i|^2}+\frac{\mu_i v^2}{|x-b_i|^2}\Big)\Big)\mathrm{d}x$$
$$-\frac{1}{2^*}\int_\Omega\big(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\big)\mathrm{d}x-\frac{1}{2}\int_\Omega\big(\sigma_1u^2+2\sigma_2uv+\sigma_3v^2\big)\mathrm{d}x.$$

Then $J\in C^1(H\times H,\mathbb{R})$ and $(u_0,v_0)\in H\times H$ is said to be a solution of (1.1) if

$$(u_0,v_0)\neq(0,0),\ \ \langle J'(u_0,v_0),(\varphi,\phi)\rangle=0,\ \ \forall(\varphi,\phi)\in H\times H,$$

where $J'(u_0,v_0)$ denotes the Fréchet derivative of J at (u_0,v_0). A solution of (1.1) is equivalent to a nonzero critical point of J, and standard elliptic argument shows that

$$u_0,v_0\in C^2(\Omega\setminus\{a_i,b_i,1\leqslant i\leqslant k\})\cap C^1(\bar{\Omega}\setminus\{a_i,b_i,1\leqslant i\leqslant k\}). \tag{1.2}$$

By (1.2), the singularities of u_0 and v_0 may occur at the points a_i and $b_i(1\leqslant i\leqslant k)$.

To study (1.1), the following Hardy inequality is used ([13]):

$$\int_{\mathbb{R}^N}\frac{|u|^2}{|x-a|^2}\mathrm{d}x\leqslant\frac{1}{\bar{\lambda}}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x,\quad\forall\,u\in C_0^\infty(\mathbb{R}^N),\ a\in\mathbb{R}^N. \tag{1.3}$$

Let $D^{1,2}(\mathbb{R}^N)=:D$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N}|\nabla\cdot|^2\mathrm{d}x)^{1/2}$. Then the following best constant is well defined by (1.3):

$$S(\lambda):=\inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}\big(|\nabla u|^2-\lambda\frac{u^2}{|x-a|^2}\big)\mathrm{d}x}{\big(\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x\big)^{\frac{2}{2^*}}},\ \ \lambda<\bar{\lambda},\ a\in\mathbb{R}^N. \tag{1.4}$$

For all $\lambda\in[0,\bar{\lambda})$, $S(\lambda)$ is achieved by the extremal functions ([23]):

$$V_{\lambda,\varepsilon}^a(x):=\varepsilon^{-\frac{N-2}{2}}U_\lambda\big(\varepsilon^{-1}|x-a|\big),\quad\forall\lambda\in[0,\bar{\lambda}),\ \varepsilon>0, \tag{1.5}$$

where

$$U_\lambda(x)=\Big(\frac{4N(\bar{\lambda}-\lambda)}{N-2}\Big)^{\frac{\sqrt{\bar{\lambda}}}{2}}\Big(|x|^{\frac{\sqrt{\bar{\lambda}}-\sqrt{\bar{\lambda}-\lambda}}{\sqrt{\bar{\lambda}}}}+|x|^{\frac{\sqrt{\bar{\lambda}}+\sqrt{\bar{\lambda}-\lambda}}{\sqrt{\bar{\lambda}}}}\Big)^{-\sqrt{\bar{\lambda}}}.$$

Similarly, for all $\lambda,\ \mu\in(-\infty,\bar{\lambda})$, $\sigma\in[0,\infty)$ and $a,b\in\mathbb{R}^N$, by the Hardy, Young and Sobolev inequalities, the following best constants are well defined on the space $\mathscr{D}:=(D^{1,2}(\mathbb{R}^N)\setminus\{0\})^2$:

$$S_{\sigma,\alpha,\beta}(\lambda,\mu):=\inf_{(u,v)\in\mathscr{D}}\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^2+|\nabla v|^2-\frac{\lambda u^2}{|x-a|^2}-\frac{\mu v^2}{|x-b|^2}\Big)\mathrm{d}x}{\Big(\int_{\mathbb{R}^N}\big(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\big)\mathrm{d}x\Big)^{\frac{2}{2^*}}}, \tag{1.6}$$

$$\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu) := \inf_{(u,v)\in\mathscr{D}} \frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^2+|\nabla v|^2-\dfrac{\lambda u^2+\mu v^2}{|x|^2}\Big)\mathrm{d}x}{\Big(\int_{\mathbb{R}^N}\big(|u|^{2^*}+|v|^{2^*}+\sigma|u|^{\alpha}|v|^{\beta}\big)\mathrm{d}x\Big)^{\frac{2}{2^*}}}. \tag{1.7}$$

The relations between two constants are verified in Theorem 1 of this paper.

In recent years, much attention has been paid to the singular problems involving the Hardy inequality and critical exponents, and many important conclusions have been established (e.g. [1], [6], [7], [9], [10], [12], [16], [18], [23] and the references therein). On the other hand, the elliptic systems involving the Hardy inequality have been also studied, several results can be found (e.g. [2], [4], [14], [15], [17], [21]), and many challenging topics remain open. Therefore it is necessary for us to investigate the related singular systems deeply.

To continue, we define

$$Q_1(u,v) := \int_{\Omega}\Big(|\nabla u|^2+|\nabla v|^2-\sum_{i=1}^{k}\Big(\frac{\lambda_i u^2}{|x-a_i|^2}+\frac{\mu_i v^2}{|x-b_i|^2}\Big)\Big)\mathrm{d}x, \tag{1.8}$$

$$Q_2(u,v) := (u,v)A(u,v)^T = \sigma_1 u^2+2\sigma_2 uv+\sigma_3 v^2,\ \ A := \begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_2 & \sigma_3\end{pmatrix}, \tag{1.9}$$

$$\bar{\Lambda}_1 := \inf_{(u,v)\in H\times H\setminus\{(0,0)\}} \frac{Q_1(u,v)}{\int_{\Omega}(u^2+v^2)\mathrm{d}x}.$$

Some assumptions are needed in this paper:

$(\mathscr{H}_1)$ $N\geqslant 3,\ k\geqslant 2,\ \sigma\geqslant 0,\ \lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k<\bar{\lambda},\ \ \mu_1\leqslant\mu_2\leqslant\cdots\leqslant\mu_k<\bar{\lambda},$ $a_i,b_i\in\Omega,\ a_i\neq a_j,\ b_i\neq b_j, i,j\in\{1,2,\cdots,k\}, i\neq j,\ \ \alpha,\beta>1,\ \alpha+\beta=2^*,$

$$\sum_{1\leqslant i\leqslant k,\,\lambda_i>0}\lambda_i<\bar{\lambda},\qquad \sum_{1\leqslant i\leqslant k,\,\mu_i>0}\mu_i<\bar{\lambda}.$$

$(\mathscr{H}_2)$ $\sigma_i>0, i=1,2,3,\ \ \sigma_1\sigma_3-\sigma_2^2>0,\ \ 0<\theta_1\leqslant\theta_2<\bar{\Lambda}_1$, where θ_1 and θ_2 are the eigenvalues of the matrix A.

Under $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, $Q_1(u,v)$ and $Q_2(u,v)$ are positive definite. Furthermore,

$$\theta_1(u^2+v^2)\leqslant Q_2(u,v)\leqslant\theta_2(u^2+v^2),\quad \forall u,v\in H.$$

Define

$$f(\tau) := \frac{1+\tau^2}{(1+\sigma\tau^{\beta}+\tau^{\alpha+\beta})^{\frac{2}{\alpha+\beta}}},\quad \tau\geqslant 0, \tag{1.10}$$

$$f(\tau_{\min}) := \min_{\tau\geqslant 0} f(\tau)>0, \tag{1.11}$$

where $\tau_{\min}\geqslant 0$ is the unique minimal point of $f(\tau)$, $\tau\geqslant 0$. When $\sigma=0$, we have $\tau_{\min}=0$ and $f(\tau_{\min})=1$.

