Differential
Equations
& Applications
Volume 5, Number 2 (2013), 205–219 doi:10.7153/dea-05-13

POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS INVOLVING NONLINEAR FRACTIONAL q–DIFFERENCE EQUATIONS

WENGUI YANG

(*Communicated by Eduardo M. Hernandez*)

Abstract. In this work, we investigate the eigenvalue intervals of nonlinear boundary value problems involving fractional q-difference equations by means of the properties of the Green function and Guo-Krasnosel'skii fixed point theorem on cones. Furthermore, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established. As applications, some examples are presented to illustrate the main results.

1. Introduction

In the past serval decades, boundary value problems for nonlinear fractional differential equations have gained considerable popularity and importance due to their application in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, engineering, signal and image processing, and so on. In consequence, the existence of solutions to fractional boundary value problems have been of great interest; for example, see [2, 3, 4, 5, 18, 24, 25, 21] and the references therein.

In [14, 15], Jackson firstly introduced the q-difference calculus or quantum calculus. For details, basic definitions and properties of q-difference calculus can be found in the book mentioned in [16]. Later, the fractional q-difference calculus has been proposed by Al-Salam [6] and Agarwal [1]. Recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus have been addressed extensively by several researchers. For example, some researcher obtained q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula, Mittage-Leffler function [7, 22, 23], and so on.

More recently, many people pay attention to boundary value problems involving nonlinear fractional q-difference equations. There have been some papers dealing with the existence and multiplicity of solutions or positive solutions for boundary value problems involving nonlinear fractional q-difference equations by the use of some well-known fixed point theorems. For some recent developments on the subject,

Mathematics subject classification (2010): 34K20, 92B20.
Keywords and phrases: fractional q-difference, boundary value problems, positive solutions, fixed point theorem, eigenvalue.

Paper DEA-05-13

see [9, 10, 11, 19] and the references therein. El-Shahed and Al-Askar [8] studied the existence of multiple positive solutions to the nonlinear q-fractional boundary value problems by using Guo-Krasnoselskii's fixed point theorem in a cone. Ma and Yang [20] considered the existence of solutions for multi-point boundary value problems of nonlinear fractional q-difference equations by means of the Banach contraction principle and Krasnoselskii's fixed point theorem.

Ferreira [12] studied the existence of positive solutions to nonlinear fractional q-difference boundary value problem as following:

$$\begin{cases} (D_q^\alpha u)(t)+f(t,u(t))=0, \quad 0<t<1, \\ u(0)=(D_q u)(0)=0, \quad (D_q u)(1)=\beta \geqslant 0, \end{cases}$$

where $2<\alpha\leqslant 3$, D_q^α is the Riemann-Liouville fractional q-derivative, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function. By using the properties of the Green function, they obtained some existence criteria for one positive solution for nonlinear fractional q-difference boundary value problems by means of the Krasnosel'skii fixed point theorem in cones.

To the best of author's knowledge, there is very little known about the existence of positive solutions for the following nonlinear boundary value problem of fractional q-difference equation:

$$\begin{cases} (D_q^\alpha u)(t)+\lambda f(u(t))=0, \quad 0<t<1, \\ u(0)=(D_q u)(0)=(D_q u)(1)=0, \end{cases} \tag{1}$$

where $2<\alpha\leqslant 3$, D_q^α is the Riemann-Liouville fractional q-derivative, λ is a positive parameter and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function.

On the one hand, the boundary value problem in [12] is the particular case of problem (1) as the case of $\lambda=1$. On the other hand, similarly as Ferreira discussed in [12], we also give some existence results by the fixed point theorem on a cone in this paper. The main purpose of this article is, by applying the properties of the Green function and Guo-Krasnosel'skii fixed point theorem in cones, to establish the eigenvalue λ intervals of the nonlinear fractional q-difference boundary value problem (1) such that, for any λ lying in this interval, the problem (1) has existence and multiplicity on positive solutions. Moreover, some sufficient conditions for the existence and nonexistence of at least one or two positive solutions for the boundary value problem are established. As applications, some examples are presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of problem (1). These details can be found in the recent literature; see [16] and references therein.

