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PERTURBATION RESULTS FOR SOME NONLINEAR EQUATIONS INVOLVING FRACTIONAL OPERATORS

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Dedicated to Antonio Ambrosetti, upon his retirement

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Abstract. By using a perturbation technique in critical point theory, we prove the existence of solutions for two types of nonlinear equations involving fractional differential operators.

1. Introduction

In this note we propose a few existence results for solutions to nonlinear elliptic equations driven by fractional operators. We will focus on two models: a pseudo-relativistic Hartree equation, and an equation involving the fractional laplacian. We refer to the next sections for more details on these problems.

Our approach relies on a perturbation technique in Critical Point Theory introduced some years ago by Ambrosetti and his collaborators. It is very useful when dealing with perturbation problems with lack of compactness. For the reader's sake, we collect in this Introduction the main ingredients of this method. The interested reader will find the complete theory in the book [2].

The first model we deal with is related to the so-called pseudo-relativistic Hartree equation

$$\sqrt{-\Delta+m^2}\, u+\mu u=\left(|x|^{-1} * |u|^2\right) u \quad \text{in } \mathbb{R}^3. \tag{1.1}$$

This problem, non-local in nature, is usually faced by direct methods like constrained minimization, as in [7, 8]. Minimizing sequences converge up to translations, but this feature is usually lost if we perturb (1.1) with a term that breaks the translation invariance. The arguments of [7] might be suitably adapted only in a *radially symmetric* setting, while those of [8] heavily rely on the autonomous structure of the problem.

The second model we deal with is related to the so-called *fractional Schrödinger equation*

$$(-\Delta)^s u+u=|u|^p u \quad \text{in } \mathbb{R}, \tag{1.2}$$

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where $0 < s < 1$. Nonlinear problems like (1.2) are now the subject of intensive research, and are often solved by means of direct variational methods: see [9, 10, 11] for very recent results. We we will perturb (1.2) with a potential term, and we will find a whole branch of solutions in the spirit of bifurcation theory. dimension are still partially unknown, the picture is rather clear in 1D, and we will perturb (1.2) with a potential term.

Although are existence results are only valid in a perturbative setting, we believe that we can cover some cases that would be really hard to treat by direct methods. In addition, what we perform is actually a bifurcation analysis, and it may give some more precise insight than a mere analysis of a minimizing sequence.

1.1. The abstract setting

Consider a (real) Hilbert space H, and a family $\{f_\varepsilon\}$ of functionals $f_\varepsilon \colon H \to \mathbb{R}$ of class C^2. We assume that

$$f_\varepsilon = f_0 + \varepsilon G, \tag{1.3}$$

where $G \colon H \to \mathbb{R}$ and $f_0 \colon H \to \mathbb{R}$ satisfies

(h_1) $f_0 \in C^2(H)$ has a smooth manifold Z of dimension $d < \infty$, such that $f_0'(z) = 0$ for every $z \in Z$.

(h_2) The linear operator $f_0''(z)$ is a Fredholm operator of index zero, for every $z \in Z$.

(h_3) $\ker f_0''(z) = T_z Z$ for every $z \in Z$. Here $T_z Z$ stands for the tangent space at z to the manifold Z.

If we look for critical points of f_ε as $\varepsilon \to 0$, i.e. for points $u \in H$ with

$$f_\varepsilon'(u) = 0, \tag{1.4}$$

we can *deform* the manifold Z to a new manifold Z_ε in such a way that Z_ε is a *natural constraint* for f_ε. Let us briefly recall the construction.

By the Implicit Function Theorem, we can construct a function $w = w(z, \varepsilon)$ with values in Z and such that

1. $w(z, 0) = 0$ for every $z \in Z$;
2. $f_\varepsilon'(z + w(z, \varepsilon)) \in T_z Z$ for every $z \in Z$;
3. $w(z, \varepsilon) \in (T_z Z)^\perp$ for every $z \in Z$.