The main results of this paper are summarized in the following theorems. To the best of our knowledge, the conclusions are new.

THEOREM 1. *Suppose that* $\lambda,\mu<\bar{\lambda}$, $a,b\in\mathbb{R}^N$, $\sigma\geqslant 0$, $\alpha,\beta>1,\alpha+\beta=2^*$. *Then the best constant* $S_{\sigma,\alpha,\beta}(\lambda,\mu)$ *defined in* (1.6) *is independent of the singular points* a *and* b *in the sense that,*

(i) $S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu),\ \forall a,b\in\mathbb{R}^N,\ \forall\lambda,\mu\in[0,\bar{\lambda});$

(ii) $S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(\lambda,0),\ \forall a,b\in\mathbb{R}^N,a\neq b,\ \forall\lambda\in[0,\bar{\lambda}),\ \mu\in(-\infty,0],$

$S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(0,\mu),\ \forall a,b\in\mathbb{R}^N,a\neq b,\ \forall\lambda\in(-\infty,0],\ \mu\in[0,\bar{\lambda});$

(iii) $S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu),\ \forall a=b\in\mathbb{R}^N,\ \forall\lambda,\mu\in(-\infty,\bar{\lambda});$

(iv) $S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(0,0),\ \forall a,b\in\mathbb{R}^N,\ \forall\lambda,\mu\in(-\infty,0].$

Theorem 1 reveals that $S_{\sigma,\alpha,\beta}(\lambda,\mu)$ depends closely on λ and μ, and is independent of the singular points a and b. The result is crucial when establishing a local $(PS)_c$ condition of the functional J by the concentration compactness arguments.

THEOREM 2. *Suppose that* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *hold,* $N\geqslant 4$, $0\leqslant\lambda_k=\mu_k\leqslant\bar{\lambda}-1$, $a_k=b_k$ *and* $\mathscr{C}_1>0$, *where*

$$\mathscr{C}_1:=\sigma_1+2\sigma_2\tau_{\min}+\sigma_3(\tau_{\min})^2+\sum_{i=1}^{k-1}\Big(\frac{\lambda_i}{|a_i-a_k|^2}+\frac{(\tau_{\min})^2\mu_i}{|b_i-b_k|^2}\Big).$$

Then the problem (1.1) *has a positive solution.*

Theorem 2 is verified by the Mountain-Pass theorem and reveals that, the existence of mountain-pass-type positive solutions to (1.1) depends mainly on the location and strength of the singular points $a_i,b_i,i=1,2,\cdots,k$. The condition $a_k=b_k$ and $\lambda_k=\mu_k\in[0,\bar{\lambda})$ means that, to ensure the existence results, the singularities of a_k and b_k must be exactly the same. Note that the condition $\mathscr{C}_1>0$ can be satisfied easily, and the location and strength of the singular points $a_i,b_i,i=1,2,\cdots,k-1$, can be chosen arbitrarily under the condition $\mathscr{C}_1>0$.

THEOREM 3. *Suppose that* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *hold,* $N\geqslant 7$, $0\leqslant\lambda_k=\mu_k\leqslant\bar{\lambda}-4$, $\sigma=0$, $a_k=b_k$ *and* $\mathscr{C}_1>0$. *Furthermore, assume that either* $\mathscr{C}_2>0$ *or* $\mathscr{C}_3>0$, *where*

$$\mathscr{C}_2:=\sigma_1+\sum_{i=1}^{k-1}\frac{\lambda_i}{|a_i-a_k|^2},\qquad \mathscr{C}_3:=\sigma_3+\sum_{i=1}^{k-1}\frac{\mu_i}{|b_i-b_k|^2}.$$

Then the problem (1.1) *has a pair of sign-changing solutions.*

Theorem 3 shows that, the existence of sign-changing solutions to (1.1) can be ensured when $\sigma=0$, i.e., when the strongly-coupled terms $|u|^{\alpha-2}|v|^{\beta}u$ and $|u|^{\alpha}|v|^{\beta-2}v$ disappear in (1.1). If $\sigma>0$ and the strongly-coupled terms appear in (1.1), the sign-changing solutions of (1.1) could not be ensured (e.g. [21]).

When $k=1$ in (1.1), the existence of both positive and sign-changing solutions has been established in [15] under the condition $a_1=b_1,\ \lambda_1=\mu_1$. However, when

$k \geqslant 2$ and there are multiple singular points in (1.1), some new questions appear, such as the independence of $S_{\sigma,\alpha,\beta}(\lambda,\mu)$ with respect to the singular points a and b, and the variant of concentration compactness principle for elliptic systems. In this paper, we overcome all these difficulties and establish the existence results. Our ideas and technics are new and can be applied to the related elliptic systems.

This paper is organized as follows: Theorem 1 and some preliminary results are verified in Section 2, Theorem 2 is proved in Section 3 and Theorem 3 is established in Section 4. In the following argument, $\|u\| = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{1/2}$ denotes the norm of the space H, $\|(u,v)\|_{H\times H} = (\|u\|^2 + \|v\|^2)^{1/2}$ is the norm of the space $H\times H$, $O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leqslant C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ is a generic infinitesimal value. In particular, the quantity $O_1(\varepsilon^t)$ means that there exist the constants $C_1, C_2 > 0$ such that $C_1\varepsilon^t \leqslant O_1(\varepsilon^t) \leqslant C_2\varepsilon^t$ as ε small. We always denote positive constants as C and omit $\mathrm{d}x$ in integrals for convenience.

2. Palais-Smale condition and the best constant

We first verify that the best constant $S_{\sigma,\alpha,\beta}(\lambda,\mu)$ defined in (1.6) is independent of the singular points a and b, and establish Theorem 1.

PROOF OF THEOREM 1. We need to consider several cases.

(i) $\lambda \geqslant 0$, $\mu \geqslant 0$, $a,b \in \mathbb{R}^N$.

For all $w \in D^{1,2}(\mathbb{R}^N)$ such that $w \geqslant 0$ a.e. in $\mathbb{R}^N$, let $w^*(x)$ be the Schwarz symmetrization of w, i.e.

$$w^*(x) = \inf\{t > 0 : |\{y \in \mathbb{R}^N, w(y) > t\}| \leqslant \omega_N |x|^N\}.$$

Suppose that $a,b \in \mathbb{R}^N, u,v \in D^{1,2}(\mathbb{R}^N)$ such that $u,v \geqslant 0$ a.e. in $\mathbb{R}^N$. From Corollary 21.7 and Theorem 21.8 in [26] it follows that

$$\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \leqslant \int_{\mathbb{R}^N} |u^*(x)|^\alpha |v^*(x)|^\beta,$$

$$\int_{\mathbb{R}^N} |u|^{2^*} = \int_{\mathbb{R}^N} |u^*(x)|^{2^*}, \quad \int_{\mathbb{R}^N} |v|^{2^*} = \int_{\mathbb{R}^N} |v^*(x)|^{2^*},$$

$$\int_{\mathbb{R}^N} \frac{u^2}{|x-a|^2} \leqslant \int_{\mathbb{R}^N} |u^*(x)|^2 \Big(\Big(\frac{1}{|x-a|} \Big)^* \Big)^2 = \int_{\mathbb{R}^N} \frac{|u^*(x)|^2}{|x|^2},$$

$$\int_{\mathbb{R}^N} \frac{v^2}{|x-a|^2} \leqslant \int_{\mathbb{R}^N} |v^*(x)|^2 \Big(\Big(\frac{1}{|x-a|} \Big)^* \Big)^2 = \int_{\mathbb{R}^N} \frac{|v^*(x)|^2}{|x|^2},$$

where we have used the fact that $\big(\frac{1}{|x-\xi|}\big)^* = \frac{1}{|x|}$ for all $\xi \in \mathbb{R}^N$. From the Pólya-Szegö inequality it follows that

