Differential
Equations
& Applications
Volume 5, Number 2 (2013), 155–184
doi:10.7153/dea-05-11

HYBRID FIXED POINT THEORY IN PARTIALLY ORDERED NORMED LINEAR SPACES AND APPLICATIONS TO FRACTIONAL INTEGRAL EQUATIONS

BAPURAO C. DHAGE

(*Communicated by Sotiris K. Ntouyas*)

Abstract. In this paper, some basic hybrid fixed point theorems of Banach and Schauder type and some hybrid fixed point theorems of Krasnoselskii type involving the sum of two operators are proved in a partially ordered normed linear spaces which are further applied to nonlinear Volterra fractional integral equations for proving the existence of solutions under certain monotonic conditions blending with the existence of either a lower or an upper solution type function.

This research is dedicated in the loving memory of my late father and mother who imbibed in me the honesty, hard-work and services for all.

1. Introduction

It is well-known that the hybrid fixed point theorems which are obtained using the mixed arguments from different branches of mathematics are very rich in applications to allied areas of mathematics, particularly to the theory of nonlinear differential and integral equations. See Aman [1], Heikkiïa and Lakshmikantham [13], Zeidler [22] and the references given therein. Recently, Ran and Reurings [20] initiated the study of hybrid fixed point theorems in partially ordered sets which is further continued in Nieto and Rodriguez-Lopez [17, 18] and proved the following hybrid fixed point theorems for the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra, analysis and geometry.

THEOREM 1.1. (Nieto and Rodriguez-Lopez [17]) *Let $(X,\preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X,d) is a complete metric space. Let $T:X\to X$ be a monotone nondecreasing mapping such that there exists a constant $k\in(0,1)$ such that*

$$d(Tx,Ty)\leqslant k\,d(x,y) \tag{1}$$

Mathematics subject classification (2010): 45G10, 45M99, 47H09, 47H10.
Keywords and phrases: Hybrid fixed point theorem, Partially ordered normed linear space, Fractional integral equation, Existence theorem.

Paper DEA-05-11

for all comparable elements $x, y \in X$. *Assume that either* T *is continuous or* X *is such that if* $\{x_n\}$ *is a nondecreasing sequence with* $x_n \to \overline{x}$ *in* X, *then* $x_n \preceq \overline{x}$ *for all* $n \in \mathbb{N}$. *Further if there is an element* $x_0 \in X$ *satisfying* $x_0 \preceq Tx_0$, *then* T *has a fixed point which is further unique if "every pair of elements in* X *has a lower and an upper bound."*

Another hybrid fixed point theorem in the above direction can be stated as follows.

THEOREM 1.2. (Nieto and Rodriguez-Lopez [18]) *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there is a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T : X \to X$ *be a monotone nondecreasing mapping satisfying (1). Assume that either* T *is continuous or* X *is such that if* $\{x_n\}$ *is a nonincreasing sequence with* $x_n \to \overline{x}$ *in* X, *then* $x_n \succeq \overline{x}$ *for all* $n \in \mathbb{N}$. *Further if there is an element* $x_0 \in X$ *satisfying* $x_0 \succeq Tx_0$, *then* T *has a fixed point which is further unique if "every pair of elements in* X *has a lower and an upper bound."*

After the publication of the above two fixed point theorems there is a huge upsurge in the development of metric fixed point theory in the partially ordered metric spaces. A good number of fixed and common fixed point theorems have been proved in the literature for two, three and four mappings in a metric space by modifying the contraction condition (1) suitably as per the requirement of the results. We claim that almost all the results proved so far along this line though not mentioned have their origin in a paper due to Heikillä and Lakshmikantham [13]. The main difference is the convergence criteria of sequence of iterations of the monotone mappings under consideration. The convergence of the sequence in Heikillä and Lakshmikantham [13] is straight forward whereas the convergence of the sequence in Nieto and Rodriguez-Lopez [17] is due mainly to the metric condition of contraction. The hybrid fixed point theorem of Heikillä and Lakshmikantham [13] for the monotone mappings in ordered metric spaces is as follows.

THEOREM 1.3. (Heikkilä and Lakshmikantham [13]) *Let* $[a, b]$ *be an order interval in a subset* Y *of the ordered metric space* X *and let* $G : [a, b] \to [a, b]$ *be a nondecreasing mapping. If the sequence* $\{Gx_n\}$ *converges in* Y *whenever* $\{x_n\}$ *is a monotone sequence in* $[a, b]$, *then the well ordered chain of* G*-iterations of* a *has the maximum* x_* *which is a fixed point of* G. *Moreover,* $x_* = \min\{y \in [a, b] \mid Gy \preceq y\}$.

The above hybrid fixed point theorem is applicable in the study of discontinuous nonlinear equations and has been used throughout the research monograph of Heikkilä and Lakshmikantham [13]. Note that the convergence of the monotone sequence in Theorem 1.3 is replaced in Theorems 1.1 and 1.2 by the Cauchy sequence $\{Gx_n\}$ and completeness of X. Further, the Cauchy nondecreasing sequence is substituted by equivalent contraction condition for comparable elements in X. Theorems 1.1 and 1.2 are the best hybrid fixed point theorems because they are derived for the mixed arguments from algebra and geometry. The main advantage of Theorems 1.1 and 1.2 is that the uniqueness of fixed point of the monotone mappings is obtained under certain additional conditions on the domain space such as lattice structure of the partially ordered space under consideration and these fixed point results are then useful in establishing the uniqueness of the solution of nonlinear differential and integral equations. Again, some hybrid fixed point theorems of Krasnoselskii type for monotone mappings are proved in Dhage [5, 6] along the lines of Tarski [21] and Heikillä and Lakshmikantham [13].

The existence part of Theorems 1.1 and 1.2 may be generalized under weaker contraction condition as follows.

THEOREM 1.4. *Let* $(X,\preceq)$ *be a partially ordered set and suppose that there is a metric* d *in* X *such that* (X,d) *is a complete metric space. Let* $T:X\to X$ *be a monotone nondecreasing mapping such that there exists a constant* $k\in(0,1)$ *such that*

$$d(Tx,T^2x)\leqslant k\,d(x,Tx) \tag{2}$$

for all elements $x\in X$ *comparable to* $Tx\in X$. *Further if* T *is continuous and there is an element* $x_0\in X$ *satisfying* $x_0\preceq Tx_0$ *or* $x_0\succeq Tx_0$, *then* T *has a fixed point.*

The proof of Theorem 1.4 is essentially same as Theorem 1.1. Note that contraction condition (2) is weaker than (1) and which is obtained by letting $y=Tx$ in the contraction condition (1).

In this paper we again improve the results of Dhage [5, 6] under weaker conditions and apply them to nonlinear Volterra type fractional integral equations for proving the existence results under certain monotonic conditions blending with the existence of either a lower or an upper solution type function for the integral equation under consideration.

2. Basic Hybrid Fixed Point Theorems

Let X be a real vector or linear space. We introduce a partial order $\preceq$ in X as follows. A relation $\preceq$ in X is said to be partial order if it satisfies the following properties:

1. Reflexivity: $a\preceq a$ for all $a\in X$,
2. Antisymmetry: $a\preceq b$ and $b\preceq a$ implies $a=b$,
3. Transitivity: $a\preceq b$ and $b\preceq c$ implies $a\preceq c$, and
4. Order linearity: $x_1\preceq y_1$ and $x_2\preceq y_2 \Rightarrow x_1+x_2\preceq y_1+y_2$; and $x\preceq y\Rightarrow tx\preceq ty$ for $t\geqslant 0$.

The linear space X together with a partial order $\preceq$ becomes a ***partially ordered linear or vector space***. Two elements x and y in a partially ordered linear space X are called ***comparable*** if either the relation $x\preceq y$ or $y\preceq x$ holds. We introduce a norm $\|\cdot\|$ in a partially ordered linear space X so that X becomes now a ***partially ordered***

normed linear space. If X is complete with respect to the metric d defined through the above norm, then it is called a ***partially ordered complete normed linear space***.

The following definitions are frequently used in the subsequent part of this paper.

DEFINITION 2.1. A mapping $T : X \to X$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

DEFINITION 2.2. A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, $\mathscr{D}$**-function** if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\psi(0) = 0$.

DEFINITION 2.3. Given a partially ordered normed linear space E, a mapping $Q : E \to E$ is called a **partially** $\mathscr{D}$**-Lipschitz** or **partially nonlinear** $\mathscr{D}$**-Lipschitz** if there is a $\mathscr{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Qx - Qy\| \leqslant \psi(\|x - y\|) \tag{3}$$

for all comparable elements $x, y \in E$. The function ψ is called a $\mathscr{D}$-function of Q on E. If $\psi(r) = kr$, $k > 0$, then Q is called **partially Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a **partially contraction** on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called a **partially nonlinear** $\mathscr{D}$**-contraction** with a $\mathscr{D}$-function ψ of Q on X.