Let $q\in(0,1)$ and define

$$[a]_q=\frac{q^a-1}{q-1}, \quad a\in\mathbb{R}.$$

The q-analogue of the power $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^{(0)} = 1, \quad (a-b)^{(n)} = \prod_{k=0}^{n-1}(a-bq^k), \quad n \in \mathbb{N}, \quad a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b=0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \ \text{ and } \ (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_q t = \int_0^b f(t)d_q t - \int_0^a f(t)d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \ \text{ and } \ (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x=0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [16]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \quad {}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$$\left({}_xD_q\int_0^x f(x,t)d_qt\right)(x)=\int_0^x {}_xD_qf(x,t)d_qt+f(qx,x).$$

We note that if $\alpha>0$ and $a\leqslant b\leqslant t$, then $(t-a)^{(\alpha)}\geqslant(t-b)^{(\alpha)}$ [11].

DEFINITION 1. ([1]). Let $\alpha\geqslant 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0f(x)=f(x)$ and

$$(I_q^\alpha f)(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-qt)^{(\alpha-1)}f(t)d_qt,\quad \alpha>0,x\in[0,1].$$

DEFINITION 2. ([23]). The fractional q-derivative of the Riemann-Liouville type of order $\alpha\geqslant 0$ is defined by $D_q^0f(x)=f(x)$ and

$$(D_q^\alpha f)(x)=(D_q^mI_q^{m-\alpha}f)(x),\quad \alpha>0,$$

where m is the smallest integer greater than or equal to α.

LEMMA 1. ([11]). *Let $\alpha>0$ and p be a positive integer. Then the following equality holds:*

$$(I_q^\alpha D_q^\alpha f)(x)=(D_q^\alpha I_q^\alpha f)(x)-\sum_{k=0}^{p-1}\frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^kf)(0).$$

LEMMA 2. ([12]). *Let $y\in C[0,1]$ and $2<\alpha\leqslant 3$, the unique solution of*

$$\begin{cases}D_q^\alpha u(t)+y(t)=0,\quad 0<t<1,\\ u(0)=(D_qu)(0)=(D_qu)(1)=0,\end{cases}$$

is given by

$$u(t)=\int_0^1 G(t,qs)y(s)d_qs,$$

where

$$G(t,s)=\begin{cases}t^{\alpha-1}(1-s)^{(\alpha-2)}-(t-s)^{(\alpha-1)}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{\alpha-1}(1-s)^{(\alpha-2)}, & 0\leqslant t\leqslant s\leqslant 1.\end{cases}\tag{2}$$

The following properties of the Green function play important roles in this paper.

LEMMA 3. ([12]). *Let Function $G(t,s)$ defined by (2) satisfies the following conditions:*

(A_1) *$G(t,qs)\geqslant 0$ and $G(t,qs)\leqslant G(1,qs)$ for all $0\leqslant t,s\leqslant 1$;*

(A_2) *$G(t,qs)\geqslant g(t)G(1,qs)$ for $0\leqslant t,s\leqslant 1$ with $g(t)=t^{\alpha-1}$.*

The following lemma is fundamental in the proofs of our main results.

LEMMA 4. ([13, 17]). *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

(B_1) $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_1$, *and* $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_2$ *or*

(B_2) $\|Sw\| \geqslant \|w\|$, $w \in P \cap \partial\Omega_1$, *and* $\|Sw\| \leqslant \|w\|$, $w \in P \cap \partial\Omega_2$

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $E = C[0,1]$ be the Banach space endowed with the norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$. Define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \geqslant g(t)\|u\|, t \in [0,1]\}.$$

Suppose that u is a solution of boundary value problem (1). Then from Lemma 2, we obtain

$$u(t) = \lambda \int_0^1 G(t,qs) f(u(s)) d_q s, \quad t \in [0,1].$$

We define an operator $S_\lambda : P \to E$ as follows:

$$(S_\lambda u)(t) = \lambda \int_0^1 G(t,qs) f(u(s)) d_q s, \quad t \in [0,1].$$

By Lemma 3, we have

$$\|S_\lambda u\| \leqslant \lambda \int_0^1 G(1,qs) f(u(s)) d_q s,$$

$$(S_\lambda u)(t) \geqslant \lambda \int_0^1 g(t) G(1,qs) f(u(s)) d_q s \geqslant g(t)\|S_\lambda u\|.$$

Thus, $S_\lambda(P) \subset P$. Then we have the following lemma.