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \leqslant \int_{\mathbb{R}^N} |\nabla u|^2, \quad \int_{\mathbb{R}^N} |\nabla v^*|^2 \leqslant \int_{\mathbb{R}^N} |\nabla v|^2.$$

Therefore, for all $u,v\in D^{1,2}(\mathbb{R}^N)$ such that $u,v\geqslant 0$ a.e. in $\mathbb{R}^N$, we have that

$$\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^2+|\nabla v|^2-\frac{\lambda u^2}{|x-a|^2}-\frac{\mu v^2}{|x-b|^2}\Big)}{\Big(\int_{\mathbb{R}^N}(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta)\Big)^{\frac{2}{2^*}}}$$
$$\geqslant\frac{\int_{\mathbb{R}^N}\Big(|\nabla u^*|^2+|\nabla v^*|^2-\frac{\lambda|u^*|^2+\mu|v^*|^2}{|x|^2}\Big)}{\Big(\int_{\mathbb{R}^N}(|u^*|^{2^*}+|v^*|^{2^*}+\sigma|u^*|^\alpha|v^*|^\beta)\Big)^{\frac{2}{2^*}}}\geqslant\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu). \quad (2.1)$$

Note that the Rayleigh quotient above remains unchanged when replacing u and v with $|u|$ and $|v|$ respectively. Then

$$S_{\sigma,\alpha,\beta}(\lambda,\mu)=\inf_{(u,v)\in\mathscr{D},u,v\geqslant 0}\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^2+|\nabla v|^2-\frac{\lambda u^2}{|x-a|^2}-\frac{\mu v^2}{|x-b|^2}\Big)}{\Big(\int_{\mathbb{R}^N}\big(u^{2^*}+v^{2^*}+\sigma u^\alpha v^\beta\big)\Big)^{\frac{2}{2^*}}},$$

which together with (2.1) implies that

$$S_{\sigma,\alpha,\beta}(\lambda,\mu)\geqslant\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu). \quad (2.2)$$

For all $u,v\in C_0^\infty(\mathbb{R}^N)$, the rescaled functions

$$u_\rho(x):=\rho^{\frac{2-N}{2}}u(x/\rho)\quad\text{and}\quad v_\rho(x):=\rho^{\frac{2-N}{2}}v(x/\rho)$$

satisfy

$$\frac{\int_{\mathbb{R}^N}\Big(|\nabla u_\rho(x)|^2+|\nabla v_\rho(x)|^2-\frac{\lambda u_\rho(x)^2}{|x-a|^2}-\frac{\mu v_\rho(x)^2}{|x-b|^2}\Big)}{\Big(\int_{\mathbb{R}^N}\big(|u_\rho(x)|^{2^*}+|v_\rho(x)|^{2^*}+\sigma|u_\rho(x)|^\alpha|v_\rho(x)|^\beta\big)\Big)^{\frac{2}{2^*}}}$$
$$=\frac{\int_{\mathbb{R}^N}\Big(|\nabla u(x)|^2+|\nabla v(x)|^2-\frac{\lambda u(x)^2}{|x-\frac{a}{\rho}|^2}-\frac{\mu v(x)^2}{|x-\frac{b}{\rho}|^2}\Big)}{\Big(\int_{\mathbb{R}^N}\big(|u(x)|^{2^*}+|v(x)|^{2^*}+\sigma|u(x)|^\alpha|v(x)|^\beta\big)\Big)^{\frac{2}{2^*}}},$$
$$=\frac{\int_{\mathbb{R}^N}\Big(|\nabla u(x)|^2+|\nabla v(x)|^2-\frac{\lambda u(x)^2}{|x|^2}-\frac{\mu v(x)^2}{|x|^2}\Big)}{\Big(\int_{\mathbb{R}^N}\big(|u(x)|^{2^*}+|v(x)|^{2^*}+\sigma|u(x)|^\alpha|v(x)|^\beta\big)\Big)^{\frac{2}{2^*}}}+o(1),\quad\text{as }\rho\to\infty.$$

Taking $\rho \to \infty$ and together with the density of $C_0^\infty(\mathbb{R}^N)$ in $D^{1,2}(\mathbb{R}^N)$, we have that

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2-\frac{\lambda u^2+\mu v^2}{|x|^2}\right)}{\left(\int_{\mathbb{R}^N}\left(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\right)\right)^{\frac{2}{2^*}}} \geqslant S_{\sigma,\alpha,\beta}(\lambda,\mu), \ \ \forall u,v\in D^{1,2}(\mathbb{R}^N),$$

which implies that

$$\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu) \geqslant S_{\sigma,\alpha,\beta}(\lambda,\mu). \tag{2.3}$$

From (2.2) and (2.3) it follows that

$$S_{\sigma,\alpha,\beta}(\lambda,\mu) = \bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu). \tag{2.4}$$

(ii) $\lambda \geqslant 0,\ \mu < 0,\ a \neq b.$

Arguing as above, for all $u,v\in D^{1,2}(\mathbb{R}^N)$ such that $u,v\geqslant 0$ a.e. in $\mathbb{R}^N$, we have

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2-\frac{\lambda u^2}{|x-a|^2}-\frac{\mu v^2}{|x-b|^2}\right)}{\left(\int_{\mathbb{R}^N}(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta)\right)^{\frac{2}{2^*}}}$$

$$\geqslant \frac{\int_{\mathbb{R}^N}\left(|\nabla u^*|^2+|\nabla v^*|^2-\lambda\frac{|u^*|^2}{|x|^2}\right)}{\left(\int_{\mathbb{R}^N}(|u^*|^{2^*}+|v^*|^{2^*}+\sigma|u^*|^\alpha|v^*|^\beta)\right)^{\frac{2}{2^*}}} \ \geqslant\ \bar{S}_{\sigma,\alpha,\beta}(\lambda,0),$$

which implies that

$$S_{\sigma,\alpha,\beta}(\lambda,\mu) \geqslant \bar{S}_{\sigma,\alpha,\beta}(\lambda,0). \tag{2.5}$$

Arguing as in case (i), for all $u,v\in C_0^\infty(\mathbb{R}^N)$, the rescaled functions

$$u_\rho(x) := \rho^{\frac{2-N}{2}}u(x/\rho) \quad \text{and} \quad v_\rho(x) := \rho^{\frac{2-N}{2}}v(x/\rho)$$

satisfy

$$\frac{\int_{\mathbb{R}^N}\left(|\nabla u_\rho(x-a)|^2+|\nabla v_\rho(x-a)|^2-\frac{\lambda u_\rho(x-a)^2}{|x-a|^2}-\frac{\mu v_\rho(x-a)^2}{|x-b|^2}\right)}{\left(\int_{\mathbb{R}^N}\left(|u_\rho(x-a)|^{2^*}+|v_\rho(x-a)|^{2^*}+\sigma|u_\rho(x-a)|^\alpha|v_\rho(x-a)|^\beta\right)\right)^{\frac{2}{2^*}}}$$

$$=\frac{\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2+|\nabla v(x)|^2-\frac{\lambda u(x)^2}{|x|^2}-\frac{\mu v(x)^2}{|x+\frac{a-b}{\rho}|^2}\right)}{\left(\int_{\mathbb{R}^N}\left(|u(x)|^{2^*}+|v(x)|^{2^*}+\sigma|u(x)|^\alpha|v(x)|^\beta\right)\right)^{\frac{2}{2^*}}}$$

$$=\frac{\int_{\mathbb{R}^N}\left(|\nabla u(x)|^2+|\nabla v(x)|^2-\lambda\frac{u(x)^2}{|x|^2}\right)}{\left(\int_{\mathbb{R}^N}\left(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\right)\right)^{\frac{2}{2^*}}}+o(1), \quad \text{as } \rho\to 0.$$