The details of different types of contraction definitions appear in the monographs of Krasnoselskii [14] and Granas and Dugundji [11]. There do exist $\mathscr{D}$-functions and the commonly used $\mathscr{D}$-functions are $\psi(r) = kr$ and $\psi(r) = \dfrac{r}{1+r}$ etc. These $\mathscr{D}$-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existence and uniqueness results via fixed point methods. See Browder [3], Deimling [4], Granas and Dugundji [11] and Krasnoselskii [14].

Other notions that we frequently need in what follows are the following definitions.

DEFINITION 2.4. An operator Q on a normed linear space E into itself is called **compact** if $Q(E)$ is a relatively compact subset of E. Q is called **totally bounded** if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called **completely continuous** on E.

DEFINITION 2.5. An operator Q on a normed linear space E into itself is called **partially compact** if $Q(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. Q is called **partially totally bounded** if for any totally ordered and bounded subset C of E, $Q(C)$ is a relatively compact subset of E. If Q is continuous and partially totally bounded, then it is called **partially completely continuous** on E.

REMARK 2.1. Note that every compact mapping in a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is continuous and partially totally bounded, but the converse may not be true.

Let (X,d) be a metric space and let $T : X \to X$ be a mapping. Given an element $x \in X$, we define an orbit $\mathscr{O}(x;T)$ of T at x by

$$\mathscr{O}(x;T) = \{x, Tx, T^2x, ..., T^nx, \dots\}.$$

Then T is called T-orbitally continuous on X if for any sequence $\{x_n\} \subseteq \mathscr{O}(x;T)$, we have that $x_n \to x^*$ implies $Tx_n \to Tx^*$ for each $x \in X$. The metric space X is called T-orbitally complete if every Cauchy sequence $\{x_n\} \subseteq \mathscr{O}(x;T)$ converses to a point x^* in X. Notice that continuity implies that T-orbitally continuity and completeness implies T-orbitally completeness of a metric space X, but the converse may not be true.

The following result is frequently used in the analytical fixed point theory of metric spaces.

LEMMA 2.1. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathscr{D}$-function satisfying $\psi(r) < r$ for $r > 0$. Then $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$.*

Proof. If $t = 0$, then the result follows immediately. So we assume that $t > 0$. By definition of ψ,

$$\psi(t) < t \Longrightarrow \psi^2(t) \leqslant \psi(t).$$

By induction principle,

$$\psi^{n+1}(t) \leqslant \psi^n(t)$$

for each $n = 0, 1, 2, \dots$. Thus $\{\psi^n(t)\}$ is a sequence of positive real numbers which is bounded below by 0. Hence, $\lim_{n\to\infty} \psi^n(t) = d$ exists. If $d \neq 0$, then by upper semi-continuity of ψ,

$$d = \lim_{n\to\infty} \psi^{n+1}(t) = \lim_{n\to\infty} \psi(\psi^n(t)) \leqslant \psi\left(\lim_{n\to\infty} \psi^n(t)\right) = \psi(d) < d$$

which is a contradiction.Hence, $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in \mathbb{R}_+$. □

Now we are well equipped to state and prove our main hybrid fixed point results of this paper. The slight generalizations of Theorems 1.1 and 1.2 are as follows.

THEOREM 2.1. *Let $(X, \preceq)$ be a partially ordered set and let $T : X \to X$ be a nondecreasing mapping. Suppose that there is a metric d in X such that (X,d) is a T-orbitally complete metric space. Assume that there exists a $\mathscr{D}$-function ψ such that*

$$d(Tx,Ty) \leqslant \psi(d(x,y)) \tag{4}$$

for all comparable elements $x, y \in X$ satisfying $\psi(r) < r$ for $r > 0$. Further assume that either T is T-orbitally continuous on X or X is such that if $\{x_n\}$ is a nondecreasing sequence with $x_n \to \overline{x}$ in X, then $x_n \preceq \overline{x}$ for all $n \in \mathbb{N}$. If there is an element $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound."

Proof. The proof is standard, but for the sake of completeness we give the details of it. Define a sequence $\{x_n\}$ of successive iterations of T at x_0 as

$$x_{n+1} = Tx_n,\ n = 0, 1, \dots. \tag{5}$$

By isotonicity of T, we obtain

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots. \tag{6}$$

If $x_n = x_{n+1}$ for some $\in \mathbb{N}$, then $u = x_n$ is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. If $x = x_{n-1}$ and $y = x_n$, then by condition (4), we obtain

$$d(x_n, x_{n+1}) \leqslant \psi(d(x_{n-1}, x_n)) \tag{7}$$

for each $n = 1, 2, \dots$.

Denote $d_n = d(x_n, x_{n+1})$. Since ψ is a $\mathscr{D}$-function, $\{d_n\}$ is a decreasing sequence of real numbers which is bounded below by 0. Hence $\{d_n\}$ is convergent and there exists a real number d such that

$$\lim_{n\to\infty} d_n = d(x_n, x_{n+1}) = d.$$

We show that $d = 0$. If $d \neq 0$, then

$$d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) \leqslant \lim_{n\to\infty} \psi(d(x_{n-1}, x_n)) \leqslant \psi(d) < d$$

which is a contradiction. Hence $d = 0$.

We show that $\{x_n\}$ is a Cauchy sequence in X. Suppose not. Then for $\varepsilon > 0$ there exists a positive integer $k \leqslant n(k) \leqslant m(k)$ such that

$$d(x_{m(k)}, x_{n(k)}) \geqslant \varepsilon$$

for all $m(k) \geqslant n(k) \geqslant k$.

Denote $r_k = d(x_{m(k)}, x_{n(k)})$. Then we have

$$\begin{aligned} \varepsilon \leqslant r_k &= d(x_{m(k)}, x_{n(k)}) \\ &\leqslant d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &< d_{m(k)-1} + \varepsilon \end{aligned}$$

and so, $\lim_{k\to\infty} r_k = \varepsilon$.

Again,

$$\begin{aligned}\varepsilon \leqslant r_k &= d(x_{m(k)}, x_{n(k)}) \\ &\leqslant d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+}, x_{n(k)}) \\ &\leqslant d_{m(k)} + \psi(r_k) + d_{n(k)}.\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\varepsilon \leqslant \psi(\varepsilon) < \varepsilon.$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in X. The metric space (X,d) being T-orbitally complete, there is a point $x^* \in X$ such that

$$\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} x_n = x^*.$$

Suppose that T is T-orbitally continuous on X. Then,

$$Tx^* = T\left(\lim_{n\to\infty} T^n x_0\right) = \lim_{n\to\infty} T^{n+1} x_0 = x^*.$$

Next, suppose that $T^n x_0 = x_n \leqslant x^*$ for all $n \in \mathbb{N}$. Then by upper semi-continuity of ψ, we obtain

$$\begin{aligned}d(Tx^*, x^*) &\leqslant d(Tx^*, T^{n+1}x_0) + d(T^{n+1}x_0, x^*) \\ &\leqslant \psi(d(x^*, T^n x_0)) + \psi(d(T^n x_0, x^*)) \\ &\leqslant \psi(\lim_{n\to\infty} d(x^*, T^n x_0)) + \psi\left(\lim_{n\to\infty} d(T^n x_0, x^*)\right) \\ &= 0.\end{aligned}$$

Hence, $Tx^* = x^*$. Thus, in both the cases T has a fixed point.

To prove uniqueness, let y^* be another fixed point of T in X and suppose that every pair of elements in X has a lower and an upper bound. Then there is a point $z \in X$ such that either $z \preceq x$ and $z \preceq y$, or $x \preceq z$ and $y \preceq z$. Thus in both the cases, by Lemma 2.1, we have

$$\begin{aligned}d(x^*, y^*) &\leqslant d(x^*, T^n z) + d(T^n z, y^*) \\ &= d(T^n x^*, T^n z) + d(T^n z, T^n y^*) \\ &\leqslant \psi^n\big(d(x^*, z)\big) + \psi^n\big(d(z, y^*)\big) \\ &= 0 \quad \text{as } n \to \infty.\end{aligned}$$

This shows that T has a unique fixed point and the proof of the theorem is complete. □

As a special case of our Theorem 2.1 we obtain Theorem 1.1 of Nieto and Rodriguez-Lopez [17] for partially contraction mappings in partially ordered complete metric spaces.

Sometimes it possibles that a mapping T is not a nonlinear $\mathscr{D}$-contraction, but some iterations of it is a nonlinear $\mathscr{D}$-contraction on X. In this connection the following hybrid fixed point theorem is noteworthy.

THEOREM 2.2. *Let $(X,\preceq)$ be a partially ordered set and let $T: X \to X$ be a nondecreasing mapping. Suppose that there is a metric d in X such that (X,d) is a T-orbitally complete metric space. Assume that there exists a $\mathscr{D}$-function ψ and a positive integer p such that*

$$d(T^p x, T^p y) \leqslant \psi(d(x,y)) \tag{8}$$

for all comparable elements $x,y \in X$ satisfying $\psi(r) < r$ for $r > 0$. Further assume that either T is T-orbitally continuous on X or X is such that if $\{x_n\}$ is a nondecreasing sequence with $x_n \to \overline{x}$ in X, then $x_n \preceq \overline{x}$ for all $n \in \mathbb{N}$. If there is an element $x_0 \in X$ satisfying $x_0 \preceq Tx_0$ and "every pair of elements in X has a lower and an upper bound," then T has a unique fixed point.

