Abstract. We consider the following quasilinear elliptic problem

\[
\begin{cases}
-\Delta_p u \pm u^q = \lambda \frac{u^{p-1}}{|x|^p} + h & \text{in } \Omega, \\
u \geq 0 \text{ and } u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where, \( 1 < p < N \), \( \Omega \subset \mathbb{R}^N \) is a bounded regular domain such that \( 0 \in \Omega \), \( q > p - 1 \) and \( h \) is a nonnegative measurable function with suitable hypotheses. The main goal of this paper is to analyze the interaction between the Hardy potential, and the term \( u^q \), in order to get existence and nonexistence of positive solution. We can summarize our main results, in the two following points:

(i) If \( u^q \) appears as a reaction term, then we show the existence of a critical exponent \( q_+(\lambda) \), such that for \( q > q_+ \), the considered problem has no positive distributional solution. If \( q < q_+ \) we find solutions under suitable hypothesis on \( h \).

(ii) If \( u^q \) appears as an absorption term, then there exists \( q_* \) such that if \( q > q_* \), the problem under consideration has a positive solution for all \( \lambda > 0 \) and for all \( h \in L^1(\Omega) \). The optimality of \( q_* \) is proved in the sense that if \( q < q_* \), then nonexistence holds if \( \lambda > \Lambda_{N,p} \).

1. Introduction and preliminaries results

In this paper we study existence and nonexistence of positive solutions to the problem

\[
(P_{\pm}) \begin{cases}
-\Delta_p u \pm u^q = \lambda \frac{u^{p-1}}{|x|^p} + h & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( 1 < p < N \), \( \Omega \subset \mathbb{R}^N \) is a bounded domain containing the origin, \( q > p - 1 \) and \( h \) is a nonnegative measurable function, with suitable hypotheses.

Problem \((P_{\pm})\) is related to the classical Hardy-Sobolev inequality

\[
\Lambda_{N,p} \int_{\mathbb{R}^N} \frac{\phi^p}{|x|^p} \, dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^p \, dx \quad \text{for all } \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N),
\]


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where $\Lambda_{N,p} = (\frac{N-p}{p})^p$ is optimal and not achieved, we refer to [14] for more details about this constant.

In the case where $u^q$ appears as a reaction term (problem $(P_\pm)$), then for $\lambda > \Lambda_{N,p}$, a strong local nonexistence result is obtained in [4].

The case $p = 2$ and $\lambda \leq \Lambda_{N,2}$ was studied in [12], the authors prove the existence of a critical exponent $q_+ (\lambda)$ such that existence holds if and only if $q < q_+$.

If $p \neq 2$ and $q \leq p^* - 1$, the problem is widely studied in the literature, we refer to [2] where the authors got the exact behavior of the solution near the origin and studied also the case where $\Omega = \mathbb{R}^N$.

In the case where $u^q$ appears as an absorption term, then if $\lambda = 0$, the existence and uniqueness of "entropy" solution is obtained in [11]. If $\lambda \leq \Lambda_{N,p}$ and $h \in L^{\frac{p}{p-1}} (\Omega)$, then existence holds in the Sobolev space $W^{1,p}_0 (\Omega)$ using variational arguments. The authors proved that for all $h \in L^1 (\Omega)$, there exists at least one distributional solution. The regularity of the solution is obtained according to the one of $h$ and the value of $q$.

If $\lambda > 0$, the situation is a quite different, in the case where $q = 0$ and $p = 2$, then an integrability condition on $h$ near the origin is needed to insure the existence of a distributional solution; see [5] for a complete discussion about this case.

The problem $(P_\pm)$ can also be seen, as the stationary case associated to the parabolic problem:

$$
\begin{cases}
  u_t - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + f, \quad u \geq 0 \text{ in } \Omega \times (0,T), \\
  u(x,t) = 0 \text{ on } \partial \Omega \times (0,T), \\
  u(x,0) = u_0(x), \quad x \in \Omega,
\end{cases}
$$

which will be studied in a forthcoming paper [3]. Notice that for the semilinear case, some related results were obtained in [7].

Since we are considering solution with data in $L^1$, then we need to use a week concept of solutions. More precisely we have the next definitions.

**Definition 1.** We say that $u$ is a nonnegative distributional solution to problem $(P_L)$ if

$$
|\nabla u|^{p-1} \in L^1_{loc} (\Omega) \quad \text{and} \quad \lambda \frac{u^{p-1}}{|x|^p}, u^q, h \in L^1_{loc} (\Omega),
$$

and for all $\phi \in C_0^\infty (\Omega)$, we have

$$
\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx = \int_\Omega \left( \lambda \frac{u^{p-1}}{|x|^p} \pm u^q + h \right) \phi \, dx. \tag{1.1}
$$

In the case where $\lambda \frac{u^{p-1}}{|x|^p}, u^q, h \in L^1 (\Omega)$ we can use the concept of entropy solution.