LEMMA 5. *$S_\lambda : P \to P$ is completely continuous.*

Proof. The operator $S_\lambda : P \to P$ is continuous in view of continuity of $G(t,s)$ and $f(u(t))$. By means of the Arzela-Ascoli theorem, $S_\lambda : P \to P$ is completely continuous.

3. Main results

In this section, we establish some sufficient conditions for the existence and nonexistence of positive solutions for boundary value problem (1).

For convenience, we denote

$$F_0 = \lim_{u\to 0^+} \sup \frac{f(u)}{u}, \quad F_\infty = \lim_{u\to +\infty} \sup \frac{f(u)}{u},$$
$$f_0 = \lim_{u\to 0^+} \inf \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to +\infty} \inf \frac{f(u)}{u},$$
$$C_1 = \int_0^1 G(1,qs)d_q s, \quad C_2 = \int_0^1 g(s)G(1,qs)d_q s.$$

THEOREM 1. *If there exists $l \in (0,1)$ such that $g(l)f_\infty C_2 > F_0 C_1$ holds, then for each*

$$\lambda \in \left((g(l)f_\infty C_2)^{-1}, (F_0C_1)^{-1}\right), \tag{3}$$

the boundary value problem (1) has at least one positive solution. Here we impose $(g(l)f_\infty C_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0C_1)^{-1} = +\infty$ if $F_0 = 0$.

Proof. Let λ satisfy (3) and $\varepsilon > 0$ be such that

$$(g(l)(f_\infty - \varepsilon)C_2)^{-1} \leqslant \lambda \leqslant ((F_0+\varepsilon)C_1)^{-1}. \tag{4}$$

By the definition of F_0, we see that there exists $r_1 > 0$ such that

$$f(u) \leqslant (F_0+\varepsilon)u, \quad \text{for} \quad 0 < u \leqslant r_1. \tag{5}$$

So if $u \in P$ with $\|u\| = r_1$, then by (4) and (5), we have

$$\begin{aligned}\|S_\lambda u\| &\leqslant \lambda \int_0^1 G(1,qs)f(u(s))d_q s \\ &\leqslant \lambda \int_0^1 G(1,qs)(F_0+\varepsilon)r_1 d_q s \\ &= \lambda(F_0+\varepsilon)r_1 C_1 \leqslant r_1 = \|u\|.\end{aligned}$$

Hence, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then we get

$$\|S_\lambda u\| \leqslant \|u\|, \quad \text{for} \quad u \in P\cap \partial\Omega_1. \tag{6}$$

Let $r_3 > 0$ be such that

$$f(u) \geqslant (f_\infty - \varepsilon)u, \quad \text{for} \quad u \geqslant r_3. \tag{7}$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, then from (4) and (7), we obtain

$$\begin{aligned}\|S_\lambda u\| \geqslant (S_\lambda u)(l) &= \lambda \int_0^1 G(l,qs)f(u(s))d_q s \\ &\geqslant \lambda \int_0^1 g(l)G(1,qs)f(u(s))d_q s\end{aligned}$$

$$\geqslant \lambda \int_0^1 g(l)G(1,qs)(f_\infty - \varepsilon)u(s)d_qs$$
$$\geqslant \lambda \int_0^1 g(l)g(s)G(1,qs)(f_\infty - \varepsilon)\|u\|d_qs$$
$$= \lambda g(l)C_2(f_\infty - \varepsilon)\|u\| \geqslant \|u\|.$$

Thus, if we set $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then we get

$$\|S_\lambda u\| \geqslant \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_2. \tag{8}$$

Now, from (6), (8), and Lemma 4,we guarantee that S_λ has a fixed-point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leqslant \|u\| \leqslant r_2$, and clearly u is a positive solution of (1). The proof is complete.

THEOREM 2. *If there exists $l \in (0,1)$ such that $g(l)f_\infty C_2 > F_0 C_1$ holds, then for each*

$$\lambda \in \left((g(l)f_0C_2)^{-1}, (F_\infty C_1)^{-1}\right), \tag{9}$$

the boundary value problem (1) has at least one positive solution. Here we impose $(g(l)f_0C_2)^{-1} = 0$ if $f_0 = +\infty$ and $(F_\infty C_1)^{-1} = +\infty$ if $F_\infty = 0$.