Proof. Set $S = T^p$. Then $S: X \to X$ is a S-orbitally continuous nondecreasing mapping. Also there exists the element $x_0 \in X$ such that $x_0 \preceq Sx_0$. Now, an application of Theorem 2.1 yields that S has a unique fixed point, that is, it is a point $u \in X$ such that $S(u) = T^p(u) = u$. Now $T(u) = T(T^p u) = S(Tu)$, showing that Tu is again a fixed point of S. By the uniqueness of u, we get $Tu = u$. The proof is complete. $\square$

THEOREM 2.3. *Let $(X,\preceq)$ be a partially ordered set and let T be a nondecreasing mapping. Suppose that there is a metric d in X such that (X,d) is a T-orbitally complete metric space. Assume that there exists a $\mathscr{D}$-function ψ satisfying he contractive condition (4). Further assume that either T is T-orbitally continuous on X or X is such that if $\{x_n\}$ is a nonincreasing sequence with $x_n \to \overline{x}$ in X, then $x_n \succeq \overline{x}$ for all $n \in \mathbb{N}$. If there is an element $x_0 \in X$ satisfying $x_0 \succeq Tx_0$, then T has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound."*

Proof. The proof is similar to Theorem 2.1 and therefore we omit the details.

THEOREM 2.4. *Let $(X,\preceq)$ be a partially ordered set and let $T: X \to X$ be nondecreasing mapping. Suppose that there is a metric d in X such that (X,d) is a T-orbitally complete metric space. Assume that there exists a $\mathscr{D}$-function ψ and a positive integer p satisfying he contractive condition (8). Assume further that either T is T-orbitally continuous on X or X is such that if $\{x_n\}$ is a nondecreasing sequence with $x_n \to \overline{x}$ in X, then $x_n \succeq \overline{x}$ for all $n \in \mathbb{N}$. If there is an element $x_0 \in X$ satisfying $x_0 \succeq Tx_0$ and "every pair of elements in X has a lower and an upper bound," then T has a unique fixed point.*

REMARK 2.2. The hypothesis concerning the conditions $x_0 \preceq Tx_0$ and $x_0 \succeq Tx_0$ in Theorems 2.2 and 2.4 may be replaced by the hypothesis with weaker conditions $x_0 \preceq T^p x_0$ and $x_0 \succeq T^p x_0$ respectively.

REMARK 2.3. The monotone hypothesis of the mapping T together with the condition $x_0 \preceq Tx_0$ or $x_0 \succeq T^p x_0$ for some $x_0 \in X$ in all above hybrid fixed point theorems may be replaced with the dominating character of T, that is, either $x \preceq Tx$ or $x \succeq Tx$ for all $x \in X$.

REMARK 2.4. The convergence condition that X is such that if $\{x_n\}$ is a nondecreasing sequence with $x_n \to \bar{x}$ in X, then $x_n \preceq \bar{x}$ for all $n \in \mathbb{N}$, holds in particular if X is a normed linear space and the order relation $\preceq$ is defined in X through the order cones $\mathscr{K}$ in it which is defined later in Section 3. The hybrid fixed point theorems with this convergence condition is useful in the study of nonlinear differential and integral equations with discontinuous nonlinearities.

REMARK 2.5. The conclusion of Theorems 2.1 and 2.3 remains true if we replace the contractive condition (4) with the following condition of generalized contraction ,

$$d(Tx,Ty) \leqslant \psi\left(\max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\}\right) \tag{9}$$

and the conclusion of Theorems 2.2 and 2.4 also remains true if we replace the contractive condition (8) with the following generalized contraction condition,

$$d(T^p x,T^p y) \leqslant \psi\left(\max\left\{ d(x,y), d(x,T^p x), d(y,T^p y), \frac{1}{2}\left[d(x,T^p y)+d(y,T^p x)\right]\right\}\right). \tag{10}$$

The contraction conditions of the type (9) have been employed in several fixed point theorems for the mappings in partially ordered metric spaces, however to the best of our knowledge, the contraction condition of the type (10) has never been used in the fixed point theorems in ordered spaces.

REMARK 2.6. The argument that every pair of elements in an ordered set X has a lower and an upper bound holds if $(X, \preceq)$ is a lattice. The details about the ordered sets and the lattice structure may be found in Birkhoff [2].

Note that Theorems 2.1, 2.2, 2.3 and 2.4 have some nice applications to various nonlinear problems modeled on nonlinear equations for proving the existence as well as uniqueness of the solutions under generalized Lipschitz conditions. The following two fixed point theorems are perhaps new to the literature. The basic principle in formulating these theorems is the same as that of Dhage [5] and Nieto and Rodriguez-Lopez [17]. Before stating these results we give a useful definition.

DEFINITION 2.6. The order relation $\preceq$ and the metric d in a non-empty set X are said to be **compatible** if $\{x_n\}$ is a monotone (order preserving), that is, monotone nondecreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(X, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leqslant$ and the norm defined by the absolute value function has this property. Similarly, the space $C(J,\mathbb{R})$ with usual order relation defined by $x \leqslant y$ if and only if $x(t) \leqslant y(t)$ for all $t \in J$ and the usual standard supremum norm $\|\cdot\|$ defined by $\|x\| = \sup_{t\in J}|x(t)|$ are compatible.

THEOREM 2.5. *Let X be a partially ordered linear space and suppose that there is a norm in X such that X is a normed linear space. Let $T : X \to X$ be a nondecreasing, partially compact and continuous mapping. Further if the order relation $\preceq$ and the norm $\|\cdot\|$ in X are compatible and if there is an element $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a fixed point.*

Proof. Define a sequence $\{x_n\}$ of successive iterations of T at x_0 as

$$x_{n+1} = Tx_n,\ n = 0,1,2,\dots. \tag{$*$}$$

Since T is nondecreasing and $x_0 \preceq Tx_0$, one has

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots. \tag{$**$}$$

Denote $C = \{x_0,x_1\dots,x_n,\dots\}$. Then C is totally ordered set in the normed linear space X and from the partially compactness of T, it follows that $T(C)$ is relatively compact subset of X. As a result $\{x_1,\dots,x_n,\dots\} \subseteq T(C)$ and the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to some point x^* in X. From the compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$, it follows that the whole sequence sequence $\{x_n\}$ converges to x^*. Finally, by the continuity of T, we obtain

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

This completes the proof. □

THEOREM 2.6. *Let X be a partially ordered linear space and suppose that there is a norm in X such that X is a normed linear space. Let $T : X \to X$ be a nondecreasing, partially compact and continuous mapping. Further if the order relation $\preceq$ and the norm $\|\cdot\|$ in X are compatible and if there is an element $x_0 \in X$ satisfying $x_0 \succeq Tx_0$, then T has a fixed point.*

Proof. The proof is similar to Theorem 2.4 and we omit the details. □

REMARK 2.7. The hypothesis of continuity on the mapping T in above Theorems 2.4 and 2.5 may be replaced with a weaker T-orbitally continuity condition of the mapping T on X.

In the following section, we try to combine Theorem 2.1 and Theorem 2.5 to yield some Krasnoselskii type hybrid fixed point theorems in a partially ordered complete normed linear space and discuss some of their applications to nonlinear Volterra type fractional integral equations.

3. Krasnoselskii Type Fixed Point Theorems

Here, in this section, first we list different Krasoselskii [14] type fixed point theorems according to their degree of generality available in the literature. The following version of Krasnoselskii's fixed point theorem due to Dhage [8] is known in the literature which do not need any order structure of the Banach space under consideration and has some nice applications to nonlinear linearly perturbed differential and integral equations.

THEOREM 3.1. (Dhage [8]) *Let S be a closed convex and bounded subset of a Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators satisfying the following conditions:*

(a) A is nonlinear $\mathscr{D}$-contraction,

(b) B is completely continuous, and

(c) $Ax + By = x$ for all $y \in S \Longrightarrow x \in S$.

Then the operator equation

$$Ax + Bx = x \tag{11}$$

has a solution.

Theorem 3.1 is very much useful and has been applied to linear perturbations of differential and integral equations by several authors in the literature for proving the existence of the solutions under mixed Lipschitz and compactness type conditions. Here we do not need any other structure of the Banach space under consideration. See Dhage [7], Dhage and Jadhav [9] and the references cited therein.

The theory of Krasnoselskii type fixed point theorems using the order relation is initiated by the present author in Dhage [6] and developed further in a series of papers. See Dhage [5, 6, 7, 8] and the references therein. Below we state these hybrid fixed point theorems as per their degree of generality. The following Krasnoselskii type fixed point theorem in a complete lattice is proved in Dhage [5].

THEOREM 3.2. (Dhage [5]) *Let S be a non-empty closed subset of a complete lattice Banach space X and let $A : S \to X$ and $A : S \to X$ be two operators satisfying the following conditions:*

(a) A is nonlinear $\mathscr{D}$-contraction,

(b) B is compact and continuous,

(c) $Ax + By \in S$ for all $x, y \in S$, and

(c) $(I - A)^{-1}B$ is isotonic increasing on S, where I denotes the identity mapping on X.