It follows without effort that $w = O(\varepsilon)$ as $\varepsilon \to 0$. Then we introduce the perturbed manifold

$$Z_\varepsilon = \{u = z + w(z, \varepsilon) \mid z \in Z\}$$

LEMMA 1. *Z_ε is a natural constraint for f_ε: if $u = z + w(z, \varepsilon) \in Z_\varepsilon$ and $f'_{\varepsilon|Z_\varepsilon}(u) = 0$, then $f_\varepsilon'(u) = 0$.*

Proof. By assumption $f'_\varepsilon(u)$ is orthogonal to $T_u z_\varepsilon$. From the properties of w it follows that $f'_\varepsilon(u) \in T_z Z$, and $T_u Z_\varepsilon$ is close to $T_z Z$ when ε is small. Therefore $f'_\varepsilon(u) = 0$. □

This Lemma allows us to replace the problem $f'_\varepsilon(u) = 0$ with the *finite-dimensional* problem $f'_{\varepsilon|Z_\varepsilon}(u) = 0$, $u \in Z_\varepsilon$.

An expansion with respect to ε,

$$\begin{aligned} f_\varepsilon(z + w(z,\varepsilon)) &= f_0(z + w(z,\varepsilon)) + \varepsilon G(z + w(z,\varepsilon)) \\ &= f_0(z) + \varepsilon G(z) + o(\varepsilon), \end{aligned}$$

shows the following existence result.[1]

PROPOSITION 1. *Under our general assumptions (h_1), (h_2) and (h_3), the functional f_ε has at least one critical point, provided that $G_{|Z}$ has a stable critical point. In particular, this happens whenever there exist an open set $A \subset Z$ and a point $z_0 \in A$ such that*

$$G(z_0) < \inf_{\partial A} G \quad (\textit{or } G(z_0) > \sup_{\partial A} G)$$

For a neat exposition of the complete theory we refer to the book [2], where the interested reader will find many applications and generalizations.

2. Pseudo-relativistic Hartree equations

As a first application of the general principle, we study a class of equations involving a fractional differential operator:

$$\sqrt{-\Delta + m^2}\, u + \mu u = \left(|x|^{-1} * |u|^2\right)(1 + \varepsilon g(x))\, u \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

We suppose that $\mu > 0$ is a given parameter.

The mean field limit of a quantum system describing many self-gravitating, relativistic bosons with rest mass $m > 0$ leads to the time-dependent pseudo-relativistic Hartree equation

$$i\frac{\partial \psi}{\partial t} = \left(\sqrt{-\Delta + m^2} - m\right)\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi, \quad x \in \mathbb{R}^3, \tag{2.2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is the wave field. Such a physical system is often referred to as a boson star in astrophysics. Solitary wave solutions $\psi(t,x) = e^{-it\lambda}\phi$, $\lambda \in \mathbb{R}$ to equation (2.2) satisfy the equation

$$\left(\sqrt{-\Delta + m^2} - m\right)\phi - \left(\frac{1}{|x|} * |\phi|^2\right)\phi = \lambda \phi \tag{2.3}$$

[1] We recall that a critical point u of a functional f is called *stable* whenever each functional g, sufficiently close to f in the C^1-norm, has itself a critical point. See [16].

boson stars (see [7, 8] for references). In [17] Lieb and Yau solved the pseudo-relativistic Hartree equation (2.3) by minimization on the sphere $\left\{\phi \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\phi|^2 = M\right\}$, and they proved that a radially symmetric ground state exists in $H^{1/2}(\mathbb{R}^3)$ whenever $M < M_c$, the so-called Chandrasekhar mass. These results have been generalized in [8]. Later Lenzmann proved in [15] that this ground state is unique (up to translations and phase change) provided that the mass M is sufficiently small; some results about the non-degeneracy of the ground-state solution are also given.

Quite recently, Coti Zelati and Nolasco (see [7]) studied the equation

$$\sqrt{-\Delta+m^2}u = \mu u + \nu |u|^{p-2}u + \sigma(W * u^2)u$$

under the assumptions that $p \in \left(2, \frac{2N}{N-1}\right)$, $N \geqslant 3$, $\mu < m$, $\nu \geqslant 0$, $\sigma \geqslant 0$ but not both zero, $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $W \geqslant 0$, $r > N/2$, W is radially symmetric and decays to zero at infinity. They proved the existence of a positive, radial solution that decays to zero at infinity exponentially fast. For the case $\sigma < 0$, $\mu < m$, we also refer to [18] where a more general nonlinear term is considered; however, radial symmetry is imposed by the author to face the lack of compactness.