Taking $\rho \to 0$ and by the density of $C_0^\infty(\mathbb{R}^N)$ in $D^{1,2}(\mathbb{R}^N)$, we have

$$\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^2+|\nabla v|^2-\lambda\frac{u^2}{|x|^2}\Big)}{\Big(\int_{\mathbb{R}^N}\big(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\big)\Big)^{\frac{2}{2^*}}} \geqslant S_{\sigma,\alpha,\beta}(\lambda,\mu),\ \ \forall u,v\in D^{1,2}(\mathbb{R}^N),$$

which implies that

$$\bar{S}_{\sigma,\alpha,\beta}(\lambda,0)\geqslant S_{\sigma,\alpha,\beta}(\lambda,\mu). \tag{2.6}$$

Then from (2.5) and (2.6) it follows that

$$S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(\lambda,0). \tag{2.7}$$

Similarly, if $\lambda<0,\ \mu\geqslant 0,\ a\neq b,$ arguing as above we have that

$$S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(0,\mu). \tag{2.8}$$

(iii) $\lambda,\mu\in(-\infty,\bar{\lambda}),\ a=b\in\mathbb{R}^N.$
For all $u,v\in D^{1,2}(\mathbb{R}^N),$ set $\bar{u}(x)=u(x+a)$ and $\bar{v}(x)=v(x+a).$ Then

$$\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^2+|\nabla v|^2-\frac{\lambda u^2+\mu v^2}{|x-a|^2}\Big)}{\Big(\int_{\mathbb{R}^N}(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta)\Big)^{\frac{2}{2^*}}}$$
$$=\frac{\int_{\mathbb{R}^N}\Big(|\nabla\bar{u}|^2+|\nabla\bar{v}|^2-\frac{\lambda\bar{u}^2+\mu\bar{v}^2}{|x|^2}\Big)}{\Big(\int_{\mathbb{R}^N}(|\bar{u}|^{2^*}+|\bar{v}|^{2^*}+\sigma|\bar{u}|^\alpha|\bar{v}|^\beta)\Big)^{\frac{2}{2^*}}}\geqslant \bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu),$$

which implies that $S_{\sigma,\alpha,\beta}(\lambda,\mu)\geqslant\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu).$ Similarly,

$$\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu)\geqslant S_{\sigma,\alpha,\beta}(\lambda,\mu).$$

Then

$$S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(\lambda,\mu). \tag{2.9}$$

(iv) $\lambda,\mu\in(-\infty,0],\ a,b\in\mathbb{R}^N.$

In this case, the argument is similar with that of case (ii), and the first part is almost the same. In the second part, we only need to use the rescaling functions $u_\rho(x-c)$ and $v_\rho(x-c)$ such that $c\in\mathbb{R}^N\setminus\{a,b\}.$ Then we have that

$$S_{\sigma,\alpha,\beta}(\lambda,\mu)=\bar{S}_{\sigma,\alpha,\beta}(0,0). \tag{2.10}$$

The proof is complete according to (2.4) and (2.7)-(2.10). □

LEMMA 1. (see [15, Theorem 1.1]) *Suppose that* $(\mathscr{H}_1)$ *holds. Then*

(i) $S_{\sigma,\alpha,\beta}(\lambda,\lambda)=f(\tau_{\min})S(\lambda),\ \ S_{0,\alpha,\beta}(\lambda)=S(\lambda),\ \ \forall\lambda\in(0,\bar{\lambda})$.

(ii) $S_{\sigma,\alpha,\beta}(\lambda,\lambda)=S_{\sigma,\alpha,\beta}(0)=f(\tau_{\min})S(0),\ \ S_{0,\alpha,\beta}(\lambda,\lambda)=S(0),\ \ \forall\lambda\in(-\infty,0]$.

LEMMA 2. *Suppose that* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *hold. Then the functional* J *satisfies the* $(PS)_c$ *condition for all* $c<c^*=\frac{1}{N}\mathscr{S}^{N/2}$, *where*

$$\mathscr{S}:=\min\left\{\begin{array}{c} S_{\sigma,\alpha,\beta}(0,0),\ S_{\sigma,\alpha,\beta}(\lambda_i,0),\ S_{\sigma,\alpha,\beta}(0,\mu_i),\\ S_{\sigma,\alpha,\beta}(\lambda_i,\mu_i),\quad i=1,2,\cdots,k \end{array}\right\}. \tag{2.11}$$

Proof. Suppose that the sequence $\{(u_n,v_n)\}\subset H\times H$ satisfies

$$J(u_n,v_n)\to c<c^*,\quad J'(u_n,v_n)\to 0\ \text{ in }\ (H\times H)^{-1},$$

where $(H\times H)^{-1}$ is the dual space of $H\times H$. It is standard to show that $\{(u_n,v_n)\}$ is bounded in $H\times H$. Up to a subsequence and for some $(u,v)\in H\times H$ we have

$$(u_n,v_n)\rightharpoonup(u,v)\text{ weakly in }H\times H,\qquad (u_n,v_n)\to(u,v)\quad\text{ a.e. in }\Omega,$$

$$(u_n,v_n)\to(u,v)\ \text{ strongly in }\ L^{q_1}(\Omega)\times L^{q_2}(\Omega),\ \forall q_1,q_2\in[1,2^*).$$

By a result of [17], which is a direct application of Lions' concentration compactness principle ([19], [20]) to systems of elliptic equations, and up to a subsequence (still denoted by $\{(u_n,v_n)\}$), there exists an at most countable set $\mathscr{J}$, a set of points $x_j\in\Omega\backslash\{a_i,b_i,i=1,2,\cdots,k\}$, real numbers ρ_{x_j}, ν_{x_j}, $j\in\mathscr{J}$, and ρ_{a_i}, ν_{a_i}, γ_{a_i}, $\tilde{\rho}_{b_i}$, $\tilde{\nu}_{b_i}$, $\tilde{\gamma}_{b_i}$, $i=1,2,\cdots,k$, such that the following convergences hold in the sense of measures:

$$|\nabla u_n|^2+|\nabla v_n|^2\rightharpoonup d\rho\geqslant|\nabla u|^2+|\nabla v|^2+\sum_{i=1}^{k}(\rho_{a_i}\delta_{a_i}+\tilde{\rho}_{b_i}\delta_{b_i})+\sum_{j\in\mathscr{J}}\rho_{x_j}\delta_{x_j},$$

$$\begin{cases} |u_n|^{2^*}+|v_n|^{2^*}+\sigma|u_n|^{\alpha}|v_n|^{\beta}\\ \rightharpoonup d\nu=|u|^{2^*}+|v|^{2^*}+\sigma|u|^{\alpha}|v|^{\beta}+\displaystyle\sum_{i=1}^{k}(\nu_{a_i}\delta_{a_i}+\tilde{\nu}_{b_i}\delta_{b_i})+\sum_{j\in\mathscr{J}}\nu_{x_j}\delta_{x_j}, \end{cases}$$

$$\begin{cases} \lambda_i\dfrac{u_n^2}{|x-a_i|^2}\rightharpoonup d\gamma_{a_i}=\lambda_i\dfrac{u^2}{|x-a_i|^2}+\gamma_{a_i}\delta_{a_i},\ \ i=1,2,\cdots,k,\\ \mu_i\dfrac{v_n^2}{|x-b_i|^2}\rightharpoonup d\tilde{\gamma}_{b_i}=\mu_i\dfrac{v^2}{|x-b_i|^2}+\tilde{\gamma}_{b_i}\delta_{b_i},\ \ i=1,2,\cdots,k, \end{cases}$$

where δ_x is the Dirac mass at x.