Then the operator equation (11) has a least and a greatest solution in S.

Other hybrid fixed point theorems concerning the solution of operator equation (11) which make use of order structure of Banach space under consideration are as follows. Before stating these hybrid fixed point theorems, we give a useful definition.

DEFINITION 3.1. A non-empty closed subset $\mathscr{K}$ of a normed linear space X is called an order cone if the following properties are satisfied.

(i) $\mathscr{K}+\mathscr{K}\subseteq\mathscr{K}$,

(ii) $\lambda\mathscr{K}\subseteq\mathscr{K}$, and

(iii) $\{-\mathscr{K}\}\cap\mathscr{K}=\theta$, where θ is the zero element of X.

The ordered Banach space X together with the order cone $\mathscr{K}$ is denoted by $(X,\mathscr{K})$. The details of order cones and their properties appear in Heikkilä and Lakshmikantham [13] and the references given therein. We define the order relation $\leqslant$ in X as follows. For any $x,y\in X$, we define

$$x\leqslant y \iff y-x\in\mathscr{K}. \tag{$*$}$$

The following lemma is sometimes useful in the study of discontinuous fractional differential and integral equations.

LEMMA 3.1. *If $\{x_n\}$ is a nondecreasing sequence in the ordered Banach space $(X,\mathscr{K})$ converging to the point $\overline{x}\in X$, then $x_n\leqslant\overline{x}$ for all $n\in\mathbb{N}$.*

Proof. Since $\{x_n\}$ is nondecreasing, we have

$$x_1\leqslant x_2\leqslant\cdots\leqslant x_n\leqslant\cdots;$$

or

$$x_m\geqslant x_n \quad \forall m\geqslant n\in\mathbb{N}.$$

By definition of $\leqslant$, we get

$$x_m-x_n\in\mathscr{K}\ \forall m\geqslant n\in\mathbb{N}.$$

As the order cone $\mathscr{K}$ is closed, one has

$$\lim_{m\to\infty}\left(x_m-x_n\right)=\lim_{m\to\infty}x_m-x_n=\overline{x}-x_n\in\mathscr{K}$$

which implies that $x_n\leqslant\overline{x}$ for each $n\in\mathbb{N}$. The proof of the lemma is complete. □

Let $x_0,y_0\in X$ be fixed elements with $x_0\leqslant y_0$. Then by an **order interval** or **vector segment** we mean a set $[x_0,y_0]$ in X defined by

$$[x_0,y_0]=\{x\in X\mid x_0\leqslant x\leqslant y_0\}.$$

Clearly, $[x_0,y_0]$ is a closed, convex set in X which is further bounded if the order cone $\mathscr{K}$ in X is normal. A few details of the order cones and their properties, a reader is referred to Heikkilä and Lakshmikatham [13] and the references therein.

THEOREM 3.3. (Dhage [6]) *Let* $[x_0, y_0]$ *be the order interval in an ordered Banach space* X *and let* $A : X \to X$ *and* $B : [x_0, y_0] \to X$ *be two nondecreasing operators satisfying the following conditions:*

(a) A *is nonlinear* $\mathscr{D}$*-contraction,*

(b) B *is compact and continuous, and*

(c) $Ax + By \in [x_0, y_0]$ *for all* $x, y \in [x_0, y_0]$.

If the order cone $\mathscr{K}$ *in a Banach space* X *is normal, then the operator equation* (11) *has a least and a greatest solution in* $[x_0, y_0]$.

Our last known Krasnoselskii type hybrid fixed point theorem which is weaker version than previous two versions is the following result.

THEOREM 3.4. (Dhage [6]) *Let* $[x_0, y_0]$ *be an order interval in an ordered Banach space* X *and let* $A, B : [x_0, y_0] \to X$ *be two operators satisfying the following conditions:*

(a) A *is nonlinear* $\mathscr{D}$*-contraction,*

(b) B *is compact and continuous,*

(c) $Ax + Bx \in [x_0, y_0]$ *for all* $x \in [x_0, y_0]$*, and*

(d) $(I - A)^{-1}B$ *is monotone nondecreasing on* $[x_0, y_0]$.

If the order cone $\mathscr{K}$ *in a Banach space* X *is normal, then the operator equation* (11) *has a least and a greatest solution in* $[x_0, y_0]$.

REMARK 3.1. Note that $(I - A)^{-1}B$ is nondecreasing if A and B are nondecreasing but the converse may not be true. Therefore, Theorem 3.4 is more general hybrid fixed point theorem than any other so far discussed Krasnoselskii type fixed point theorem in an ordered Banach space X in view of hypothesis (c).

Here, we obtain other interesting versions of Krasnoselskii type fixed point theorem in a partially ordered complete normed linear space under weaker hypotheses (a), (b) and (c) of Theorems 3.1 through 3.4.

THEOREM 3.5. *Let* $\big(X, \preceq, \|\cdot\|\big)$ *be a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A, B : X \to X$ *be two nondecreasing operators such that*

(a) A *is continuous and partially nonlinear* $\mathscr{D}$*-contraction,*

(b) B *is continuous and partially compact,*

(c) there exists an element $x_0 \in X$ *such that* $x_0 \preceq Ax_0 + By$ *for all* $y \in X$*, and*

(d) every pair of elements $x, y \in X$ has a lower and an upper bound in X.

Then the operator equation (11) *has a solution in X.*

Proof. Define the operator $T : X \to X$ by

$$T = (I - A)^{-1}B. \tag{12}$$

Clearly, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = Ax + By. \tag{13}$$

A_y is nondecreasing and by hypothesis (c), there is a point $x_0 \in X$ such that $x_0 \preceq Tx_0$. Now, for any two comparable elements $x_1, x_2 \in X$, one has

$$\|A_y(x_1) - A_y(x_2)\| = \|Ax_1 - Ax_2\| \leqslant \psi(\|x_1 - x_2\|) \tag{14}$$

where, ψ is a $\mathscr{D}$-function of T on X. So A_y is a partially nonlinear $\mathscr{D}$-contraction on X. Hence, by an application of a fixed point Theorem 2.1, A_y has a unique fixed point, say $x^* \in X$. Thus, we have a unique element x^* in X such that

$$A_y(x^*) = Ax^* + By = x^*$$

which implies that

$$(I - A)^{-1}By = x^*$$

or,

$$Ty = x^*.$$

Thus the mapping $T : X \to X$ is well defined. Now define a sequence $\{x_n\}$ of iterates of T at x_0, that is, $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. From hypothesis (c), it follows that $x_0 \leqslant Tx_0$. Again, by Remark 3.1, we obtain that the mapping T is monotone nondecreasing on X. So we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{15}$$

Since B is partially compact and $(I - A)^{-1}$ is continuous, the composition mapping $T = (I - A)^{-1}B$ is partially compact and continuous on X into X. Therefore the sequence $\{x_n\}$ has a convergent subsequence converging to some point, say $x^* \in X$ and from the compatibility of order relation $\preceq$ and the norm $\|\cdot\|$ in X, it follows that the whole sequence $\{x_n\}$ converges to the point x^* in X. Hence, an application of Theorem 2.5 implies that T has a fixed point. This further implies that $(I - A)^{-1}Bx^* = x^*$ or $Ax^* + Bx^* = x^*$. This completes the proof. □

THEOREM 3.6. *Let $\big(X, \preceq, \|\cdot\|\big)$ be a partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in X are compatible. Let $A, B : X \to X$ be two nondecreasing mappings satisfying* *(a) A is linear and bounded and A^p is partially nonlinear $\mathscr{D}$-contraction for some positive integer p,*

(b) B is continuous and partially compact,

(c) there exists an element $x_0 \in X$ such that $x_0 \preceq Ax_0 + By$ for all $y \in X$.

(d) every pair of elements $x, y \in X$ has a lower and an upper bound in X.

Then the operator equation (11) has a solution in X.

Proof. Define the operator T on X by

$$T = (I - A)^{-1}B. \tag{16}$$

Now the mapping $(I - A)^{-1}$ exists in view of the relation

$$(I - A)^{-1} = (I - A^p)^{-1} \sum_{j=1}^{p-1} A^j, \tag{17}$$

where $\sum_{j=1}^{p-1} A^j$ is bounded and $(I - A^p)^{-1}$ exists in view of a Theorem 2.2 . Hence, $(I - A)^{-1}$ exists and is continuous on X.

Next, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = Ax + By. \tag{18}$$

Clearly, A_y is nondecreasing on X into itself. Again, by hypothesis (c), there is a point $x_0 \in X$ such that

$$x_0 \preceq A_y(x_0) \preceq A_y^2(x_0) \preceq \ldots \preceq A_y^p(x_0).$$

Now for any two comparable elements $x_1, x_2 \in X$, one has

$$\|A_y^p(x_1) - A_y^p(x_2)\| \leqslant \psi(\|x_1 - x_2\|).$$

Hence, by Theorem 2.2, there exists a unique element x^* such that

$$A_y^p(x^*) = A^p(x^*) + By = x^*.$$

This further implies that $A_y(x^*) = x^*$ and x^* is a unique fixed point of A_y. Thus, we have $A_y(x^*) = x^* = Ax^* + By$, or, $(I - A)^{-1}By = x^*$. As a result, $Ty = x^*$ and so T is well defined. The rest of the proof is similar to Theorem 3.5 and we omit the details. The proof is complete. □

Below we state two hybrid fixed point theorems in a partially ordered complete normed linear space by reverting the inequality given in hypothesis (c) of the above two fixed point theorems. The proofs of these theorems are similar to Theorems 3.5 and 3.6. Hence we omit the details.