For $k > 0$, define

$$
T_k (s) = \begin{cases}
  s, & \text{if } |s| \leq k; \\
  k \frac{s}{|s|}, & \text{if } |s| > k;
\end{cases}
$$
then we have the following definitions.

**Definition 2.** Let \( u \) be a measurable function, we say that \( u \in \mathcal{T}_{0}^{1,p}(\Omega) \) if \( T_{k}(u) \in W_{0}^{1,p}(\Omega) \) for all \( k > 0 \). Let \( F \in L^{1}(\Omega) \), then \( u \in \mathcal{T}_{0}^{1,p}(\Omega) \) is an entropy solution to

\[
\begin{align*}
-\Delta_{p} u &= F \text{ in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} &= 0,
\end{align*}
\]

if for all \( k > 0 \) and all \( v \in W_{0}^{1,p}(\Omega) \cap L^{\infty}(\Omega) \), we have

\[
\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla (T_{k}(u-v)) \rangle dx = \int_{\Omega} F T_{k}(u-v) dx.
\]

Hence we say that \( u \) is an entropy solution to problem \((P \pm)\) if

\[
\lambda \frac{u^{p-1}}{|x|^p}, u^{q}, h \in L^{1}(\Omega)
\]

and the above definition holds with

\[
F(x) \equiv \lambda \frac{u^{p-1}}{|x|^p} \pm u^{q} + h.
\]

From the results of [10], we know that if \( u \) is an entropy solution, then \( |\nabla u|^{p-1} \in L^{s}(\Omega) \) for all \( s < \frac{N}{N+1} \). Hence, we conclude that if \( u \) is an entropy solution, then \( u \) is also a distributional solution.

We recall the following existence result obtained in [10].

**Theorem 1.** Assume that \( 1 < p \) and \( F \in L^{1}(\Omega) \). Let \( \{f_{n}\}_{n} \subset L^{\infty}(\Omega) \) be such that \( f_{n} \to F \) strongly in \( L^{1}(\Omega) \). Consider \( u_{n} \in W_{0}^{1,p}(\Omega) \), the unique solution to problem

\[
\begin{align*}
-\Delta_{p} u_{n} &= f_{n} \text{ in } \Omega, \\
u_{n} &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

then there exits \( u \in \mathcal{T}_{0}^{1,p}(\Omega) \) such that \( u \) is the unique entropy solution of (1.2), \( T_{k}(u_{n}) \to T_{k}(u) \) strongly in \( W_{0}^{1,p}(\Omega) \), \( u_{n}^{p-1} \to u^{p-1} \) strongly in \( L^{\sigma}(\Omega) \) for all \( \sigma < \frac{N}{N+2} \) and \( |\nabla u_{n}|^{p-1} \to |\nabla u|^{p-1} \) strongly in \( L^{s}(\Omega) \) for all \( s < \frac{N}{N-1} \).

The paper is organized as follows. In Section 2 we deal with the problem \((P_{-})\). In Subsection 2.1 we prove the existence of a critical exponent \( q_{+}(\lambda) \) such that a strong non existence result holds if \( q > q_{+}(\lambda) \). As a consequence we prove some complete Blow-up results for approximated problems.

The case \( q < q_{+}(\lambda) \) is treated in Subsection 2.2, then, under suitable hypothesis on \( h \), problem \((P_{-})\) has a positive solution. This prove the optimality of \( q_{+}(\lambda) \).

The case of absorption term is considered in Section 3, we find an exponent \( q_{*} \) such that if \( q > q_{*} \), then problem \((P_{+})\) has an entropy solution for all \( \lambda > 0 \) and \( h \in L^{1}(\Omega) \).
Notice that, without the absorption term $u^q$, existence holds if and only if $\lambda \leq \Lambda_{N,p}$ with strong condition on $h$. Thus this shows the strong effect of the absorption term $u^q$ in order to break down any resonant effect of the reaction term $\lambda \frac{u^{p-1}}{|x|^p}$.

The optimality of $q_*$ is proved by showing that if $q < q_+$, then for $\lambda > \Lambda_{N,p}$, problem $(P_+)$ has no positive solution. Some extensions are given at the end of the section.

2. Problem with reaction term

In this section we consider the following problem

$$
\begin{cases}
-\Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q + h & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(2.1)

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ containing the origin, $1 < p < N$ and $q > p - 1$.

First, let us consider the equation

$$
-\Delta_p w = \lambda \frac{w^{p-1}}{|x|^p}.
$$

(2.2)

By setting $w(x) = C|x|^{-\alpha}$, there results

$$
-\frac{1}{r^{N-1}} \left( |w'|^{p-2} w' r^{N-1} \right)' = \lambda w^{p-1} r^{-p}.
$$

Hence, we get the next algebraic equation

$$
D(\alpha) \equiv (p - 1) \alpha^p - (N - p) \alpha^{p-1} + \lambda = 0
$$

(2.3)

Under the hypothesis $\lambda < \Lambda_{N,p} \equiv \left( (N - p)/p \right)^p$, equation (2.3) possesses exactly two solutions $\alpha_1 < (N - p)/p < \alpha_2$ (see the computation details in [2]).