Proof. Let λ satisfy (9) and $\varepsilon > 0$ be such that

$$(g(l)(f_0 - \varepsilon)C_2)^{-1} \leqslant \lambda \leqslant ((F_\infty + \varepsilon)C_1)^{-1}. \tag{10}$$

From the definition of f_0, we see that there exists $r_1 > 0$ such that

$$f(u) \geqslant (f_0 - \varepsilon)u, \quad \text{for} \quad 0 < u \leqslant r_1.$$

Further, if $u \in P$ with $\|u\| = r_1$, then similar to the second part of Theorem 1, we can obtain that $\|S_\lambda u\| \geqslant \|u\|$. Thus, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|S_\lambda u\| \geqslant \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_1. \tag{11}$$

Next, we may choose $R_1 > 0$ such that

$$f(u) \leqslant (F_\infty + \varepsilon)u, \quad \text{for} \quad u \geqslant R_1. \tag{12}$$

We consider two cases.

Case 1. Suppose f is bounded. Then there exists some $M > 0$, such that

$$f(u) \leqslant M, \quad \text{for} \quad u \in (0, +\infty).$$

We define $r_3 = \max\{2r_1, \lambda M C_1\}$, and $u \in P$ with $\|u\| = r_3$, then

$$\|S_\lambda u\| \leqslant \lambda \int_0^1 G(1,qs)f(u(s))d_qs$$

$$\leqslant \lambda M \int_0^1 G(1,qs)d_qs$$
$$= \lambda MC_1 \leqslant r_3 = \|u\|.$$

Hence,

$$\|S_\lambda u\| \leqslant \|u\|, \quad \text{for} \quad u \in P_{r_3} = \{u \in P : \|u\| \leqslant r_3\}.$$

Case 2. Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1, R_1\}$ such that

$$f(u) \leqslant f(r_4), \quad \text{for} \quad 0 < u \leqslant r_4.$$

Let $u \in P$ with $\|u\| = r_4$. Then by (9) and (12), we have

$$\begin{aligned}\|S_\lambda u\| &\leqslant \lambda \int_0^1 G(1,qs)f(u(s))d_qs \\ &\leqslant \lambda \int_0^1 G(1,qs)(F_\infty+\varepsilon)\|u\|d_qs \\ &= \lambda C_1(F_\infty+\varepsilon)\|u\| \leqslant \|u\|.\end{aligned}$$

Thus,

$$\|S_\lambda u\| \leqslant \|u\|, \quad \text{for} \quad u \in P_{r_4} = \{u \in P : \|u\| \leqslant r_4\}.$$

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in P : \|u\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|S_\lambda u\| \leqslant \|u\|, \quad \text{for} \quad u \in P \cap \partial\Omega_2. \tag{13}$$

Now that we obtain (11), (13), it follows from Lemma 4 that S_λ has a fixed-point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leqslant \|u\| \leqslant r_2$. It is clear that u is a positive solution of (1). The proof is complete.

THEOREM 3. *If there exists $l \in (0,1)$ and $r_2 > r_1 > 0$ such that $g(l) > r_1/r_2$ and f satisfy*

$$\min_{g(l)r_1 \leqslant u \leqslant r_1} f(u) \geqslant \frac{r_1}{\lambda g(l)C_1}, \quad \max_{0 \leqslant u \leqslant r_2} f(u) \leqslant \frac{r_2}{\lambda C_1}.$$

the boundary value problem (1) *has a positive solution u with $r_1 \leqslant \|u\| \leqslant r_2$.*

Proof. Choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$; then for $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}\|S_\lambda u\| &\geqslant (S_\lambda u)(l) = \lambda \int_0^1 G(l,qs)f(u(s))d_qs \\ &\geqslant \lambda \int_0^1 g(l)G(1,qs)f(u(s))d_qs\end{aligned}$$

$$= \lambda g(l)C_1 \frac{r_1}{\lambda g(l)C_1} = r_1 = \|u\|.$$

On the other hand, choose $\Omega_2 = \{u \in E : \|u\| < r_2\}$; then for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|S_\lambda u\| &\leqslant \lambda \int_0^1 G(1,qs) f(u(s)) d_q s \\
&\leqslant \lambda \int_0^1 G(1,qs) \max_{0 \leqslant u \leqslant r_2} f(u(s)) d_q s \\
&\leqslant \lambda C_1 \frac{r_2}{\lambda C_1} = r_2 = \|u\|.
\end{aligned}$$

Thus, by Lemma 4, the boundary value problem (1) has a positive solution u with $r_1 \leqslant \|u\| \leqslant r_2$. The proof is complete.