THEOREM 3.7. *Suppose that* $\left(X,\preceq,\|\cdot\|\right)$ *is a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A,B: X \to X$ *be two nondecreasing operators such that*

(a) A *is continuous and partially nonlinear* $\mathscr{D}$*-contraction,*

(b) B *is continuous and partially compact,*

(c) there exists an element $x_0 \in X$ *such that* $Ax_0 + By \succeq x_0$ *for all* $y \in X$*, and*

(d) every pair of elements $x,y \in X$ *has a lower and an upper bound in* X*.*

Then the operator equation (11) *has a solution.*

THEOREM 3.8. *Let* $\left(X,\preceq,\|\cdot\|\right)$ *be a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A,B: X \to X$ *be two nondecreasing mappings such that*

(a) A *is linear and bounded and* A^p *is partially nonlinear* $\mathscr{D}$*-contraction for some positive integer* p*,*

(b) B *is a continuous and partially compact,*

(c) there exists an element $x_0 \in X$ *such that* $Ax_0 + By \succeq x_0$ *for all* $y \in X$*,*

(d) every pair of elements $x,y \in X$ *has a lower and an upper bound in* X*.*

Then the operator equation (11) *has a solution.*

REMARK 3.2. The hypothesis (d) of Theorems 3.5 and 3.6 holds if the partially ordered set X is a lattice. Furthermore, the space $C(J,\mathbb{R})$ of continuous real-valued functions on the closed and bounded interval $J=[a,b]$ is a lattice, where the order relation $\leqslant$ is defined as follows. For any $x,y \in C(J,\mathbb{R})$, $x \leqslant y$ if and only if $x(t) \leqslant y(t)$ for all $t \in J$. The real variable operations show that $\min\{x,y\}$ and $\max\{x,y\}$ are respectively the lower and upper bounds for the pair of elements x and y in X.

4. Fractional Integral Equations

In this section we apply the hybrid fixed point theorems proved in the previous two sections to some nonlinear fractional integral equations Volterra type for proving the existence and uniqueness theorems under certain mixed arguments from algebra, geometry and topology. First we deal with the nonlinear Volterra fractional integral equations for existence as well as uniqueness results under certain mixed conditions. Let us recall some basic definitions of fractional calculus [15, 19] which we need in what follows.

DEFINITION 4.1. The **Riemann-Liouville fractional derivative** of a n-times differential function $x : J \to \mathbb{R}$ of fractional order q is defined as

$$D^q_{0+}x(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1}x(s)\,ds, \quad n-1<q<n, n=[q]+1,$$

provided that the integral exists, where $[q]$ denotes the integer part of the real number q.

Similarly, **Caputo fractional derivative** of a n-times differential function $x : J \to \mathbb{R}$ fractional order q is defined as

$$^cD^q_{0+}x(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}x^{(n)}(s)\,ds, \quad n-1<q<n, n=[q]+1,$$

provided that the integral exists, where $[q]$ denotes the integer part of the real number q.

DEFINITION 4.2. The **Riemann-Liouville fractional integral** of a function $x : J \to \mathbb{R}$ of order $q>0$ is defined as

$$I^q_{0+}x(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{x(s)}{(t-s)^{1-q}}\,ds,$$

provided the integral exists.

The equation involving the unknown function and its fractional derivatives is called a fractional differential equation and an equation involving the presence of unknown function under fractional integral is called the fractional integral equation. The topic of fractional differential and integral equation is of current interest and several authors all over the contributed to this area of investigation. The details of fractional calculus and fractional differential equations may be found in Kilbas *et. al.* [15], Lakshmikantham *et al.*[16] and Podlubny [19] etc. In the following section we discuss some Volterra type fractional integral equations for the existence results via the abstract results developed in the previous sections.

4.1. The Volterra fractional integral equations

Given $0<q<1$ and $p=1-q$, and given a closed and bounded interval $J=[t_0, t_0+a]$ in $\mathbb{R}$, denote, by $C(J,\mathbb{R})$ and $C_p(J,\mathbb{R})$ the spaces of continuous real-valued functions on J and

$$C_p(J,\mathbb{R}) = \big\{u \in C(J,\mathbb{R}) \mid (t-t_0)^p u(t) \in C(J,\mathbb{R})\big\}. \tag{19}$$

Consider the following nonlinear Volterra fractional integral equation (in short VFIE)

$$x(t) = h(t) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}g(s,x(s))\,ds \tag{20}$$

for all $t \in J$, where Γ is a gamma function, $h : J \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous functions.

The special case when $h(t) = \dfrac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$, where $x^0 = x(t)(t-t_0)^{1-q}\Big|_{t=t_0}$, we obtain the Volterra fractional integral equation considered in Lakshmikantham *et al.*[16]. Thus, equation (19) is a more general Volterra type fractional integral equation that has been studied in the literature on fractional calculus and applications.

By a ***solution*** of the VFIE (19) we mean a function $x \in C_p(J,\mathbb{R})$ that satisfies the equation (19) on J.

In the following we shall discuss the basic existence result for the VFIE (19) on J. We need the following set of assumptions in what follows.

(H_1) g is bounded on $J \times \mathbb{R}$ with bound M_g.

(H_2) $g(t,x)$ is nondecreasing in x for each $t \in J$.

(H_3) There exists an element $u_0 \in C(J,\mathbb{R})$ such that

$$u_0(t) \leqslant h(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,u_0(s))\, ds$$

for all $t \in J$.

Note that all the hypotheses (H_1) through (H_3) are standard and frequently used in the theory of nonlinear differential and integral equations.

THEOREM 4.1. *Assume that the hypotheses (H_1) through (H_3) hold. Then the VFIE (19) has a solution defined on J.*

Proof. Set $X = C(J,\mathbb{R})$ with the supremum norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|. \tag{$**$}$$

Define an order relation $\leqslant$ in X as follows. Let any $x, y \in X$. Then $x \leqslant y$ if and only if $x(t) \leqslant y(t)$ for all $t \in J$. Clearly, as mentioned earlier, $C(J,\mathbb{R})$ is a lattice. Therefore, the lower and upper bounds for every pair of elements $x, y \in C(J,\mathbb{R})$ exist. Furthermore, the order relation $\leqslant$ and the norm $\|\cdot\|$ in X are compatible. Define the operator T on X by

$$Tx(t) = h(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\, ds, \ t \in J. \tag{21}$$

From the continuity of the functions involved in (21) it follows that T defines a mapping $T : X \to X$. We show that the operator T satisfies all the conditions of Theorem 2.5 on the Banach space X. First we note that $x_0 \leqslant Tx_0$ in view of hypothesis

(H_3). Therefore, it is enough to that T ia a compact and continuous operator on X in view of Remark 2.4. This will be done in the following three steps:

Step 1: *T is uniformly bounded*.

First we show that T is a uniformly bounded on $C(J,\mathbb{R})$. Let $x \in C(J,\mathbb{R})$ be an arbitrary element. Then

$$\begin{aligned}
|Tx(t)| &\leqslant |h(t)| + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} |g(s,x(s))|\, ds \\
&\leqslant |h(t)| + \frac{M_g}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, ds \\
&\leqslant \|h\| + \frac{M_g}{q\Gamma(q)} (t-t_0)^q \\
&\leqslant \|h\| + \frac{M_g a^q}{\Gamma(q+1)} = M
\end{aligned}$$

for all $t \in J$ and for all $x \in X$. This shows that T is a uniformly bounded operator on X.

Step 2: *$T(X)$ is an equicontinuous set.*

Let $x \in X$ be arbitrary. Then, by definition of T, we have

$$\begin{aligned}
|Tx(t_1) - Tx(t_2)| &\leqslant \big|h(t_1) - h(t_2)\big| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_{t_0}^{t_1} (t_1-s)^{q-1} g(s,x(s))\, ds - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2-s)^{q-1} g(s,x(s))\, ds \right| \\
&\leqslant \big|h(t_1) - h(t_2)\big| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_{t_0}^{t_1} (t_1-s)^{q-1} g(s,x(s))\, ds - \int_{t_0}^{t_2} (t_1-s)^{q-1} g(s,x(s))\, ds \right| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_{t_0}^{t_2} (t_1-s)^{q-1} g(s,x(s))\, ds - \int_{t_0}^{t_2} (t_2-s)^{q-1} g(s,x(s))\, ds \right| \\
&\leqslant \big|h(t_1) - h(t_2)\big| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1-s)^{q-1} |g(s,x(s))|\, ds \right| \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t_0+a} \left| (t_1-s)^{q-1} - (t_2-s)^{q-1} \right| |g(s,x(s))|\, ds \\
&\leqslant \big|h(t_1) - h(t_2)\big| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1-s)^{q-1} M_g\, ds \right| \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t_0+a} \left| (t_1-s)^{q-1} - (t_2-s)^{q-1} M_g\, ds \right| \\
&\leqslant \big|h(t_1) - h(t_2)\big| + \frac{M_g}{\Gamma(q+1)} |t_1 - t_2|^q
\end{aligned}$$

$$
\begin{aligned}
&+\frac{M_g}{\Gamma(q)}\int_{t_1}^{t_0+a}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right| \\
&=\left|h(t_1)-h(t_2)\right|+\frac{M_g}{\Gamma(q+1)}\,|t_1-t_2|^q \\
&+\frac{M_g}{\Gamma(q)}\int_{t_0}^{t_0+a}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|ds \\
&\to 0 \quad \text{as} \quad t_1\to t_2
\end{aligned}
$$

uniformly for all $t_1,t_2\in J$. This shows that the family $T(X)$ is an equicontinuous set in X. Now we apply Arzelá-Ascoli theorem to yield that T is a compact operator on X.