For the reader's sake, we recall that the operator

$$\sqrt{-\Delta+m^2}$$

can be defined on $f \in H^1(\mathbb{R}^3)$ by the following formula:

$$\mathscr{F}\left(\left(\sqrt{-\Delta+m^2}\, f\right)\right)(\xi) = \sqrt{|\xi|^2+m^2}\,\mathscr{F} f(k),$$

where $\mathscr{F}$ is the Fourier transform and $\xi \in \mathbb{R}^3$. An alternative approach to this fractional operator is through a *local realization*: given any $u \in C_0^\infty(\mathbb{R}^3)$, there exists a unique solution $v \in C_0^\infty(\mathbb{R}^{3+1}_+)$ of the Dirichlet problem

$$\begin{cases} -\Delta v + m^2 y = 0 & \text{in } \mathbb{R}^{3+1}_+ \\ v(0,y) = u(y) & \text{for } y \in \mathbb{R}^3 = \partial\mathbb{R}^{3+1}_+. \end{cases} \tag{2.4}$$

Here $\mathbb{R}^{3+1}_+ = \left\{(x,y) \mid x \in \mathbb{R}^3,\ y > 0\right\}$. Setting

$$Tu(y) = -\frac{\partial v}{\partial x}(0,y),$$

the function $w(x,y) = -\frac{\partial v}{\partial x}(x,y)$ solves the problem

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{3+1}_+ \\ w(0,y) = Tu(y) = -\frac{\partial v}{\partial x}(0,y) & \text{for } y \in \mathbb{R}^3 = \partial\mathbb{R}^{3+1}_+, \end{cases} \tag{2.5}$$

and this implies that

$$T(Tu)(y) = -\frac{\partial w}{\partial x}(0,y) = \frac{\partial^2 w}{\partial x^2}(0,y) = \left(-\Delta_y v + m^2 v\right)(0,y)$$

and therefore $T^2 = \left(-\Delta_y v + m^2 v\right)$.

Since the group of translations acts on solutions to (2.1), we face here a non-trivial lack of compactness: generally speaking, Palais-Smale sequences need not be (relatively) compact. Let us see how the perturbation technique presented in the introduction may help us to overcome this issue.

To embed this problem into our abstract scheme, we set $H = H^{\frac{1}{2}}(\mathbb{R})$, the usual Sobolev space of fractional order that can also be seen as the trace space of $H^1(\mathbb{R}^{3+1}_+)$, and

$$f_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(\sqrt{-\Delta + m^2}\,|u|^2 + \mu|u|^2\right) - \frac{1}{4}\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right)(1+\varepsilon g(x))\,|u|^2\,dx.$$

We assume that $g \in L^\infty(\mathbb{R}^3)$. Since the convolution kernel $x \mapsto |x|^{-1}$ belongs to the Lorentz space $L^3_w(\mathbb{R}^3)$, we can invoke the following *weak Young inequality* to conclude that f_ε is well-defined:

$$\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right)|u|^2 \leqslant C\||x|^{-1}\|_{L^3_w}\|u\|_{L^2}\|u\|_{L^3},$$

where $C > 0$ is a universal constant independent of $u \in H^{\frac{1}{2}}(\mathbb{R}^3)$. We recall that the Lorentz space (or weak L^3 space) $L^3_w(\mathbb{R}^3)$ is the set of those functions f for which the quasi-norm[2]

$$\|f\|^3_{L^3_w} = \sup_{t>0} t^3 \mathscr{L}\left(\left\{x \in \mathbb{R}^3 \mid |f(x)| > t\right\}\right)$$

is finite. Moreover, we remark that $|\cdot|^{-1} \in L^r(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, as is immediately seen by writing, for $R > 0$,

$$\frac{1}{|x|} = \frac{1}{|x|}\chi_{B(0,R)} + \frac{1}{|x|}\chi_{\mathbb{R}^3\setminus B(0,R)}.$$

It is also easy to check that $f_\varepsilon \in C^2(H)$. For our setting we define

$$f_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(\sqrt{-\Delta + m^2}\,|u|^2 + \mu|u|^2\right) - \frac{1}{4}\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right)|u|^2\,dx$$
$$G(u) = -\frac{1}{4}\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right) g(x)|u|^2\,dx$$

Let us recall some important facts proved in [15]. If

$$\mathscr{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \sqrt{-\Delta + m^2}\,|u|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right)|u|^2\,dx,$$

we consider the variational problem

$$E(N) = \inf\left\{\mathscr{E}(u) \mid u \in H^{\frac{1}{2}}(\mathbb{R}^3),\ \int_{\mathbb{R}^3} |u|^2 = N\right\}.$$

Then there exists a number (called the Chandrasekar mass) $N_c > 4/\pi$ such that

[2] $\mathscr{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$.