For the reminder of this section, we will need the following condition.

(H) $(\min_{u \in [g(l)r,r]} f(u))/r > 0$, where $l \in (0,1)$.

For convenience, we denote

$$\lambda_1 = \sup_{r>0} \frac{r}{C_1 \max_{u \in [0,r]} f(u)}, \quad \lambda_2 = \inf_{r>0} \frac{r}{C_1 \min_{u \in [g(l)r,r]} f(u)}. \tag{14}$$

In view of the continuity of $f(u)$ and (H), we have $0 < \lambda_1 \leqslant +\infty$ and $0 \leqslant \lambda_2 < +\infty$.

THEOREM 4. *Assume (H) holds. If $f_0 = f_\infty = +\infty$, then the boundary value problem (1) has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.*

Proof. Define

$$a(r) = \frac{r}{C_1 \max_{u \in [0,r]} f(u)}.$$

By the continuity of $f(u)$ and $f_0 = f_\infty = +\infty$, we have that $a(r) : (0,+\infty) \to (0,+\infty)$ is continuous and

$$\lim_{r \to 0} a(r) = \lim_{r \to +\infty} a(r) = 0.$$

From (14), there exists $r_0 \in (0,+\infty)$ such that

$$a(r_0) = \sup_{r>0} a(r) = \lambda_1;$$

then for $\lambda \in (0,\lambda_1)$, there exist constants c_1, c_2 $(0 < c_1 < r_0 < c_2 < +\infty)$ with

$$a(c_1) = a(c_2) = \lambda;$$

Thus,

$$f(u) \leqslant \frac{c_1}{\lambda C_1}, \quad \text{for} \quad u \in [0, c_1], \tag{15}$$

$$f(u) \leqslant \frac{c_2}{\lambda C_1}, \quad \text{for} \quad u \in [0, c_2]. \tag{16}$$

On the other hand, applying the conditions $f_0 = f_\infty = +\infty$, there exist constants d_1, d_2 $(0 < d_1 < c_1 < r_0 < c_2 < d_2 < +\infty)$ with

$$\frac{f(u)}{u} \geqslant \frac{1}{g^2(l)\lambda C_1}, \quad \text{for} \quad u \in (0, d_1) \cup (g(l)d_2, +\infty).$$

Then

$$\min_{g(l)d_1 \leqslant u \leqslant d_1} f(u) \leqslant \frac{d_1}{\lambda g(l) C_1}, \tag{17}$$

$$\min_{g(l)d_2 \leqslant u \leqslant d_2} f(u) \leqslant \frac{d_2}{\lambda g(l) C_1}. \tag{18}$$

By (15) and (17), (16) and (18), combining with Theorem 3 and Lemma 4, we can complete the proof.

COROLLARY 1. *Assume* (H) *holds. If* $f_0 = +\infty$ *or* $f_\infty = +\infty$*, then the boundary value problem* (1) *has at least one positive solution for each* $\lambda \in (0, \lambda_1)$.

THEOREM 5. *Assume* (H) *holds. If* $f_0 = f_\infty = 0$*, then the boundary value problem* (1) *has at least two positive solutions for each* $\lambda \in (\lambda_2, +\infty)$.

Proof. Define

$$b(r) = \frac{r}{C_1 \min_{u \in [g(l)r, r]} f(u)}.$$

By the continuity of $f(u)$ and $f_0 = f_\infty = 0$, we can easily see that $b(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r \to 0} b(r) = \lim_{r \to +\infty} b(r) = +\infty.$$

From (14), there exists $r_0 \in (0, +\infty)$ such that

$$b(r_0) = \sup_{r > 0} b(r) = \lambda_2;$$

then for $\lambda \in (0, \lambda_1)$, there exist constants d_1, d_2 $(0 < d_1 < r_0 < d_2 < +\infty)$ with

$$b(d_1) = b(d_2) = \lambda;$$

Therefore,

$$f(u) \leqslant \frac{d_1}{\lambda g(l)C_1}, \quad \text{for} \quad u \in [g(l)d_1, d_1], \tag{19}$$

$$f(u) \leqslant \frac{d_2}{\lambda g(l)C_1}, \quad \text{for} \quad u \in [g(l)d_2, d_2]. \tag{20}$$