Since $\lambda > 0$, then using the strong maximum principle and a suitable comparison function we can show that $u(x) \to \infty$ as $|x| \to 0$. The next result gives a more precise information about the behavior of any supersolution of (2.1) near the origin, the proof can be seen in [2].

**Lemma 1.** Assume that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a nonnegative supersolution to problem (2.2), then if $u \not\equiv 0$, there exist a positive constant $C$ and a small ball $B_\eta(0) \subset \subset \Omega$ such that

$$
u \geq C|x|^{-\alpha_1} \text{ in } B_\eta(0)
$$

(2.4)

where $\alpha_1$ is defined above.
2.1. Nonexistence results: the optimal exponent.

Assume that $\lambda < \Lambda_{N,p}$, we look now for radial solutions to the elliptic equation

$$-\Delta pw = \lambda \frac{w^{p-1}}{|x|^p} + w^q, \ x \in \mathbb{R}^N. \quad (2.5)$$

Then by setting $w(x) = |x|^{-\alpha}$, it follows that

$$(p-1)\alpha^p - (N-p)\alpha^{p-1} - \lambda r^{-\alpha(p-1)-p} = r^{-\alpha q} \quad (2.6)$$

so by identification, one have that

$$\alpha = \frac{p}{1+q-p}, \ q > p-1, \ \text{and} \ \alpha < \frac{N-p}{p-1}.$$ 

Since

$$\frac{w^{p-1}}{|x|^p} \in L^1_{loc}(\Omega) \ \text{and} \ w^q \in L^1_{loc}(\Omega),$$

then by the result of Lemma 1 we obtain that $\alpha_1 < \alpha < \alpha_2$ which is equivalent to

$$\frac{p}{\alpha_2} + p - 1 < q < \frac{p}{\alpha_1} + p - 1, \quad (2.7)$$

We set

$$q_+(\lambda) \equiv \frac{p}{\alpha_1} + p - 1 \ \text{and} \ q_-(\lambda) \equiv \frac{p}{\alpha_2} + p - 1, \quad (2.8)$$

then

$$p - 1 < q_-(\lambda) < p^* - 1 < q_+(\lambda).$$

with $p^* = Np/(N-p)$.

Since we are considering an equation with right hand side in $L^1$, then we will use the concept of entropy solutions given in Definition 2

We are now able to prove the next nonexistence result.

**THEOREM 2.** Assume that $q > q_+(\lambda) \equiv (p-1) + p/\alpha_1$. Then for all $\lambda > 0$, the problem (2.1) has no positive entropy solution.

To prove this theorem we need the following well known inequality [8].

**THEOREM 3.** (Picone inequality) Let $v \in W_{0}^{1,p}(\Omega)$ be such that $-\Delta_pv \geq 0$ is a bounded Radon measure $v \geq 0$, then for all $u \in W_{0}^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^p \, dx \geq \int_{\Omega} \frac{|u|^p}{\nu^{p-1}}(-\Delta_p v) \, dx.$$
Proof of Theorem 2.

If \( \lambda > \Lambda_{N,p} \), then the nonexistence result is obtained in [4]. Let us consider the case \( \lambda \leq \Lambda_{N,p} \).

We argue by contradiction. Let \( u \) be an entropy solution to (2.1), then using an approximation argument as in [4], we get the existence of a minimal entropy solution obtained as a limit of approximation problems. We note \( u \) the minimal solution. Let \( \varphi \in C^\infty_0(B_r(0)) \), then using Picone inequality of Theorem 3 to \( u \), it follows that

\[
\int_{B_r(0)} |\nabla \varphi|^p dx \geq \lambda \int_{B_r(0)} \frac{-\Delta_p u}{u^{p-1}} |\varphi|^p dx \geq \lambda \int_{B_r(0)} \frac{|\varphi|^p}{|x|^p} dx + \int_{B_r(0)} u^q - p + 1 \frac{|\varphi|^p}{|x|^\alpha(q-p+1)} dx.
\]

If \( q > q_+(\lambda) \), then \( \alpha(q-p+1) > p \), thus we get a contradiction with the Hardy inequality, hence non existence holds.

**Remark 1.** Since the arguments used in the proof of the nonexistence result, are local, then we conclude that problem (2.1) has no non-trivial supersolution in the sense that \( \lambda u^{p-1} + u^q \in L^1_{loc}(\Omega) \) in any domain containing the origin.

**Theorem 4.** Assume that \( g : \mathbb{R} \to [0, \infty) \) is a continuous function such that \( g(s) > 0 \) if \( s > 0 \) and

\[
\liminf_{s \to \infty} \frac{g(s)}{s^q} = c > 0 \text{ for some } q > q_+(\lambda).
\]

Then we have:

(i) if \( g(0) = 0 \), then the unique entropy solution to problem

\[
-\Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + g(u), \text{ in } \Omega, u|_{\partial\Omega} = 0,
\]

is \( u = 0 \);

(ii) if \( g(0) > 0 \) than problem (2.9), does not admit any entropy positive solution.