- $E(N)$ is attained if and only if $0 < N < N_*$; the corresponding minimizer $Q \in H^{\frac{1}{2}}(\mathbb{R}^3)$ solves
$$\sqrt{-\Delta + m^2}\, Q + \mu Q = \left(|x|^{-1} * |Q|^2\right) Q$$
for some Lagrange multiplier $\mu \in \mathbb{R}$.

- Any minimizer Q of $E(N)$ belongs to $H^s(\mathbb{R}^3)$ for all $s \geqslant 0$ and Q decays exponentially fast at infinity.

- Any minimizer Q is radially decreasing and positive everywhere.

- If $N \ll 1$, then there exists one and only one minimizer Q (up to translations), which is *non-degenerate* in the following sense: the linearized operator
$$L_+ \xi = \left(\sqrt{-\Delta + m^2} + \mu\right) \xi - \left(|x|^{-1} * |Q|^2\right) \xi - 2Q\left(|x|^{-1} * (Q\xi)\right)$$
satisfies the condition
$$\ker L_+ = \operatorname{span}\left\{ \frac{\partial Q}{\partial x_1}, \frac{\partial Q}{\partial x_2}, \frac{\partial Q}{\partial x_3} \right\}.$$

REMARK 1. The Lagrange multiplier μ cannot be discarded. From a technical viewpoint this is due to the lack of scaling properties for $\sqrt{-\Delta + m^2}$. But this also reflects the fact that $E(N)$ is attained only if N is strictly smaller than the Chandrasekar mass.

From this moment we fix $N \ll 1$ and its Lagrange multiplier μ in such a way that $E(N)$ is uniquely solvable by a non-degenerate element Q. Without loss of generality, we can assume $m = 1$.

We define the manifold
$$Z = \left\{ Q(\cdot - \xi) \mid \xi \in \mathbb{R}^3 \right\}.$$
Since the Euler-Lagrange equation associated to critical points of f_0 is invariant under translations, each element of Z is a critical point of f_0.

LEMMA 2. *The linear operator $f_0''(z)$ is Fredholm of index zero at every $z \in Z$.*

Proof. Since
$$f_0(u) = \frac{1}{2}\|u\|_H^2 - \frac{1}{4}\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right) |u|^2 \, dx,$$
it suffices to check that $f_0''(z)$ is a compact perturbation of the identity. Therefore, we need to check that
$$K(v_n, w_n) = \int_{\mathbb{R}^3} \left(|x|^{-1} * |Q|^2\right) v_n w_n + 2Q\left(|x|^{-1} * (Qv_n)\right) w_n \to 0$$

whenever $\{v_n\}$ and $\{w_n\}$ are bounded sequences in $H^{\frac{1}{2}}(\mathbb{R}^3)$. Of course, we can assume without loss of generality that $v_n \rightharpoonup 0$ and $w_n \rightharpoonup 0$.

The first term in $K(v_n, w_n)$ goes to zero because it can be seen as a multiplication operator with $|x|^{-1} * |Q|^2$, and

$$\lim_{|x| \to +\infty} |x|^{-1} * |Q|^2 = 0.$$

This was proved in [6, Lemma 2.13] in a slightly different context. We recall the short argument in [7]: given $\varepsilon > 0$, fix $\rho > 0$ such that

$$\sup\left\{\frac{1}{|y|} : |y| > \rho\right\} < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{Q(y)^2}{|y-\zeta|}\,dy &= \int_{B(\zeta,\rho)} \frac{Q(y)^2}{|y-\zeta|}\,dy + \int_{\mathbb{R}^3 \setminus B(\zeta,\rho)} \frac{Q(y)^2}{|y-\zeta|}\,dy \\ &\leqslant \| |x|^{-1} \|_{L^r} \left(\int_{B(\zeta,\rho)} Q(y)^{2r'}\,dy \right)^{1/r'} + \frac{\varepsilon}{2} \|Q\|_{L^2}^2 \end{aligned} \tag{2.6}$$

for $r > 3/2$, and we conclude by letting $|\zeta| \to +\infty$.