(i) We first consider the concentration at $x_j\in\mathbb{R}^N\backslash\{a_i,\ b_i,1\leqslant i\leqslant k\}$, $j\in\mathscr{J}$.

For $\varepsilon > 0$ small enough, take $\phi_\varepsilon^j \in C_0^\infty(B_\varepsilon(x_j))$ such that $\phi_\varepsilon^j = 1$ in $B_{\varepsilon/2}(x_j)$, $0 \leqslant \phi_\varepsilon^j \leqslant 1$ and $|\nabla \phi_\varepsilon^j| \leqslant \frac{4}{\varepsilon}$ in $B_\varepsilon(x_j)$. Then

$$\begin{aligned}
&\langle J'(u_n,v_n),(u_n\phi_\varepsilon^j, v_n\phi_\varepsilon^j)\rangle \\
&= \int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)\phi_\varepsilon^j - \int_\Omega \sum_{i=1}^k \Big(\frac{\lambda_i u_n^2}{|x-a_i|^2} + \frac{\mu_i v_n^2}{|x-b_i|^2}\Big)\phi_\varepsilon^j \\
&\quad + \int_\Omega \big(u_n \nabla u_n + v_n \nabla v_n\big)\nabla \phi_\varepsilon^j - \int_\Omega \big(\sigma_1 u_n^2 + 2\sigma_2 u_n v_n + \sigma_3 v_n^2\big)\phi_\varepsilon^j \\
&\quad - \int_\Omega \big(|u_n|^{2^*} + |v_n|^{2^*} + \sigma |u_n|^\alpha |v_n|^\beta\big)\phi_\varepsilon^j .
\end{aligned}$$

Standard argument shows that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (u_n\nabla u_n + v_n \nabla v_n)\nabla\phi_\varepsilon^j = 0, \tag{2.12}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (|\nabla u_n|^2 + |\nabla v_n|^2)\phi_\varepsilon^j = \lim_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon^j d\rho \geqslant \rho_{x_j}, \tag{2.13}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (|u_n|^{2^*}| + |v_n|^{2^*} + \sigma|u_n|^\alpha |v_n|^\beta)\phi_\varepsilon^j = \lim_{\varepsilon\to 0}\int_\Omega \phi_\varepsilon^j d\nu = \nu_{x_j}, \tag{2.14}$$

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty}\int_\Omega \Big(\frac{\lambda_i u_n^2}{|x-a_i|^2} + \frac{\mu_i v_n^2}{|x-b_i|^2}\Big)\phi_\varepsilon^j \\ = \lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} O_1\Big(\int_\Omega (u_n^2 + v_n^2)\phi_\varepsilon^j\Big) = 0, \quad 1 \leqslant i \leqslant k, \end{cases} \tag{2.15}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega (\sigma_1 u_n^2 + 2\sigma_2 u_n v_n + \sigma_3 v_n^2)\phi_\varepsilon^j = 0. \tag{2.16}$$

From (2.12)-(2.16) it follows that

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n,v_n),(u_n\phi_\varepsilon^j, v_n\phi_\varepsilon^j)\rangle \geqslant \rho_{x_j} - \nu_{x_j}. \tag{2.17}$$

The Sobolev inequality implies that

$$S_{\sigma,\alpha,\beta}(0,0)(\nu_{x_j})^{\frac{2}{2^*}} \leqslant \rho_{x_j}, \ \forall j \in \mathscr{J}. \tag{2.18}$$

By (2.17) and (2.18), we deduce that

$$\nu_{x_j} = 0 \ \text{ or } \ \nu_{x_j} \geqslant (S_{\sigma,\alpha,\beta}(0,0))^{N/2}, \ \forall j \in \mathscr{J}, \tag{2.19}$$

which implies that the set $\mathscr{J}$ is finite.

(ii) Next we consider the concentration at the points a_i and b_i $(1 \leqslant i \leqslant k)$.

If $a_i = b_i$, for $\varepsilon > 0$ small enough, take $\varphi_\varepsilon^i(x) \in C_0^\infty(B_\varepsilon(a_i))$ such that $\varphi_\varepsilon^i(x) = 1$ in $B_{\varepsilon/2}(a_i)$, $0 \leqslant \varphi_\varepsilon^i(x) \leqslant 1$ and $|\nabla\varphi_\varepsilon^i| \leqslant \frac{4}{\varepsilon}$ in $B_\varepsilon(a_i)$. Then arguing as in (i) we have

$$\begin{cases} 0 = \lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty}\langle J'(u_n,v_n),(u_n\varphi_\varepsilon^i, v_n\varphi_\varepsilon^i)\rangle \\ \geqslant \rho_{a_i} + \tilde\rho_{a_i} - (\gamma_{a_i} + \tilde\gamma_{a_i}) - (\nu_{a_i} + \tilde\nu_{a_i}). \end{cases} \tag{2.20}$$

By (1.6) we have

$$S_{\sigma,\alpha,\beta}(\lambda_i,\mu_i)(\nu_{a_i}+\tilde{\nu}_{a_i})^{\frac{2}{2^*}} \leqslant \rho_{a_i}+\tilde{\rho}_{a_i}-(\gamma_{a_i}+\tilde{\gamma}_{a_i}). \tag{2.21}$$

From (2.20) and (2.21) it follows that

$$\nu_{a_i}=\tilde{\nu}_{a_i}=0 \ \text{ or } \ \nu_{a_i}+\tilde{\nu}_{a_i} \geqslant (S_{\sigma,\alpha,\beta}(\lambda_i,\mu_i))^{N/2}. \tag{2.22}$$

If $a_i \neq b_i$, since $\mathscr{J}$ is finite and $\{x_j\}_{j\in\mathscr{J}} \subset \Omega\backslash\{a_i,b_i,1\leqslant i\leqslant k\}$, we can take $\varepsilon>0$ small such that $b_i,x_j \notin B_\varepsilon(a_i),\ 1\leqslant i\leqslant k,\ j\in\mathscr{J}$. Choose $\varphi_{a_i}(x)\in C_0^\infty(B_\varepsilon(a_i))$ such that $\varphi_{a_i}(x)=1$ in $B_{\varepsilon/2}(a_i),\ 0\leqslant\varphi_{a_i}(x)\leqslant 1$ and $|\nabla\varphi_{a_i}|\leqslant\frac{4}{\varepsilon}$ in $B_\varepsilon(a_i)$. Then

$$0=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\langle J'(u_n,v_n),(u_n\varphi_\varepsilon^i,v_n\varphi_\varepsilon^i)\rangle \geqslant \rho_{a_i}-\gamma_{a_i}-\nu_{a_i}. \tag{2.23}$$

By (1.6) we have

$$S_{\sigma,\alpha,\beta}(\lambda_i,0)(\nu_{a_i})^{\frac{2}{2^*}} \leqslant \rho_{a_i}-\gamma_{a_i},\ \ 1\leqslant i\leqslant k. \tag{2.24}$$

From (2.23) and (2.24) it follows that

$$\nu_{a_i}=0 \quad \text{or} \quad \nu_{a_i}\geqslant (S_{\sigma,\alpha,\beta}(\lambda_i,0))^{N/2}. \tag{2.25}$$