Step 3 : T *is continuous.*

Let $\{x_n\}$ be a sequence of points in X converging to a point x in X. Then, by dominated convergence theorem and the continuity of the function g, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} Tx_n(t) &= h(t)+\frac{1}{\Gamma(q)}\lim_{n\to\infty}\int_{t_0}^{t}(t-s)^{q-1}g(s,x_n(s))\,ds \\
&= h(t)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\lim_{n\to\infty}g(s,x_n(s))\,ds \\
&= h(t)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds \\
&= Tx(t)
\end{aligned}
$$

for all $t\in J$. This shows that Tx is a pointwise continuous function on J. Since $T(X)$ is equicontinuous, the sequence $\{Tx_n\}$ is equicontinuous. Now, it can be shown as in Guenther *et.al.* [12] (see aslo Granas and Dugundji [11]) that $\{Tx_n\}$ converges to Tx uniformly on J. Hence T a is continuous operator on X. Thus all the conditions of Theorem 2.5 are satisfied by the operator T on X. Hence we apply Theorem 2.5 and conclude that T has a fixed point in X. This further implies that the VFIE (19) has a solution defined on J and the proof of the theorem is complete. □

Our next result is about the uniqueness theorem for the VFIE (19) under a certain partial Lipschitz condition called the one-sided Lipschitz condition together with the application of a basic hybrid fixed point theorem. We need the following hypothesis in the sequel.

(B_1) There exist constants $L>0$ and $K>0$ such that

$$
0\leqslant g(t,x)-g(t,y)\leqslant\frac{L(x-y)}{K+(x-y)}
$$

for all $x,y\in\mathbb{R}$ with $x\geqslant y$.

THEOREM 4.2. *Assume that the hypotheses (B_1) and (H_2)-(H_3) hold. Further, if* $\dfrac{La}{\Gamma(q+1)}\leqslant K$, *then the VFIE (19) has a unique solution defined on* J.

Proof. Set $X = C(J,\mathbb{R})$ with supremum norm $\|\cdot\|$ defined by $\|x\| = \sup_{t\in J}|x(t)|$. Define an order relation $\leqslant$ in X as follows. Let any $x,y \in X$. Then $x \leqslant y$ if and only if $x(t) \leqslant y(t)$ for all $t \in J$. Clearly, as mentioned earlier, $C(J,\mathbb{R})$ is a lattice. Therefore, the lower and upper bounds for every pair of elements $x,y \in C(J,\mathbb{R})$ exist. Define an operator T on X into itself by (21). From hypothesis (H$_2$) it follows that T is a monotone nondecreasing operator on X. We show that T is a partial nonlinear $\mathscr{D}$-contraction on X. Let $x,y \in X$ be such that $x \geqslant y$. Then, by hypothesis (B$_1$), we obtain

$$\begin{aligned}
\big|Tx(t)-Ty(t)\big| &= \frac{1}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds - \int_{t_0}^{t}(t-s)^{q-1}g(s,y(s))\,ds\right| \\
&\leqslant \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\big|g(s,x(s))-g(s,y(s))\big|\,ds \\
&\leqslant \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\frac{L||x-y||}{K+||x-y||}\,ds \\
&\leqslant \frac{La}{\Gamma(q+1)}\cdot\frac{||x-y||}{K+||x-y||}
\end{aligned} \tag{22}$$

Taking supremum over t, we obtain

$$\|Tx-Ty\| \leqslant \frac{La}{\Gamma(q+1)}\cdot\frac{||x-y||}{K+||x-y||} = \psi(||x-y||) \tag{23}$$

for all $x,y \in X$ with $x \geqslant y$, where $\psi(r) = \dfrac{La}{\Gamma(q+1)}\cdot\dfrac{r}{K+r} < r$ for $r > 0$ since $\dfrac{La}{\Gamma(q+1)} \leqslant K$. Clearly, ψ is a $\mathscr{D}$-function for the operator A on X. Consequently, A is a partially $\mathscr{D}$-nonlinear contraction on X. Now the desired conclusion follows by a direct application of Theorem 2.1. □

If the nonlinearity g involved in VFIE (19) is not continuous on the domain of its definition, then we consider the following hypothesis:

(B$_2$) The function $s \mapsto (t-s)^{q-1}g(s,x(s))$ is Riemann integrable on J for each $x \in C(J,\mathbb{R})$ with $t > s$.

THEOREM 4.3. *Assume that the hypotheses (B$_1$) and (H$_3$) hold. Suppose that the function g is not continuous but the hypothesis (B$_2$) holds. Further if $\dfrac{La}{\Gamma(q+1)} \leqslant K$ holds, then the VFIE (19) has a unique solution defined on J.*

Proof. Set $X = C(J,\mathbb{R})$ and define an order relation $\leqslant$ and the norm $\|\cdot\|$ in X as in the proof of Theorem 4.2. Define the order cone $\mathscr{K}_C$ in the Banach space X as

$$\mathscr{K}_C = \big\{x \in X \mid x(t) \geqslant 0 \text{ for all } t \in J\big\}.$$

Then the order relation $\leqslant$ is identical with the order relation defined by (**) with the help of order cone $\mathscr{K}_C$ in X . The rest of the proof is similar to that of Theorem 4.2 and now the conclusion follows directly by an application of Lemma 3.1. □

Notice that the Reimann integrability of the function $s \mapsto (t-s)^{q-1}g(s,x(s))$ for $t > s$ is guaranteed in the following three cases.

1. g is continuous on $J \times \mathbb{R}$.

2. $g(s,x)$ is Carathéodory, i.e., $g(s,x)$ is measurable in s for each $x \in \mathbb{R}$ and continuous in x for each $t \in J$ and it is bounded on $J \times \mathbb{R}$.

3. The map $s \mapsto g(s,x)$ is monotone increasing for all $x \in \mathbb{R}$.

We mention that above stated cases have been discussed in the literature on the theory of nonlinear continuous and discontinuous differential equations.

REMARK 4.1. The conclusion of Theorems 4.1, 4.2 and 4.4 holds if we replace the hypothesis (H_3) by the following one.

(H_3') There exists an element $u_0 \in C(J,\mathbb{R})$ such that

$$u_0(t) \geqslant h(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,u_0(s))\,ds.$$

for all $t \in J$.

4.2. Fractional integral equations of mixed type

Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$, $\mathbb{R}$ the set of real numbers, for some $t_0 \in \mathbb{R}$ and $a \in \mathbb{R}$ with $a > 0$ and given a real number $0 < q < 1$, consider the nonlinear hybrid fractional integral equation (in short HFIE)

$$x(t) = f(t,x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\,ds,\; t \in J, \tag{24}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : J \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous.

We seek the solutions of HFIE (24) in the space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We consider the following set of hypotheses in what follows.

(H_4) There exist constants $L > 0$ and $K > 0$ such that

$$0 \leqslant f(t,x) - f(t,y) \leqslant \frac{L(x-y)}{K + (x-y)}$$

for all $x, y \in \mathbb{R}$ with $x \geqslant y$. Moreover $L \leqslant K$.

(H_5) There exists an element $u_0 \in X = C(J,\mathbb{R})$ such that

$$u_0(t) \leqslant f(t,u_0(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,y(s))ds$$

for all $t \in J$ and $y \in X$.

REMARK 4.2. The condition given in hypothesis (H_5) is a little more than that of a lower solution for the HFIE (24). It is clear that u_0 is a lower solution of the FIE (24), however the converse is not true.

THEOREM 4.4. *Assume that hypotheses (H_1) through (H_5) hold. Then the HFDE (24) admits a solution.*

Proof. Set $X = C(J,\mathbb{R})$, the Banach space of continuous real-valued functions on J with usual supremum norm $\|\cdot\|$ given by $\|x\| = \sup_{t\in J} |x(t)|$.