As a consequence we get the next blow-up result.

**Theorem 5.** Fix \( q > q_+(\lambda) \) and \( \lambda < \Lambda_{N,p} \). Define

\[
a_n(x) = \min\{n, \frac{1}{|x|^p}\} \quad \text{and} \quad D_n(s) = \min\{n, s^q\}, \ s \geq 0.
\]

Let \( \{h_n\}_n \subset L^\infty(\Omega) \) be such that \( h_n \geq 0 \) and \( h_n \uparrow h \in L^1(\Omega) \). Let \( u_n \) be the minimal solution to problem

\[
\begin{cases}
-\Delta_p u_n = \lambda a_n(x) u_n^{p-1} + g_n(u_n) + h_n \text{ in } \Omega, \\
u_n \geq 0 \text{ in } \Omega, \\
u_n = 0 \text{ on } \partial\Omega.
\end{cases}
\]

Then \( u_n(x) \to \infty \) as \( n \to \infty \) uniformly in \( x \in \Omega \).
We first recall the following Lemma proved in [4], which will be useful in the proof of Theorem 5.

**Lemma 2.** Let \( u \) be the unique positive energy solution to problem

\[
\begin{align*}
-\Delta_p u &= f \text{ in } \Omega, \\
0 &< u \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \( f \in L^\infty(\Omega) \) and \( f \geq 0 \). Then for all ball \( B_r \subset \Omega \) such that \( \overline{B}_{4r} \subset \Omega \), there exist a positive constant \( c = c(r, N, p) \) such that,

\[
\frac{u^{p-1}(x)}{(d(x, \partial \Omega))^{p-1}} \geq c \int_{B_{2r}} f(y) dy \text{ for all } x \in \Omega.
\]

**Proof of Theorem 5.**

Since \( \lambda < \Lambda_{N,p} \), then we get easily the existence of a minimal solution \( u_n \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \) to (2.10). Using the fact that \( \lambda a_n(x) u_n^{p-1} + g_n(u_n) + h_n \) is increasing in \( n \), then we conclude that \( \{u_n\} \) is an increasing sequence in \( n \).

Assume by contradiction that there exist \( x_0 \in \Omega \), such that \( \sup_n u_n(x_0) = c_0 < \infty \), then using Lemma (2) for a ball satisfying \( 0 \in B_{2r} \subset \Omega \), we obtain that

\[
\int_{B_r} (\lambda a_n(x) u_n^{p-1} + g_n(u_n) + h_n) dx \leq c_1 \text{ for all } n.
\]

Since \( F_n(x) := \lambda a_n(x) u_n^{p-1} + g_n(u_n) + h_n \) is increasing we obtain that \( F_n \to w, \ n \to \infty \) in \( L^1(B_{2r}) \) to some \( w \geq 0 \). Starting from \( v_0 = 0 \), we define the sequence,

\[
\begin{align*}
-\Delta_p v_{n+1} &= \lambda a_{n+1}(x) v_n^{p-1} + g_n(v_n) + h_n \text{ in } B_r, \\
v_{n+1}|_{\partial B_r} &= 0.
\end{align*}
\]

Since \( \lambda a_n(x)s^{p-1} + g_n(s) + h_n \) is increasing in \( n \), by comparison we obtain that \( v_n \leq v_{n+1} \).

**Claim:** \( v_n \leq u_n \) in \( B_r(0) \) for all \( n \in \mathbb{N} \).

We prove the claim by induction. We have \( -\Delta_p v_1 = h_1 \leq -\Delta_p u_1 \) and since \( u_1|_{\partial B_r} > 0 = v_1|_{\partial B_r} \) we conclude that \( v_1 \leq u_1 \). Suppose \( v_n \leq u_n \). Recalling \( u_n \leq u_{n+1} \), then using the fact that

\[
\lambda a_{n+1}(x) v_n^{p-1} + g_n(v_n) + h_n \leq a_{n+1}(x) u_{n+1}^{p-1} + g_n(u_{n+1}) + h_{n+1},
\]

we conclude that

\[
-\Delta_p v_{n+1} \leq -\Delta_p u_{n+1} \text{ and } u_{n+1} \geq v_{n+1} \text{ on } \partial B_r.
\]

Thus \( v_{n+1} \leq u_{n+1} \) and the claim follows.

Moreover we have

\[
\int_{B_r} |\nabla T_k(v_n)|^p dx \leq k \int_{B_r} (a_{n+1}(x) u_{n+1}^{p-1} + g_n(u_{n+1}) + h_{n+1}) dx \leq ck.
\]
Therefore we reach that $T_k(v_n)$ is bounded in $W_0^{1,p}(B_r)$ for all $k > 0$. Thus $T_k(v_n) \rightharpoonup T_k(v)$ weakly in $W_0^{1,p}(B_r)$.