Similarly, at the point $b_i(b_i\neq a_i)$, we deduce that

$$\tilde{\nu}_{b_i}=0 \quad \text{or} \quad \tilde{\nu}_{b_i}\geqslant (S_{\sigma,\alpha,\beta}(0,\mu_i))^{N/2}. \tag{2.26}$$

Note that

$$\begin{aligned} c &= J(u_n,v_n)-\frac{1}{2}\langle J'(u_n,v_n),(u_n,v_n)\rangle+o(1) \\ &= \frac{1}{N}\int_\Omega(|u_n|^{2^*}+|v_n|^{2^*}+\sigma|u_n|^\alpha|v_n|^\beta))+o(1) \\ &= \frac{1}{N}\Big(\int_\Omega(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta)+\sum_{i=1}^k(\nu_{a_i}+\tilde{\nu}_{b_i})+\sum_{j\in\mathscr{J}}\nu_{x_j}\Big). \end{aligned}$$

Since $c<c^*$, from (2.19), (2.22), (2.25) and (2.26) it follows that

$$\nu_{a_i}=\tilde{\nu}_{b_i}=0,\ i=1,2,\cdots,k; \qquad \nu_{x_j}=0,\ \forall j\in\mathscr{J}.$$

Up to a subsequence, $(u_n,v_n)\to(u,v)$ strongly in $H\times H$. $\square$

Let $\rho>0$ be a constant small enough and $V^{a_k}_{\lambda_k,\varepsilon}(x)$ be the extremal defined as in (1.5). Set $u^{a_k}_{\lambda_k,\varepsilon}(x)=\psi_{a_k}(x)V^{a_k}_{\lambda_k,\varepsilon}(x)$, where $\psi_{a_k}(x)\in C_0^\infty(B_\rho(a_k))$ is a cut-off function such that $\psi_{a_k}(x)\equiv 1$ in $B_{\rho/2}(a_k)$, $0\leqslant\psi_{a_k}\leqslant 1$ and $\nabla\psi_{a_k}\leqslant\frac{4}{\varepsilon}$ in $B_\rho(a_k)$.

LEMMA 3. (see [18, Theorem 1.1]) *As $\varepsilon \to 0$, the following estimates hold:*

$$\int_\Omega \Big(|\nabla u^{a_k}_{\lambda_k,\varepsilon}|^2 - \lambda_k \frac{|u^{a_k}_{\lambda_k,\varepsilon}|^2}{|x-a_k|^2} \Big) = S(\lambda_k)^{\frac{N}{2}} + O(\varepsilon^{2\sqrt{\bar\lambda-\lambda_k}}),$$

$$\int_\Omega |u^{a_k}_{\lambda_k,\varepsilon}|^{2^*} = S(\lambda_k)^{\frac{N}{2}} + O(\varepsilon^{2^*\sqrt{\bar\lambda-\lambda_k}}),$$

$$\int_\Omega |u^0_{\lambda_k,\varepsilon}|^2 = \begin{cases} \varepsilon^2 \int_{\mathbb{R}^N} |U_{\lambda_k}(x)|^2 + o(\varepsilon^2), \ \lambda_k < \bar\lambda - 1, \\ C^2_{\lambda_k} \omega_N \varepsilon^2 |\ln\varepsilon| + O(\varepsilon^2), \ \ \lambda_k = \bar\lambda - 1, \end{cases}$$

$$\int_\Omega \frac{|u^0_{\lambda_k,\varepsilon}|^2}{|x+\xi|^2} = \begin{cases} \frac{\varepsilon^2}{|\xi|^2} \int_{\mathbb{R}^N} |U_{\lambda_k}(x)|^2 + o(\varepsilon^2), \quad \lambda_k < \bar\lambda - 1, \\ \frac{1}{|\xi|^2} C^2_{\lambda_k} \omega_N \varepsilon^2 |\ln\varepsilon| + O(\varepsilon^2), \ \lambda_k = \bar\lambda - 1, \end{cases}$$

where $\xi \in \mathbb{R}^N \backslash \{0\}$, $C_{\lambda_k} = \big(\frac{4N(\bar\lambda-\lambda_k)}{N-2}\big)^{\frac{N-2}{4}}$ *and* ω_N *is the volume of the unit ball in* $\mathbb{R}^N$.

LEMMA 4. *Under the assumptions of Theorem 2, there exists a pair of functions* $(\tilde u, \tilde v) \in H \times H \setminus \{(0,0)\}$ *such that* $\sup_{t\geqslant 0} J(t\tilde u, t\tilde v) < c^*$.

Proof. Suppose that

$$N \geqslant 4, \ a_k = b_k, \ \lambda_k = \mu_k \in [0, \bar\lambda - 1], \ \mathscr{C}_1 > 0.$$

Since $S_{\sigma,\alpha,\beta}(\lambda,\mu)$ is decreasing with respect to λ and μ, by $(\mathscr{H}_1)$ we have that

$$c^* = \frac{1}{N} S_{\sigma,\alpha,\beta}(\lambda_k, \lambda_k).$$

Consider the function

$$g(t) = J(t u^{a_k}_{\lambda_k,\varepsilon}, t\tau_{\min} u^{a_k}_{\lambda_k,\varepsilon}).$$

Note that

$$\lim_{t\to+\infty} g(t) = -\infty \quad \text{and} \quad g(t) > 0 \ \text{ as } t \to 0.$$

Thus $\sup_{t\geqslant 0} g(t)$ is attained at some finite $t_\varepsilon > 0$ with $g'(t_\varepsilon) = 0$. As ε small enough, from Lemma 3 we derive that $\bar c_1 < t_\varepsilon < \bar c_2$, where $\bar c_1$ and $\bar c_2$ are the positive constants independent of ε. Note that $\int_\Omega |u^{a_k}_{\lambda_k,\varepsilon}|^2 = \int_\Omega |u^0_{\lambda_k,\varepsilon}|^2$, and

$$\begin{cases} \displaystyle\int_\Omega \sum_{i=1}^{k-1} \Big(\frac{\lambda_i |u^{a_k}_{\lambda_k,\varepsilon}|^2}{|x-a_i|^2} + \frac{\mu_i (\tau_{\min})^2 |u^{a_k}_{\lambda_k,\varepsilon}|^2}{|x-b_i|^2} \Big) \\ \displaystyle = \int_\Omega \sum_{i=1}^{k-1} \Big(\frac{\lambda_i |u^0_{\lambda_k,\varepsilon}|^2}{|x+(a_i-a_k)|^2} + \frac{\mu_i (\tau_{\min})^2 |u^0_{\lambda_k,\varepsilon}|^2}{|x+(b_i-b_k)|^2} \Big), \end{cases}$$

$$\max_{t\geqslant 0} \Big(\frac{t^2}{2} A_1 - \frac{t^{2^*}}{2^*} A_2 \Big) = \frac{1}{N} \Big(A_1 A_2^{-\frac{2}{2^*}} \Big)^{\frac{N}{2}}, \ A_1 > 0, \ A_2 > 0. \tag{2.27}$$