Define an order relation $\leqslant$ in X as follows. Let any $x,y \in X$. Then $x \leqslant y$ if and only if $x(t) \leqslant y(t)$ for all $t \in J$. Clearly, as mentioned earlier, X is a lattice. Therefore, the lower and upper bounds for every pair of elements $x,y \in X$ exist. Furthermore, the order relation $\leqslant$ and the norm $\|\cdot\|$ in X are compatible. Define two operators $A,B : X \to X$ by

$$Ax(t) = f(t,x(t)),\ t \in J \tag{25}$$

and

$$Bx(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))ds,\ t \in J. \tag{26}$$

Then the given hybrid fractional integral equation (24) is transformed into an equivalent operator equation as

$$Ax(t) + Bx(t) = x(t),\ t \in J. \tag{27}$$

From the continuity of the functions involved in the right hand side of (25) and (26) it follows that A and B define the operators $A,B : X \to X$. We show that the operators A and B satisfy all the conditions of Theorem 3.5 on X. First, we show that A is a nonlinear contraction on X. Let $x,y \in X$ be such that $x \geqslant y$. Then, by hypothesis (H_4), we obtain

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leqslant \frac{L|x(t)-y(t)|}{K + |x(t)-y(t)|} \\ &\leqslant \frac{L||x-y||}{K+||x-y||}. \end{aligned} \tag{28}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$||Ax - Ay|| \leqslant \frac{L||x-y||}{K+||x-y||} = \psi(||x-y||) \tag{29}$$

for all $x,y \in X$ with $x \geqslant y$, where $\psi(r) = \frac{Lr}{K+r} < r$ for $r > 0$. Clearly, ψ is a $\mathscr{D}$-function for the operator A on X. Consequently, A is a partially nonlinear $\mathscr{D}$-contraction on X.

Next, we show that B is a compact continuous operator on X. To finish, we show that $B(X)$ is a uniformly bounded and equi-continuous set in X. Now for any $x \in X$, one has

$$|Bx(t)| \leqslant \frac{1}{\Gamma(q)} \int_{t_0}^{t} |t-s|^{q-1} |g(s,x(s))|\, ds \\ \leqslant \frac{M_g}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}\, ds \\ \leqslant \frac{a^q M_g}{\Gamma(q+1)} \tag{30}$$

for all $t \in J$ which shows that B is a uniformly bounded set in X. Now, let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned} |Bx(t_1) - Bx(t_2)| &\leqslant \frac{M_g}{\Gamma(q)} \int_{t_0}^{t_2} |(t_1-s)^{q-1} - (t_2-s)^{q-1}| M_g\, ds \\ &\quad + \frac{M_g}{\Gamma(q+1)} |t_1 - t_2|^q \\ &\leqslant \frac{M_g}{\Gamma(q)} \int_{t_0}^{t_0+a} |(t_1-s)^{q-1} - (t_2-s)^{q-1}|\, ds \\ &\quad + \frac{M_g}{\Gamma(q+1)} |t_1 - t_2|^q \\ &\to 0 \quad \text{as} \quad t_1 \to t_2, \end{aligned} \tag{31}$$

uniformly for all $x \in X$. Hence $B(X)$ is an equi-continuous set in X. Now we apply Arzelá-Ascoli theorem to yield that $B(X)$ is a relatively compact set in X. The continuity of B follows from the continuity of the function g on $J \times \mathbb{R}$.

Finally, since $f(t,x)$ and $g(t,x)$ are nondecreasing in x for each $t \in J$, the operators A and B are nondecreasing on X. Also the hypothesis (H_5) yields that $u_0 \leqslant Au_0 + By$ for all $y \in X$. Thus, all the conditions of Theorem 3.5 are satisfied and we conclude that the hybrid fractional integral equation (24) admits a solution. This completes the proof. □

REMARK 4.3. The conclusion of Theorem 4.4 also remains true if we replace the hypothesis (H_5) concerning the existence of lower solution type function by the following one.

(H_5') There exists an element $u_0 \in X = C(J,\mathbb{R})$ such that

$$u_0(t) \geqslant f(t,u_0(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,y(s))ds$$

for all $t \in J$ and $y \in X$.

Next, we consider the following nonlinear fractional integral equation of mixed type, viz.,

$$x(t) = h(t) + \int_{t_0}^{t} v(t,s)f(s,x(s))\,ds + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds \tag{32}$$

for all $t \in J$ and $0 < q < 1$, where the functions $h : J \to \mathbb{R}$, $v : J \times J \times \mathbb{R}$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

We consider the following set of hypotheses in what follows.

(H_6) The function $v : J \times J \to \mathbb{R}_+$ is continuous. Moreover, $V = \sup_{t,s\in J}|v(t,s)|$.

(H_7) $f(t,x)$ is linear and nondecreasing in x for each $t \in J$.

(H_8) f is bounded on $J \times \mathbb{R}$ and there exists a constant $L > 0$ such that $|f(t,x)| \leqslant L|x|$ for all $t \in J$ and $x \in \mathbb{R}$.

(H_9) There exists an element $u_0 \in X = C(J,\mathbb{R})$ such that

$$u_0(t) \leqslant h(t) + \int_{t_0}^{t} v(t,s)f(s,u_0(s))\,ds + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,y(s))\,ds$$

for all $t \in J$ and $y \in X$.

REMARK 4.4. The condition given in hypothesis (H_8) is a little more than that of a lower solution for the HFIE (32) defined on J.

THEOREM 4.5. *Assume that the hypotheses (H_1)-(H_2) and (H_6) through (H_9) hold. Then the HFIE* (32) *admits a solution.*

Proof. Set $X = C(J,\mathbb{R})$ and define an order relation $\leqslant$ in X as follows. Let any $x, y \in X$ be arbitrary. Then $x \leqslant y$ if and only if $x(t) \leqslant y(t)$ for all $t \in J$. Clearly, as mentioned earlier, $C(J,\mathbb{R})$ is a lattice. Therefore, the lower and upper bounds for every pair of elements $x, y \in X$ exist. Define the two operators A and B on X by

$$Ax(t) = \int_{t_0}^{t} v(t,s)f(s,x(s))\,ds,\ t \in J, \tag{33}$$

and

$$Bx(t) = h(t) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds,\ t \in J. \tag{34}$$

From the continuity of the integrals involved in the right hand side of (33) and (34) it follows that A and B define the operators $A, B : X \to X$. We show that A and B satisfy all the conditions of Theorem 4.5 on X.

Clearly, the operator A is linear and bounded in view of the hypotheses (H$_7$) and (H$_8$). We only show that the operator A^n is partially $\mathscr{D}$-contraction on X for large positive integer n. Let $x,y \in X$ be such that $x \geqslant y$. Then by (H$_7$)-(H$_8$),

$$\begin{aligned}|Ax(t)-Ay(t)| &\leqslant \int_{t_0}^{t} |V|\,|f(s,x(s))-f(s,y(s))|\,ds\\ &\leqslant V\int_{t_0}^{t_0+a} L|x(s)-y(s)|\,ds\\ &\leqslant LVa\|x-y\|.\end{aligned}$$

Taking supremum over t, we obtain

$$\|Ax-Ay\| \leqslant LVa\|x-y\|.$$

Similarly, it can be proved that

$$\begin{aligned}\|A^2x-A^2y\| &= |A(Ax(t)-A(Ay(t))|\\ &\leqslant LV\int_{t_0}^{t_0+a}\left(\int_{t_0}^{t}|Ax(s)-Ay(s)|\,ds\right)ds\\ &\leqslant \frac{L^2V^2a^2}{2!}\|x-y\|.\end{aligned}$$

In general, proceeding in the same way, for any positive integer n, we have

$$\|A^nx-A^ny\| \leqslant \frac{L^nV^na^n}{n!}\|x-y\|. \tag{35}$$

for all $x,y \in X$ with $x \geqslant y$. Therefore, for large n, A^n is a partially contraction on X. The rest of proof is similar to that of Theorem 4.4 and now the desired result follows by an application of Theorem 3.6. This completes the proof. □

REMARK 4.5. The conclusion of Theorem 4.5 holds if we replace the condition (H$_9$) by

(H$'_9$) There exists an element $u_0 \in X = C(J,\mathbb{R})$ such that

$$u_0(t) \geqslant h(t)+\int_{t_0}^{t} v(t,s)f(s,u_0(s))\,ds+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,y(s))\,ds$$

for all $t \in J$ and $y \in X$.

The existence theorem for the nonlinear fractional integral equations of mixed type

$$x(t) = h(t)+\int_{t_0}^{t} v(t,s)x(s)\,ds+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds,\ t \in J, \tag{36}$$

can also be proved by an application of Theorem 3.5 along the lines similar to that of Theorem 4.5 with appropriate modifications.

We remark that we have applied our hybrid fixed point theorems to initial value problems of fractional Volterra integral equations, however the idea can be extended to nonlinear problems of fractional Fredolm integral equations.

5. Applications

As the applications of the existence results proved in the previous section, we consider the initial value problems of nonlinear fractional differential equations.

EXAMPLE 1. Given a closed and bounded interval $J = [t_0, t_0 + a] \subset \mathbb{R}$, for some $t_0, a \in \mathbb{R}$ with $a > 0$ and given a real number $0 < q < 1$, consider the IVP of nonlinear fractional differential equation (in short FDE)

$$\left.\begin{aligned} D^q x(t) &= f(t, x(t)),\ t \in J, \\ x(t_0) &= x^0 = x(t)(t - t_0)^{1-q}\Big|_{t=t_0}, \end{aligned}\right\} \tag{37}$$

where D^q is the Riemann-Louville fractional derivative of order q and $f : J \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous function.