Since $\{T_k(v_n)\}_n$ is increasing in $n$, then using a simple variation of the compactness argument of [10] we can prove that and then $T_k(v_n) \rightarrow T_k(v)$ strongly in $W_0^{1,p}(B_r)$. Hence, $v$ is an entropy solution to problem

\[
\begin{aligned}
-\Delta_p v &= \frac{\lambda v^{p-1}}{|x|^p} + v^q + h \quad \text{in } B_r, \\
v &\geq 0 \text{ in } B_r \text{ and } v \neq 0, \\
v|_{\partial B_r} &= 0
\end{aligned}
\]

(2.14)

with $q > q_+ (\lambda)$. This is a contradiction with the nonexistence result of Theorem (2).

Hence for all $x_0 \in \Omega$, $u_n(x_0) \rightarrow \infty$ as $n \rightarrow \infty$ and the proof is complete.

2.2. Existence result for $q < q_+ (\lambda)$.

To show the optimality of the exponent $q_+ (\lambda)$ we will prove the next existence result.

**Theorem 6.** Assume that $\lambda \leq \Lambda_{N,p}$ and $q < q_+ (\lambda)$, then:

(i) if $q < p^* - 1$ and $\lambda < \Lambda_{N,p}$, then for $h \equiv 0$, problem (2.1) has a positive solution $u \in W_0^{1,p}(\Omega)$;

(ii) if $q < p^* - 1$ and $\lambda = \Lambda_{N,p}$, then for $h \equiv 0$, problem (2.1) has a positive solution $u \in W_0^{1,s}(\Omega)$ for all $s < p$;

(iii) if $p^* - 1 \leq q < q_+ (\lambda)$ and $\lambda < \Lambda_{N,p}$, then there exists a positive constant $c$ such that if $h(x) \leq c/|x|^p$, then problem (2.1) has an entropy positive solution $u$ such that $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$.

**Proof.** We divide the proof in several steps.

The first case: $q < p^* - 1$ and $\lambda < \Lambda_{N,p}$.

In this case problem (2.1) has a variational structure in the space $W_0^{1,p}(\Omega)$, then we can find a solution as a critical points of the functional

\[
J_\lambda (u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u|^p}{|x|^p} dx - \frac{1}{q+1} \int_{\Omega} u^{q+1} dx.
\]

By a direct application of the Mountain-Pass theorem [9], we reach the existence of positive solution as a mountain pass point.

The second case $q < p^* - 1$ and $\lambda = \Lambda_{N,p}$.

To get the existence result in this case we use the following improved Hardy-Sobolev inequality obtained in [1], for any $s < p$, there exists a positive constant $C \equiv C(N, p, s, \Omega)$ such that

\[
\int_{\Omega} \left( \left| \nabla u \right|^p - \Lambda_{N,p} \frac{|u|^p}{|x|^p} \right) dx \geq C \|u\|^{p}_{W_0^{1,s}(\Omega)} \quad \text{for all } u \in C_0^{\infty}(\Omega).
\]

(2.15)
Let now \( \{\lambda_n\}_n \) be a strictly increasing sequence of positive constants, such that \( \lambda_n \uparrow \Lambda_{N,p} \) as \( n \to \infty \). Using the result of the first case, we reach that the problem

\[
-\Delta_p u_n = \lambda_n \frac{u_n^{p-1}}{|x|^p} + u_n^q, \quad \text{in } \Omega, u_n \in W^{1,p}_0(\Omega).
\]

has a positive solution \( u_n \) obtained using the Mountain-Pass Theorem [9]. Notice that

\[
J_{\lambda_n}(u_n) = \left( \frac{1}{p} - \frac{1}{q+1} \right) \left( \int_{\Omega} |\nabla u_n|^p \, dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u_n|^p}{|x|^p} \, dx \right) \equiv C_n
\]

where

\[
C_n = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\lambda_n}(\gamma(t))
\]

with

\[
\Gamma = \{ \gamma \in C([0,1], W^{1,p}_0(\Omega)), \gamma(0) = 0, \gamma(1) = v \}
\]

and \( v \in W^{1,p}_0(\Omega) \) is such that

\[
\frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \frac{1}{q+1} \int_{\Omega} v^{q+1} \, dx < 0.
\]

It is clear that \( J_{\lambda_n}(v) < 0 \) uniformly for \( \lambda_n \in [0, \Lambda_{N,p}] \). If \( \gamma(t) = tv \), then \( \gamma \in \Gamma \), and

\[
C_n \leq \max_{t \in [0,1]} J_{\lambda_n}(tv) \leq A,
\]

where

\[
A = \max_{t \in [0,1]} \left( \frac{t^p}{p} \int_{\Omega} |\nabla v|^p \, dx - \frac{t^{q+1}}{q+1} \int_{\Omega} v^{q+1} \, dx \right).
\]

Hence we conclude that

\[
\left( \frac{1}{p} - \frac{1}{q+1} \right) \left( \int_{\Omega} |\nabla u_n|^p \, dx - \frac{\lambda}{p} \int_{\Omega} \frac{|u_n|^p}{|x|^p} \, dx \right) \leq A.
\]