If $0 \leqslant \lambda_k < \bar{\lambda} - 1$, from (2.27) and Lemmas 1 and 3 it follows that

$$g(t_\varepsilon) \leqslant \frac{1}{N}\left(\frac{\left(1+(\tau_{\min})^2\right)S(\lambda_k)^{\frac{N}{2}} + O(\varepsilon^{2\sqrt{\bar{\lambda}-\lambda_k}}) - \mathscr{C}_1\int_{\mathbb{R}^N}|U_{\lambda_k}(x)|^2\varepsilon^2 + o(\varepsilon^2)}{\left(1+\sigma(\tau_{\min})^\beta + (\tau_{\min})^{\alpha+\beta}\right)^{\frac{2}{2^*}}\left(S(\lambda_k)^{\frac{N}{2}} + O(\varepsilon^{2^*\sqrt{\bar{\lambda}-\lambda_k}})\right)^{\frac{2}{2^*}}}\right)^{\frac{N}{2}}$$

$$= \frac{1}{N}(f(\tau_{\min})S(\lambda_k))^{\frac{N}{2}} + O(\varepsilon^{2\sqrt{\bar{\lambda}-\lambda_k}}) - O_1(\varepsilon^2) + o(\varepsilon^2)$$

$$< \frac{1}{N}(S_{\sigma,\alpha,\beta}(\lambda_k,\lambda_k))^{\frac{N}{2}}.$$

If $0 \leqslant \lambda_k = \bar{\lambda} - 1$, we have that

$$g(t_\varepsilon) \leqslant \frac{1}{N}(f(\tau_{\min})S(\lambda_k))^{\frac{N}{2}} + O(\varepsilon^{2\sqrt{\bar{\lambda}-\lambda_k}}) - O_1(\varepsilon^2|\ln\varepsilon|) + O(\varepsilon^2) < c^*.$$

The proof is thus complete. □

PROOF OF THEOREM 2. For any $(u,v) \in H \times H \setminus \{(0,0)\}$, from the Hardy and Sobolev inequalities it follows that

$$J(u,v) \geqslant C(\|u\|^2 + \|v\|^2 - \|u\|^{2^*} - \|v\|^{2^*} - \|(u,v)\|_{H\times H}^{2^*})$$
$$\geqslant C\|(u,v)\|_{H\times H}^2 - C\|(u,v)\|_{H\times H}^{2^*},$$

and there exists a positive number ρ small enough such that

$$b := \inf_{\|(u,v)\|_{H\times H}=\rho} J(u,v) > 0 = J(0,0).$$

Set $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], H\times H) \,|\, \gamma(0) = (0,0),\ J(\gamma(1)) < 0,\ \|\gamma(1)\| > \rho\}.$$

Since $J(tu,tv) \to -\infty$ as $t \to \infty$, there exists $t_0 > 0$ such that

$$\|(t_0u, t_0v)\|_{H\times H} > \rho \ \text{ and } \ J(t_0u,t_0v) < 0.$$

By the Mountain-Pass theorem ([3], [5]), there exists a sequence $\{(u_n,v_n)\} \subset H\times H$ such that $J(u_n,v_n) \to c$ and $J'(u_n,v_n) \to 0$ as $n \to \infty$.

Let $(\tilde{u},\tilde{v})$ be the testing functions obtained as in Lemma 4, then

$$0 < c \leqslant \sup_{t\in[0,1]} J(t\,t_0\,\tilde{u}, t\,t_0\,\tilde{v}) \leqslant \sup_{t\geqslant 0} J(t\,\tilde{u}, t\,\tilde{v}) < c^*.$$

From Lemma 2 it follows that there exists a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, such that $(u_n,v_n) \to (u,v)$ strongly in $H\times H$. We thus get a critical point (u,v) of J satisfying (1.1) and c is the corresponding critical value. Set $u^+ = \max\{u,0\}$. In order to obtain positive solution of (1.1), replacing u and v by u^+ and v^+ respectively in the terms at the right hand side of equations in (1.1), and repeating the above process, we can get a nonnegative solution (u_0,v_0) of (1.1). By the maximum principle([24]), we obtain that $u_0, v_0 > 0$ in Ω. □

3. Sign-changing solutions

Let (u_0,v_0) be the positive solution of (1.1) obtained as in Theorem 2 and set $c_0 := J(u_0,v_0)$. Then c_0 can be characterized by $c_0 = \min_{(u,v)\in\mathscr{B}} J(u,v)$ ([25]), where

$$\mathscr{B} := \Big\{(u,v)\,|\,(u,v)\in H\times H\setminus\{(0,0)\},\ (u,v)\geqslant 0,\ \langle J'(u,v),(u,v)\rangle = 0\Big\}$$
$$= \left\{\begin{array}{l}(u,v)|(u,v)\in H\times H\setminus\{(0,0)\},\ (u,v)\geqslant 0,\\ \dfrac{\int_\Omega\big(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\big)}{Q_1(u,v)-Q_2(u,v)}=1\end{array}\right\},$$

and Q_1 and Q_2 are defined as in (1.8) and (1.9). Define the functional and sets

$$f(u,v) := \begin{cases}\dfrac{\int_\Omega\big(|u|^{2^*}+|v|^{2^*}+\sigma|u|^\alpha|v|^\beta\big)}{Q_1(u,v)-Q_2(u,v)}, & (u,v)\neq(0,0),\\ 0, & (u,v)=(0,0),\end{cases}$$

$$M := \big\{(u,v)\in H\times H|f(u^+,v^+)=f(u^-,v^-)=1\big\},$$

$$N := \Big\{(u,v)\in H\times H|\,|f(u^+,v^+)-1|<\frac{1}{2}\ \text{and}\,|f(u^-,v^-)-1|<\frac{1}{2}\Big\},$$

where $u^+=\max\{u,0\}$, $u^-=\max\{-u,0\}$. Then it's not difficult to see that $M\neq\emptyset$. Arguing as in [8], let $\mathscr{P}=\{(u,v)\in H\times H|\,(u,v)\geqslant 0\}$ and Σ be the set of maps σ such that

(i) $\sigma\in C(\mathscr{D},\ H\times H)$, where $\mathscr{D}=[0,1]\times[0,1]$.

(ii) $\sigma(s_1,0)=0$, $\sigma(0,s_2)\in\mathscr{P}$, $\sigma(1,s_2)\in-\mathscr{P}$, $\forall(s_1,s_2)\in D$.

(iii) $(J\cdot\sigma)(s_1,1)\leqslant 0$, $(f\cdot\sigma)(s_1,1)\geqslant 2$, $\forall(s_1,s_2)\in D$.

We claim that $\Sigma\neq\emptyset$. In fact, for any $(u,v)\in H\times H$ with $(u^+,v^+)\neq(0,0)$ and $(u^-,v^-)\neq(0,0)$, define

$$\sigma=\sigma(s_1,s_2)=ks_2(1-s_1)(u^+,v^+)-ks_1s_2(u^-,v^-),\quad (s_1,s_2)\in\mathscr{D}.$$

when $k>0$ is large enough, it is easy to know that $\sigma\in\Sigma$.

Let $\bar N$ be the closure of N. We have the following Lemmas 4 and 5.

LEMMA 5. *There exists a sequence* $\{(u_n,v_n)\}\subset\bar N$ *such that*

$$J(u_n,v_n)\to c_1 := \inf_{(u,v)\in M} J(u,v),\quad J'(u_n,v_n)\to 0,\quad n\to\infty.$$

Furthermore,

$$\inf_{\sigma\in\Sigma}\ \sup_{(u,v)\in\sigma(\mathscr{D})} J(u,v) = \inf_{(u,v)\in M} J(u,v).$$

Proof. The argument is similar to [22] and the details are omitted. □

LEMMA 6. *Suppose that* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *hold. Assume that* $c_1 < c_0 + c^*$ *and* $\{(u_n, v_n)\} \subset \bar{N}$ *satisfies*

$$J(u_n, v_n) \to c_1, \quad J'(u_n, v_n) \to 0, \quad n \to \infty.$$

Then $\{(u_n, v_n)\}$ *is relatively compact in* $H \times H$.