It is known that the FDE (37) is equivalent to the Volterra fractional integral equation (VFIE)

$$x(t) = x^0 (t - t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x(s))\, ds,\ t \in J. \tag{38}$$

The details of Riemann-Louville fractional derivative and its relations with IVPs may be found in Lakshmikantham, Leela and Vasundhara Devi [16] and the references cited therein. The above VFIE (38) is valid for all functions $x \in C_p(J, \mathbb{R})$. Hence, if hypotheses (H_1), (H_2) and (H_3) hold, then by Theorem 4.1 with $h(t) = x^0 (t - t_0)^{q-1}$, the FDE (37) has a solution defined on J.

EXAMPLE 2. Given a closed and bounded interval $J = [t_0, t_0 + a] \subset \mathbb{R}$, for some $t_0, a \in \mathbb{R}$ with $a > 0$ and given a real number $0 < q < 1$, consider the IVP of nonlinear fractional differential equation (in short FDE)

$$\left.\begin{aligned} {}^c D^q x(t) &= f(t, x(t)),\ t \in J, \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{39}$$

where ${}^c D^q$ is the Caputo fractional derivative of order q and $f : J \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous function.

It is known that the FDE (39) is equivalent to the Volterra fractional integral equation (VFIE)

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x(s))\, ds,\ t \in J. \tag{40}$$

Again, the details of Caputo fractional derivative and its relations with IVPs may be found in Lakshmikantham, Leela and Vasundhara Devi [16] and the references cited therein. The above VFIE (40) is valid for all functions $x \in C_p(J, \mathbb{R})$. Hence, if hypotheses (H_1), (H_2) and (H_3) hold with $h(t) = x_0$, then by Theorem 4.1, the FDE (39) has a solution defined on J.

EXAMPLE 3. Given a closed and bounded interval $J = [t_0, t_0 + a] \subset \mathbb{R}$, for some $t_0, a \in \mathbb{R}$ with $a > 0$ and given a real number $0 < q < 1$, consider the IVP of nonlinear hybrid fractional differential equation (in short HFDE)

$$\left.\begin{aligned} {}^{c}D^{q}[x(t) - f(t,x(t))] &= g(t,x(t)),\ t \in J, \\ x^0 &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{41}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order q, $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : J \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous function.

It is easy to verify that the HFDE (41) is equivalent to the hybrid fractional integral equation (in short HFIE)

$$x(t) = F(t,x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,x(s))\,ds,\ t \in J, \tag{42}$$

where $F(t,x(t)) = x_0 - f(t_0,x_0) + f(t,x(t))$ for $t \in J$. The details of Caputo fractional derivatives and its relations with IVPs may be found in Podlubny [19] and the references cited therein.

Now if $x_0 - f(t_0,x_0) \geqslant 0$ and the hypotheses (H_1) and (H_5) hold, then all the conditions of Theorem 4.4 are satisfied with the function f replaced by F. Hence, by Theorem 4.4, the HFDE (41) has a solution defined on J.

EXAMPLE 4. Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$, consider the following hybrid fractional differential equation with a linear perturbation of second type,

$$\left.\begin{aligned} D^{q}[x(t) - f(t,x(t))] &= g(t,x(t)),\ t \in J, \\ X^0 &= X(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \end{aligned}\right\} \tag{43}$$

where D^q is the Riemann-Louville fractional derivative of order q, $0 < q < 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous and $g : J \times \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous function and $X(t) = x(t) - f(t,x(t))$, $t \in J$.

The existence theorem for the HFDE (43) is proved using the hybrid fixed point result embodied in Theorem 3.5 under Lipschitz and compactness type conditions(see Dhage and Mugale [10]). We mention that the existence theorem for the HFDE (43) can also be obtained under weaker partially Lipschitz and partially compactness type conditions. Note that the HFDE (43) is equivalent to the hybrid fractional integral equation (42), where the function F is defined by

$$F(t,x(t)) = X^0(t-t_0)^{q-1} + f(t,x(t)),\ t \in J.$$

Therefore, if $X_0(t-t_0)^{q-1} \geqslant 0$ and hypotheses (H_1) and (H_5) hold, then all the conditions of Theorem 4.4 are satisfied with the function f replaced by F. Hence, by Theorem 4.4, the HFDE (43) has a solution defined on J.

REMARK 5.1. Lastly, we mention that the existence theorems for the hybrid fractional differential equations with the linear perturbation of first kind, namely,

$$\left.\begin{aligned} D^q x(t) &= f(t,x(t)) + g(t,x(t)),\ t \in J, \\ x^0 &= x(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \end{aligned}\right\} \tag{44}$$

and

$$\left.\begin{aligned} {}^cD^q x(t) &= f(t,x(t)) + g(t,x(t)),\ t \in J, \\ x(t_0) &= x_0, \end{aligned}\right\} \tag{45}$$

can also be proved along the similar lines with appropriate modifications.

6. Conclusion

In this paper we have proved a very fundamental hybrid fixed point theorems in partially ordered normed linear spaces. However, more general hybrid fixed point theorems under weaker conditions may be proved along the similar lines with appropriate modifications. Further these hybrid fixed point theorems have some nice applications to hybrid differential and integral equations for proving the existence and uniqueness theorems under weaker hypotheses than earlier ones discussed in the literature. Here, in this work we have discussed only the dynamic systems with continuous nonlinearity, however the results on hybrid fixed point theorems of this paper can also be applied to study the dynamic systems with discontinuous nonlinearities on the domains of their definition. This can be accomplished by defining the order relation through the appropriate order cone in a normed linear space and following the lines of arguments given in Hekkillä and Lakshmikantham [13]. Therefore, the work presented in this paper opens a new vistas for the research work in the area of nonlinear analysis and applications. Finally, while concluding we mention that some of the results in the stated direction under weaker hypotheses than that presented here will be reported elsewhere.

Acknowledgements. The author is thankful to the referee for pointing out some misprints and giving some suggestions for the improvement of this paper.

REFERENCES

[1] H. AMANN, *Order Structures and Fixed Points*, ATTI 2^o Sem. Anal. Funz. Appl., Itali, (1977).
[2] G. BIRKHOFF, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. 1947.
[3] F. E. BROWDER, *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces*, Proc. Symp. Pure Math. Amer. Math. Soc., Providence, Rhode Island 1976.
[4] K. DEIMLING, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.
[5] B. C. DHAGE, *On extension of Tarski's fixed point theorem and applications*, Pure Appl. Math. Sci., **25** (1987), 37–42.
[6] B. C. DHAGE, *Some fixed point theorems for in ordered Banach spaces and applications*, Mathematics Student, **61** (1992), 81–88.
[7] B. C. DHAGE, *Fixed point theorems in ordered Banach algebras and applications*, Pan Amer. Math. J., **9**, (4) (1999), 93–102.

[8] B. C. DHAGE, *A nonlinear alternative with applications to nonlinear perturbed differential equations*, Nonlinear Studies, **13**, (4) (2006), 343–354.

[9] B. C. DHAGE AND N. S. JADHAV, *Basic results on hybrid differential equations with linear perturbation of second type*, Tamkang J. Math., **43**, (4) (2013), 00–00.

[10] B. C. DHAGE AND P. V. MUGALE, *On Some Basic Results for Nonlinear First Order Hybrid Fractional Differential Equations*, Comm. Appl. Nonlinear Anal., **19** (2012), 83–96.

[11] A. GRANAS AND J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[12] A. GRANAS, R. B. GUENTHER AND J. W. LEE, *Some general existence principles in the Carathéodary theory of nonlinear differential systems*, J. Math. Pures Appl., **70** (1991), 153–198.

[13] S. HEIKKILÄ AND V. LAKSHMIKANTHAM, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[14] M. A. KRASNOSELSKII, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press 1964.

[15] A. A. KILBAS, H. M. SRIVASTAVA AND JUAN TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North Holland Mathematics Series, 204, Elsevier Science B. V. Amsterdam, 2006.

[16] V. LAKSHMIKANTHAM, S. LEELA AND J. VASUNDHARA DEVI, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, (2009).

[17] J. J. NIETO AND R. RODRIGUEZ-LOPEZ, *Contractive mappings theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239.

[18] J. J. NIETO AND R. RODRIGUEZ-LOPEZ, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica (English Series), **23** (2007), 2205–2212.

[19] I. PODLUBNY, *Fractional Differential Equations*, Academic Press New York 1999.

[20] A. C. M. RAN, M.C. R. REURINGS, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2003), 1435–1443.

[21] A. TARSKI, *A lattice theoretical fixed-point theorem and its applications*, Pacific J. Math., **5** (1955), 285–310.

[22] E. ZEIDLER, *Nonlinear Functinal Analysis and its Applications: Part I*, Springer Verlag 1985.

(Received December 14, 2012)

Bapurao C. Dhage
Kasubai, Gurukul Colony
Ahmedpur-413 515, Dist: Latur
Maharashtra
India
e-mail: bcdhage@gmail.com