Now using the improved Hardy-Sobolev inequality stated in (2.15), it follows that

\[
\|u_n\|_{W^{1,s}_0(\Omega)}^p \leq C \text{ for all } s < p \text{ and for all } n \geq 1.
\]

Since \( q + 1 < p^* \), we get the existence of \( 1 < s_0 < p \), such that

\[
q + 1 < s_0^* = \frac{s_0N}{N - s_0}.
\]

Fix \( s_0 \) to get the above estimate, then \( \|u_n\|_{W^{1,s_0}_0(\Omega)}^p \leq C \). In the same way and using (2.15), we get the existence of a positive constant \( a \) such that \( J_{\lambda_n}(u_n) \geq a \). This follows using the fact that \( a^p - Ca^{q+1} > 0 \) if \( a \) is small enough.
Hence we get the existence of \( u_0 \in W_0^{1,s_0}(\Omega) \) such that \( u_n \rightharpoonup u \) weakly in \( W_0^{1,s_0}(\Omega) \) and \( u_n \rightarrow u \) strongly in \( L^{q+1}(\Omega) \).

Since
\[
J_{\lambda_n}(u_n) = \left( \frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega (u_n)^{q+1} \, dx \rightarrow \left( \frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega (u_0)^{q+1} \, dx,
\]
then \( u_0 \geq 0 \) and \( u_0 \) solves
\[
-\Delta_p u = \Lambda_{N,p} \frac{u^{p-1}}{|x|^p} + u^q \text{ in } \Omega, \ u_n \in W_0^{1,s_0}(\Omega)
\]
at least in the distributional sense. It is clear that, by the above computation, \( u_0 \in W_0^{1,s}(\Omega) \) for all \( s < p \). Hence the existence result follows.

The third case \( q_-(\lambda) \leq q < q_+(\lambda) \) and \( \lambda < \Lambda_{N,p} \).

Recall that \( p - 1 < q_- (\lambda) < p^* - 1 < q_+ (\lambda) \). Let \( R > 0 \) be such that \( \Omega \subset \subset B_R(0) \), then using a dilatation argument, without loss of generality one can put \( R = 1 \). Assume that \( h(x) \leq c/|x|^p \), where \( c > 0 \) will be chosen later.

By a continuity argument, we get the existence of \( \lambda_1 > \lambda \) such that \( q_- (\lambda_1) < q < q_+ (\lambda_1) \). Define
\[
w(x) = |x|^{-\alpha} - 1 \text{ for } x \in B_1(0), \text{ with } \alpha = \frac{p}{q - (p-1)},
\]
then
\[
\frac{w^{p-1}}{|x|^p} + w^q \in L^1(B_1(0))
\]
and \( w \) solves
\[
-\Delta_p w = \lambda_1 \frac{(w + 1)^{p-1}}{|x|^p} + (w + 1)^q \text{ in } \mathcal{D}'(B_1(0)).
\]
Using the fact that \( \lambda < \lambda_1 \) we get the existence of a positive constant \( c_1 > 0 \) such that
\[
\lambda_1 \frac{(w + 1)^{p-1}}{|x|^p} \geq \lambda \frac{w^{p-1}}{|x|^p} + \frac{c_1}{|x|^p}.
\]
Choosing \( c \leq c_1 \), then we obtain a supersolution to problem (2.1).

Let \( w_0 \) be the unique solution to the problem
\[
\begin{cases}
-\Delta_p w_0 = h \text{ in } \Omega, \\
w_0 = 0 \text{ on } \partial\Omega,
\end{cases}
\]
it is clear that \( w_0 \) is subsolution to problem (2.1) with \( w_0 \leq w \). Thus using a monotonicity argument we get the existence result.
3. Problem with absorption term: breaking of resonance

In this section we deal with the existence of a nonnegative solutions to the problem

\[
\begin{align*}
-\Delta_p u + u^q &= \lambda \frac{u^{p-1}}{|x|^p} + h \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (3.1)

**Theorem 7.** Assume that \( q > q_* \equiv \frac{N(p-1)}{N-p} \), then for all \( \lambda > 0 \) and for all \( h \in L^1(\Omega) \), problem (3.1) has a minimal positive entropy solution.

We will also need the following comparison principle, proved in [4]

**Theorem 8.** (Comparison Principle) Assume that \( 1 < p \) and let \( f \) be a nonnegative continuous function such that \( f(x,s)/s^{p-1} \) is decreasing for \( s > 0 \). Suppose that \( u, v \in W^{1,p}_0(\Omega) \) are such that

\[
\begin{align*}
-\Delta_p u &\geq f(x,u), \quad u > 0 \text{ in } \Omega, \\
-\Delta_p v &\leq f(x,v), \quad v > 0 \text{ in } \Omega.
\end{align*}
\] (3.2)

Then \( u \geq v \) in \( \Omega \).