Proof. According to Lemma 2 and following the same lines as that of [22], we can get the desired result. The details are omitted. □

LEMMA 7. (see[14, Theorem 1.2]) *Suppose that* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_2)$ *hold,*

$$a_k = b_k,\ \lambda_k = \mu_k \in [0, \bar{\lambda}),\ \gamma_k' := \sqrt{\bar{\lambda}} - \sqrt{\bar{\lambda} - \lambda_k},$$

and $(u_0, v_0) \in H \times H$ *is a positive solution of* (1.1). *Then there exist* $\rho > 0$ *small enough such that* $B_\rho(a_k) \subset \Omega$ *and*

$$u_0(x) = O_1\left(|x - a_k|^{-\gamma_k'}\right),\ \ v_0(x) = O_1\left(|x - a_k|^{-\gamma_k'}\right),\ \ \forall x \in B_\rho(a_k) \setminus \{a_k\}.$$

LEMMA 8. *Under the assumptions of Theorem* 3, *we have* $c_1 < c_0 + c^*$.

Proof. We only verify the case $\mathscr{C}_2 > 0$. Another case $\mathscr{C}_3 > 0$ can be proved similarly and the details are omitted. Since $\sigma = 0$, by Lemma 1 we infer that $\tau_{\min} = 0$, $S_{0,\alpha,\beta} = S(\lambda_k)$ and therefore $c^* = \frac{1}{N} S(\lambda_k)^{\frac{N}{2}}$. According to Lemma 4, it suffices to show that

$$\sup_{s_1, s_2 \in \mathbb{R}} J\left(s_1(u_0, v_0) + s_2(u^{a_k}_{\lambda_k,\varepsilon}, 0)\right) < c_0 + \frac{1}{N} S(\lambda_k)^{\frac{N}{2}}.$$

Since

$$\lim_{|s_1|+|s_2| \to 0} J\left(s_1(u_0, v_0) + s_2(u^{a_k}_{\lambda_k,\varepsilon}, 0)\right) = 0,$$

$$\lim_{|s_1|+|s_2| \to \infty} J\left(s_1(u_0, v_0) + s_2(u^{a_k}_{\lambda_k,\varepsilon}, 0)\right) = -\infty,$$

we may assume $|s_1| = O_1(1)$ and $|s_2| = O_1(1)$. Since (u_0, v_0) is a positive solution of (1.1), we have

$$\langle J'(u_0, v_0), (u^{a_k}_{\lambda_k,\varepsilon}, 0)\rangle = 0,$$

i.e.

$$\int_\Omega \left(\nabla u_0 \nabla u^{a_k}_{\lambda_k,\varepsilon} - \sum_{i=1}^{k} \frac{\lambda_i u_0 u^{a_k}_{\lambda_k,\varepsilon}}{|x - a_i|^2} - u_0^{2^*-1} u^{a_k}_{\lambda_k,\varepsilon} - \sigma_1 u_0 u^{a_k}_{\lambda_k,\varepsilon} - \sigma_2 v_0 u^{a_k}_{\lambda_k,\varepsilon}\right) = 0. \quad (3.1)$$

From (3.1) it follows that

$$\begin{aligned}
&J\big(s_1(u_0,v_0)+s_2(u^{a_k}_{\lambda_k,\varepsilon},0)\big)\\
&=J(s_1u_0,s_1v_0)+J(s_2u^{a_k}_{\lambda_k,\varepsilon},0)\\
&\quad+s_1s_2\int_\Omega\Big(\nabla u_0\nabla u^{a_k}_{\lambda_k,\varepsilon}-\sum_{i=1}^{k}\frac{\lambda_i u_0u^{a_k}_{\lambda_k,\varepsilon}}{|x-a_i|^2}-\sigma_1u_0u^{a_k}_{\lambda_k,\varepsilon}-\sigma_2v_0u^{a_k}_{\lambda_k,\varepsilon}\Big)\\
&\quad+\frac{1}{2^*}\int_\Omega\big(|s_1u_0|^{2^*}+|s_2u^{a_k}_{\lambda_k,\varepsilon}|^{2^*}-|s_1u_0+s_2u^{a_k}_{\lambda_k,\varepsilon}|^{2^*}\big)\\
&\leqslant J(s_1(u_0,v_0))+J(s_2(u^{a_k}_{\lambda_k,\varepsilon},0))+C\int_\Omega\big(u_0(u^{a_k}_{\lambda_k,\varepsilon})^{2^*-1}+u_0{}^{2^*-1}u^{a_k}_{\lambda_k,\varepsilon}\big),
\end{aligned}\tag{3.2}$$

where the following inequality is used:

$$\big||a+b|^q-|a|^q-|b|^q\big|\leqslant C\big(|a|^{q-1}|b|+|a|\,|b|^{q-1}\big),\quad \forall a,b\in\mathbb{R},\ \ q\geqslant 1.$$

By Lemma 6 and arguing as in [15], we have that

$$\int_\Omega|u_0|^{2^*-1}u^{a_k}_{\lambda_k,\varepsilon}=O(\varepsilon^{\sqrt{\bar\lambda-\lambda_k}}),\qquad \int_\Omega|u_0|^{2^*-1}u^{a_k}_{\lambda_k,\varepsilon}=O(\varepsilon^{\sqrt{\bar\lambda-\lambda_k}}).\tag{3.3}$$

Let $\mathscr{C}_2>0$ be defined as in Theorem 3. Note that

$$0\leqslant\lambda_k<\bar\lambda-4\iff\sqrt{\bar\lambda-\lambda_k}>2,\qquad 0<\mathscr{C}_2\int_{\mathbb{R}^N}|U_{\lambda_k}(x)|^2<\infty.$$

Taking $\varepsilon\to0$ and arguing as in Lemma 4, from (3.2) and (3.3) it follows that

$$\begin{aligned}
&\sup_{s_1,s_2\in\mathbb{R}}J\big(s_1(u_0,v_0)+s_2(u^{a_k}_{\lambda_k,\varepsilon},0)\big)\\
&\qquad\leqslant\sup_{s_1\in\mathbb{R}}J(s_1(u_0,v_0))+\sup_{s_2\in\mathbb{R}}J(s_2(u^{a_k}_{\lambda_k,\varepsilon},0))+O(\varepsilon^{\sqrt{\bar\lambda-\lambda_k}})\\
&\qquad\leqslant c_0+\frac1N S(\lambda_k)^{\frac N2}-\mathscr{C}_2\varepsilon^2\int_{\mathbb{R}^N}|U_{\lambda_k}(x)|^2+o(\varepsilon^2)+O\big(\varepsilon^{\sqrt{\bar\lambda-\lambda_k}}\big)\\
&\qquad<c_0+\frac1N S(\lambda_k)^{\frac N2}.
\end{aligned}$$

The proof is complete. □

PROOF OF THEOREM 3. By Lemmas 5-8, there exists a sequence $\{(u_n,v_n)\}\subset\bar N$, such that

$$J(u_n,v_n)\to c_1<c_0+\frac1N S(\lambda_k)^{\frac N2},\quad J'(u_n,v_n)\to0,\quad n\to\infty.$$

Passing to a subsequence if necessary, $(u_n,v_n)\to(u,v)$ strongly in $H\times H$ as $n\to\infty$. Therefore (u,v) is a critical point of J and solves (1.1). Since $(u_n,v_n)\in\bar N$, we deduce that $(u,v)\in\bar{N}$. Furthermore, $u\neq 0$ and $v\neq 0$. From the Hölder and Young inequalities it follows that there exists a constant $\delta>0$, such that

$$\|(u^+,v^+)\|_{H\times H}\geqslant\delta,\quad \|(u^-,v^-)\|_{H\times H}\geqslant\delta,$$

(u,v) is thus a sign-changing solution of (1.1) and $(-u,-v)$ is also a solution. □

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