On the other hand, applying the conditions $f_0 = 0$, there exist constants c_1 $(0 < c_1 < d_1)$ with

$$\frac{f(u)}{u} \leqslant \frac{1}{\lambda C_1}, \quad \text{for} \quad u \in (0, c_1).$$

Then

$$\max_{0 \leqslant u \leqslant c_1} f(u) \leqslant \frac{c_1}{\lambda C_1}. \tag{21}$$

In view of $f_\infty = 0$, there exists a constant $c_2 \in (d_2, +\infty)$ such that

$$\frac{f(u)}{u} \leqslant \frac{1}{\lambda C_1}, \quad \text{for} \quad u \in (c_2, +\infty).$$

Let $M = \max_{0 \leqslant u \leqslant c_2} f(u)$, $c_2 \geqslant \lambda C_1 M$. It is easily seen that

$$\max_{0 \leqslant u \leqslant c_2} f(u) \leqslant \frac{c_2}{\lambda C_1}. \tag{22}$$

By (19) and (21), (20) and (22), combining with Theorem 3 and Lemma 4, we can complete the proof.

COROLLARY 2. *Assume (H) holds. If $f_0 = 0$ or $f_\infty = 0$, then the boundary value problem (1) has at least one positive solution for each $\lambda \in (\lambda_2, +\infty)$.*

By the above theorems, we can obtain the following results.

COROLLARY 3. *Assume (H) holds. If $f_0 = +\infty$, $f_\infty = d$ or $f_\infty = +\infty$, $f_0 = d$, then the boundary value problem (1) has at least one positive solution for each $\lambda \in (0, (dC_1)^{-1})$.*

COROLLARY 4. *Assume (H) holds. If $f_0 = 0$, $f_\infty = d$ or $f_\infty = 0$, $f_0 = d$, then the boundary value problem (1) has at least one positive solution for each $\lambda \in ((g(l)dC_2)^{-1}, +\infty)$.*

THEOREM 6. *Assume (H) holds. If $F_0 < +\infty$ and $F_\infty < +\infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, the boundary value problem (1) has no positive solution.*

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers m_1, m_2, r_1 and r_2 such that $r_1 < r_2$ and

$$f(u) \leqslant m_1 u, \quad \text{for} \quad u \in [0, r_1]; \quad f(u) \leqslant m_2 u, \quad \text{for} \quad u \in [r_2, +\infty).$$

Let

$$m = \max\Big\{m_1, m_2, \max_{u \in [r_1, r_2]} \{f(u)/u\}\Big\}.$$

Then we have

$$f(u) \leqslant mu, \quad \text{for} \quad u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of (1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0; = (mC_1)^{-1}$. Since $S_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|S_\lambda v\| \leqslant \lambda \int_0^1 G(1, qs) f(v(s)) d_q s \leqslant m\lambda \|v\| \int_0^1 G(1, qs) d_q s < \|v\|,$$

which is a contradiction. Therefore, (1) has no positive solution. The proof is complete.

THEOREM 7. *Assume (H) holds. If $f_0 > 0$ and $f_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, the boundary value problem (1) has no positive solution.*

Proof. Since $f_0 > 0$ and $f_\infty > 0$, there exist positive numbers n_1, n_2, r_1 and r_2 such that $r_1 < r_2$ and

$$f(u) \geqslant n_1 u, \quad \text{for} \quad u \in [0, r_1]; \quad f(u) \geqslant n_2 u, \quad \text{for} \quad u \in [r_2, +\infty).$$

Let $n = \min\{n_1, n_2, \min_{u \in [r_1, r_2]}\{f(u)/u\}\}$. Then we have

$$f(u) \geqslant nu, \quad \text{for} \quad u \in [0, +\infty).$$

Assume $v(t)$ is a positive solution of (1). We will show that this leads to a contradiction for $\lambda > \lambda_0; = (g(l)nC_2)^{-1}$. Since $S_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|S_\lambda v\| \geqslant \lambda \int_0^1 g(l) G(1, qs) f(v(s)) d_q s > \|v\|,$$

which is a contradiction. Therefore, (1) has no positive solution. The proof is complete.