**Proof of Theorem 7.** Let

\[
h_n \equiv T_n(h) \quad \text{and} \quad a_n(x) = \frac{1}{|x|^p + \frac{1}{n}}.
\]

Using the sub-supersolution argument, we get the existence of a unique positive solution to problem

\[-\Delta_p w_n + w_n^q = h_n, w_n \in W^{1,p}_0(\Omega).\]

Notice that the positivity of \( w_n \) follows from the strong maximum principle obtained in [16], moreover the uniqueness is obtained using the comparison principle of Theorem 8. We claim that the approximated problem

\[
\begin{align*}
-\Delta_p u_n + u_n^q &= \lambda a_n(x) T_n(u_n^{p-1}) + h_n, \\
u_n &\in W^{1,p}_0(\Omega), \quad u_n \geq 0
\end{align*}
\] (3.3)

has a unique positive solution \( u_n \), such that \( u_n \leq u_{n+1} \) for all \( n \geq 1 \). Let us begin by showing the existence. Define \( v_n \) as the unique positive solution to problem

\[-\Delta_p v_n = n\lambda a_n(x) + h_n, v_n \in W^{1,p}_0(\Omega), \]

then \( v_n \) is a supersolution to problem (3.3). Since \( u \equiv 0 \) is a subsolution to (3.3), then using an iteration argument, we get the existence of a solution \( u_n \) such that \( u_n \leq v_n \). The positivity of \( u_n \) follows using the result of [16]. To get the uniqueness we use Lemma 8. Let \( D_n(x,s) = \lambda a_n(x) T_n(s^{p-1}) + h_n(x) - s^q, s \geq 0 \), it is clear that, for \( s > 0 \),
Let \( D_n(x,s)/s^{p-1} \) be a decreasing function, then the uniqueness result follows. Using the fact that \( D_n(x,s) \leq D_{n+1}(x,s) \), it results that \( u_{n+1} \) is a supersolution to (3.3), thus \( u_n \leq u_{n+1} \) and the claim follows.

Let \( k > 0 \) fixed, using \( T_k(u_n) \) as a test function in (3.1) we get

\[
\int_{\Omega} |\nabla T_k u_n|^p \, dx + \int_{\Omega} u_n^q T_k u_n \, dx = \lambda \int_{\Omega} a_n(x) T_n(u_n^{p-1}) T_k(u_n) \, dx + \int_{\Omega} h_n(x) T_k u_n \, dx.
\]

Using Hölder inequality we reach that

\[
\lambda \int_{\Omega} a_n(x) T_n(u_n^{p-1}) \, dx \leq \lambda \left( \int_{\Omega} u_n^q \, dx \right)^{\frac{p-1}{q}} \left( \int_{\Omega} \frac{1}{|x|^\frac{pq}{q-(p-1)}} \, dx \right)^{\frac{q-(p-1)}{q}}.
\]

Recall that \( q > \frac{N(p-1)}{N-p} \), then \( \frac{pq}{q-(p-1)} < N \), hence

\[
\lambda \int_{\Omega} a_n(x) T_n(u_n^{p-1}) T_k(u_n) \, dx \leq C \lambda \left( \int_{\Omega} u_n^q \, dx \right)^{\frac{p-1}{q}}.
\]

Thus

\[
\int_{\Omega} |\nabla T_k u_n|^p \, dx + \int_{\Omega} u_n^q T_k u_n \, dx \leq C k \lambda \left( \int_{\Omega} u_n^q \, dx \right)^{\frac{p-1}{q}} + k ||h||_{L^1}.
\]

Notice that

\[
\int_{\Omega} u_n^q T_k u_n \, dx \geq \int_{\Omega} u_n^q \, dx - C(k).
\]

Hence

\[
\int_{\Omega} |\nabla T_k u_n|^p \, dx + \int_{\Omega} u_n^q \, dx \leq C(k, \lambda, ||h||_{L^1}).
\]

Therefore we conclude that

\[
\int_{\Omega} u_n^q \, dx \leq C \text{ uniformly in } n,
\]

\[
\int_{\Omega} a_n(x) T_n(u_n^{p-1}) \, dx \leq C \text{ uniformly in } n.
\]

Using the monotonicity of the sequence \( \{u_n\} \) we get the existence of a measurable function \( u \) such that

\[
u_n^q \uparrow u \quad \text{and} \quad a_n(x) T_n(u_n^{p-1}) \uparrow \frac{u^{p-1}}{|x|^p} \quad \text{strongly in } L^1(\Omega).
\]

Setting \( f_n = a_n(x) T_n(u_n^{p-1}) - u_n^q \), then \( f_n \to f \equiv \frac{u^{p-1}}{|x|^p} - u^q \) strongly in \( L^1(\Omega) \). Thus following the arguments of [10], we reach that \( u \) is an entropy solution to (3.1). It
is not difficult to show that if \( v \) is another positive entropy solution to (3.1), then \( v \geq u_n \) for all \( n \geq 0 \), thus \( v \geq u \).

To show the optimality of the condition imposed in the Theorem 7 we prove the next non existence result.