4. Examples

In this section, we will present some examples to illustrate the main results.

EXAMPLE 1. Consider the fractional q-difference boundary value problem

$$\begin{cases} (D_{0.5}^{2.5}u)(t) + \lambda u^a = 0, \quad 0 < t < 1, \quad a > 1, \\ u(0) = (D_q u)(0) = (D_q u)(1) = 0. \end{cases} \tag{23}$$

Since $\alpha = 2.5$ and $q = 0.5$, we have [12]

$$C_1 = \int_0^1 G(1,qs)d_q s \leqslant \frac{1-(1-q)^{(\alpha-1)}}{\Gamma_q(\alpha)} \approx 0.48636,$$

$$\begin{aligned} C_2 &= \int_0^1 g(s)G(1,qs)d_q s \geqslant \int_{\frac{1}{2}}^1 s^{\alpha-1}G(1,qs)d_q s \\ &\geqslant \left(\frac{1}{2}\right)^{\alpha-1} \int_{\frac{1}{2}}^1 G(1,qs)d_q s \\ &\geqslant \left(\frac{1}{2}\right)^{\alpha-1} \frac{(1-q/2)^{(\alpha-1)} - (1-q)^{(\alpha-1)}}{2\Gamma_q(\alpha)} \approx 0.06250. \end{aligned}$$

Let $f(u) = u^a$, $a > 1$. Then from [26], we have $F_0 = 0$ and $f_\infty = +\infty$. Choose $l = 0.5$. Then $g(0.5) = 0.5^{1.5} \approx 0.35355$. So $g(l)C_2 f_\infty > F_0 C_1$ holds. Thus, by Theorem 1, the boundary value problem (23) has a positive solution for each $\lambda \in (0, +\infty)$.

EXAMPLE 2. Consider the fractional q-difference boundary value problem

$$\begin{cases} (D_{0.5}^{2.5}u)(t) + \lambda u^b = 0, \quad 0 < t < 1, \quad 0 < b < 1, \\ u(0) = (D_q u)(0) = (D_q u)(1) = 0. \end{cases} \tag{24}$$

Since $\alpha = 2.5$ and $q = 0.5$, we have $C_1 \leqslant 0.48636$ and $C_2 \geqslant 0.06250$. Let $f(u) = u^b$, $0 < b < 1$. Then from [26], we have $F_\infty = 0$ and $f_0 = +\infty$. Choose $l = 0.5$. Then $g(0.5) = 0.5^{1.5} \approx 0.35355$. So $g(l)C_2 f_0 > F_\infty C_1$ holds. Thus, by Theorem 2, the boundary value problem (24) has a positive solution for each $\lambda \in (0, +\infty)$.

EXAMPLE 3. Consider the fractional q-difference boundary value problem

$$\begin{cases} (D_{0.5}^{2.5}u)(t) + \lambda \dfrac{(200u^2 + u)(2 + \sin u)}{u+1} = 0, \quad 0 < t < 1, \\ u(0) = (D_q u)(0) = (D_q u)(1) = 0. \end{cases} \tag{25}$$

Since $\alpha = 2.5$ and $q = 0.5$, we have $C_1 \leqslant 0.48636$ and $C_2 \geqslant 0.06250$. Let

$$f(u) = \frac{(200u^2 + u)(2 + \sin u)}{u+1}.$$

Then from [26], we have $F_0 = f_0 = 2$, $F_\infty = 600$, $f_\infty = 200$, and $2u < f(u) < 600u$.

(i) Choose $l = 0.5$. Then $g(0.5) = 0.5^{1.5} \approx 0.35355$. So $g(l)C_2 f_\infty > F_0 C_1$ holds. Thus, by Theorem 1, the boundary value problem (25) has a positive solution for each $\lambda \in (0.22628, 1.02805)$.

(ii) By Theorem 6, the boundary value problem (25) has no positive solution for all $\lambda \in (0, 0.00342)$.

(iii) By Theorem 7, the boundary value problem (25) has no positive solution for all $\lambda \in (22.6276, +\infty)$.

REFERENCES

[1] R.P. AGARWAL, *Certain fractional q-integrals and q-derivatives*, Proc. Cambridge Philos. Soc., **66** (1969), 365–370.