**THEOREM 9.** Assume that \( q < q_* \). If \( \lambda > \Lambda_{N,p} \), then problem (3.1) has no very weak positive supersolution in the sense that \( u^q, u^{p-1}/|x|^p \in L^1_{\text{loc}}(\Omega) \) and

\[
\int \left( (-\Delta_p u) \phi + |u|^q \phi \right) dx \geq \lambda \int \frac{u^{p-1}}{|x|^p} dx + \int h(x) \phi dx, \text{ for all } \phi \in C^\infty_0(\Omega).
\]

**Proof.** Without loss of generality we can assume that \( h \in L^\infty(\Omega) \). We argue by contradiction. Suppose that for some \( \lambda > \Lambda_{N,p} \), problem (3.1) has a nonnegative very weak supersolution \( u \) in the sense defined above. Let \( \Omega_1 \subset \subset \Omega \), then \( u \) is a supersolution to problem (3.1) in \( \Omega_1 \). Thus using an iteration argument we get the existence of \( u_1 \), the minimal entropy positive solution to (3.1) in \( \Omega_1 \). It is clear that \( u_1^{p-1} \in L^s(\Omega_1) \) for all \( s < N/(N-p) \).

Let \( B_\eta(0) \subset \subset \Omega_1 \) where \( \eta \) is a small constant to be chosen later. Consider \( \phi \in C^\infty_0(B_\eta(0)) \), since \( u_1 > 0 \) in \( B_\eta(0) \), then using Picone inequality it follows that

\[
\int_{B_\eta(0)} |\nabla \phi|^p dx \geq \lambda \int_{B_\eta(0)} \frac{|\phi|^p}{|x|^p} dx - \int_{B_\eta(0)} u_1^{q-(p-1)} |\phi|^p.
\]

Since \( q < q_* \), then

\[
(q - (p-1)) \frac{p^*}{p^* - p} < \frac{N(p-1)}{N-p},
\]

thus using Hölder and Sobolev inequality inequalities we obtain that

\[
\int_{B_\eta(0)} u^{q-(p-1)} |\phi|^p dx \leq \left( \int_{B_\eta(0)} |\phi|^p dx \right)^{\frac{p^*}{p}} \left( \int_{B_\eta(0)} u^{q-(p-1) - \frac{p^*}{p^* - p}} dx \right)^{\frac{p^* - p}{p}}
\]

\[
\leq S^{-1} \int_{B_\eta(0)} |\nabla \phi|^p dx \left( \int_{B_\eta(0)} u^{q-(p-1) - \frac{p^*}{p^* - p}} dx \right)^{\frac{p^* - p}{p}}.
\]

Using the fact that

\[
(q - (p-1)) \frac{p^*}{p^* - p} < \frac{N(p-1)}{N-p},
\]

there result that

\[
\int_{\Omega_1} u^{q-(p-1) - \frac{p^*}{p^* - p}} dx < \infty. \tag{3.5}
\]

We claim that

\[
\lim_{\eta \to 0} \int_{B_\eta(0)} u^{q-(p-1) - \frac{p^*}{p^* - p}} dx = 0. \tag{3.6}
\]

To prove (3.6), we set

\[
k_\eta(x) \equiv u^{q-(p-1) - \frac{p^*}{p^* - p}} 1_{B_\eta(0)}.
\]
then
\[ k_\eta \leq u^{(q-(p-1))\frac{p^*}{p-p}} \] and \[ ||k_\eta||_{L^1(\Omega_1)} \leq ||u^{(q-(p-1))\frac{p^*}{p-p}}||_{L^1(\Omega_1)} \]
for all \( \eta << 1 \).

Using the fact that \( k_\eta \to 0 \) a.e. in \( \Omega_1 \), then by the dominated convergence Theorem we reach that
\[ k_\eta \to 0 \text{ strongly in } L^1. \]

Hence the claim follows.

Since \( \lambda > \Lambda_{N,p} \), we get the existence of \( \varepsilon > 0 \) such that choosing \( \eta \) small enough we obtain that
\[
\frac{\lambda}{1 + S^{-1}\left(\int_{B_\eta(0)} u^{(q-(p-1))\frac{p^*}{p-p}} \, dx\right)^\frac{p-p}{p}} \geq \Lambda_{N,p} + \varepsilon.
\]

Hence, back to (3.4) we obtain
\[
\int_{B_\eta(0)} |\nabla \phi|^p \, dx \geq (\Lambda_{N,p} + \varepsilon) \int_{B_\eta(0)} |\phi|^p \frac{1}{|x|^p} \, dx
\]
a contradiction with Hardy inequality.

**Remark 2.** Fix \( q > p - 1 \) and let \( g \) be a measurable function such that \( g \geq 0 \) and \( g^{\frac{q}{q-(p-1)}} \in L^1(\Omega) \), then using the same arguments as in the proof of Theorem 7 we can prove that problem
\[
\begin{cases}
-\Delta_p u + u^q = \lambda g(x)u + h(x) \text{ in } \Omega, \\
u > 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]
has an entropy positive solution for all \( \lambda > 0 \) and for all \( h \in L^1(\Omega) \). In this case we say that \( g \) is an admissible weight related to the problem (3.7).

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**References**


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