[2] R.P. AGARWAL, B. AHMAD, *Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions*, Comput. Math. Appl., **62** (2011), 1200–1214.

[3] R.P. AGARWAL, V. LAKSHMIKANTHAM, J.J. NIETO, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., **72** (2010), 2859–2862.

[4] R.P. AGARWAL, D. O'REGAN, S. STANĚK, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations*, J. Math. Anal. Appl., **371** (2010), 57–68.

[5] B. AHMAD, J.J. NIETO, J. PIMENTEL, *Some boundary value problems of fractional differential equations and inclusions*, Comput. Math. Appl., **62** (2011), 1238–1250.

[6] W.A. AL-SALAM, *Some fractional q-integrals and q-derivatives*, Proc. Edinb. Math. Soc., **15**, 2 (1966-1967), 135–140.

[7] F.M. ATICI, P.W. ELOE, *Fractional q-calculus on a time scale*, J. Nonlinear Math. Phys., **14** (2007), 333–344.

[8] M. EL-SHAHED, F.M. AL-ASKAR, *Positive solutions for boundary value problem of nonlinear fractional q-difference equation*, ISRN Math. Anal., **2011** (2011), Art. ID 385459, 12 pages.

[9] M. EL-SHAHED, M. AL-YAMI, *On the existence and uniqueness of solutions for q-fractional boundary value problem*, Int. J. Math. Anal., **5** (2011), 1619–1630.

[10] M. EL-SHAHED, H.A. HASSAN, *Positive solutions of q-difference equation*, Proc. Amer. Math. Soc., **138** (2010), 1733–1738.

[11] R.A.C. FERREIRA, *Nontrivial solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equ., **70** (2010), 1–10.

[12] R.A.C. FERREIRA, *Positive solutions for a class of boundary value problems with fractional q-differences*, Comput. Math. Appl., **61** (2011), 367–373.

[13] D.J. GUO, V. LAKSHMIKANTHAM, *Nonlinear problems in abstract cones*, Academic Press, Boston, MA, 1988.

[14] F.H. JACKSON, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edinburgh, **46** (1908), 253–281.

[15] F.H. JACKSON, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41** (1910), 193–203.

[16] V. KAC AND P. CHEUNG, *Quantum Calculus*, Springer, New York, NY, USA, 2002.

[17] M.A. KRASNOSEL'SKII, *Positive solutions of operator equations*, Noordhoff, Groningen, The Netherlands, 1964.

[18] C. LI, X. LUO, Y. ZHOU, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations*, Comput. Math. Appl., **59** (2010), 1363–1375.

[19] S. LIANG, J. ZHANG, *Existence and uniqueness of positive solutions for three-point boundary value problem with fractional q-differences*, J. Appl. Math Comput., **40** (2012), 277–288.

[20] J. MA, J. YANG, *Existence of solutions for multi-point boundary value problem of fractional q-difference equation*, Electron. J. Qual. Theory Differ. Equ., **92** (2011), 1–10.

[21] G. WANG, L. ZHANG, S.K. NTOUYAS, *Existence of multiple positive solutions of a nonlinear arbitrary order boundary value problem with advanced arguments*, Electron. J. Qual. Theory Differ. Equ., **15** (2012), 1–13.

[22] P.M. RAJKOVIĆ, S.D. MARINKOVIĆ, AND M.S. STANKOVIĆ, *On q-analogues of caputo derivative and Mittag-Leffler function*, Fract. Calc. Appl. Anal., **10** (2007), 359–373.

[23] P.M. RAJKOVIĆ, S.D. MARINKOVIĆ, AND M.S. STANKOVIĆ, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math., **1** (2007), 311–323.

[24] X. YANG, Z. WEI, W. DONG, *Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 85–92.

[25] C. YUAN, D. JIANG, D. O'REGAN, R.P. AGARWAL, *Multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions*, Electron. J. Qual. Theory Differ. Equ., **13** (2012), 1–17.

[26] Y. ZHAO ET AL., *Positive solutions to boundary value problems of nonlinear fractional differential equations*, Abstr. Appl. Anal., **2011** (2011), Art. ID 390543, 16 pages.

(Received May 31, 2012)
(Revised April 17, 2013)

Wengui Yang
Ministry of Public Education
Sanmenxia Polytechnic
Sanmenxia 47220
China
e-mail: wgyang0617@yahoo.com