As for the second term, fix $\delta > 0$ and $R > 0$. Define

$$\mathscr{K}_\delta(x) = \begin{cases} 1/|x| & \text{if } |x| \leqslant 1/\delta \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathscr{K}_\delta^R(x) = \max\{|\mathscr{K}_\delta(x) - R, 0\}\,\chi_{B(0,R)}(x) + \mathscr{K}_\delta(x)\chi_{\mathbb{R}^3 \setminus B(0,R)}(x).$$

It is clear that, given $\delta > 0$, $\lim_{R \to +\infty} \|\mathscr{K}_\delta^R\|_{L^r} = 0$ for any $r \in [1,3)$. Finally, set $\Theta_\delta^R = \mathscr{K}_\delta - \mathscr{K}_\delta^R$, and remark that $\operatorname{supp} \Theta_\delta^R \subset B(0,R)$.

We apply Young's inequality and get

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(|x|^{-1} * |Qv_n|\right) |Qw_n| \\ &\qquad \leqslant \int_{\mathbb{R}^3} \left(\left(|x|^{-1} - \mathscr{K}_\delta\right) * |Qv_n|\right) |Qw_n| + \int_{\mathbb{R}^3} \left(\mathscr{K}_\delta^R * |Qv_n|\right) |Qw_n| \\ &\qquad\qquad + \int_{\mathbb{R}^3} \left(\Theta_\delta^R * |Qv_n|\right) |Qw_n| \\ &\qquad \leqslant \delta \|Qv_n\|_{L^1} \|Qw_n\|_{L^1} + \|\mathscr{K}_\delta^R\|_{L^{3/2}} \|Qv_n\|_{L^{3/2}} \|Qw_n\|_{L^{3/2}} \\ &\qquad\qquad + \int_{\mathbb{R}^3} \left(\Theta_\delta^R * |Qv_n|\right) |Qw_n| \\ &\qquad \leqslant C\left(\delta + \|\mathscr{K}_\delta^R\|_{L^{3/2}}\right) + \int_{\mathbb{R}^3} \left(\Theta_\delta^R * |Qv_n|\right) |Qw_n|.\end{aligned}$$

We need to prove that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left(\Theta_\delta^R * |Qv_n|\right)|Qw_n| = 0. \tag{2.7}$$

To this aim, pick any $\varepsilon>0$ and choose a radius $R_1>0$ such that

$$\int_{\mathbb{R}^3\setminus B(0,R_1)}|Q|^2<\varepsilon.$$

Then write $R_2=R_1+R$, so that $\Theta_\delta^R(z-y)=0$ whenever $|y|<R_1$ and $|z|\geqslant R_2$. Now,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(\Theta_\delta^R * |Qv_n|\right)|Qw_n| \\
&\quad=\int_{B(0,R_1)}\left(\Theta_\delta^R * \left(\chi_{B(0,R_2)}|Qv_n|\right)\right)|Qw_n|+\int_{\mathbb{R}^3\setminus B(0,R_1)}\left(\Theta_\delta^R * |Qv_n|\right)|Qw_n| \\
&\quad\leqslant R\left(\int_{B(0,R_2)}|Qv_n|\right)\left(\int_{B(0,R_1)}|Qw_n|\right) \\
&\qquad\qquad+\|\Theta_\delta^R * |Qv_n|\|_\infty\left(\int_{\mathbb{R}^3\setminus B(0,R_1)}|w_n|^2\right)^{\frac12}\left(\int_{\mathbb{R}^3\setminus B(0,R_1)}|Q|^2\right)^{\frac12} \\
&\quad\leqslant R\|Q\|_{L^2}\|w_n\|_{L^2}\left(\left(\int_{B(0,R_2)}|v_n|^2\right)^{\frac12}\|Q\|_{L^2}+R\|v_n\|_{L^2}\left(\int_{\mathbb{R}^3\setminus B(0,R_1)}|Q|^2\right)^{\frac12}\right) \\
&\quad\leqslant CR\left(\left(\int_{B(0,R_2)}|v_n|^2\right)^{\frac12}+\left(\int_{\mathbb{R}^3\setminus B(0,R_1)}|Q|^2\right)^{\frac12}\right),
\end{aligned}$$

and the right-hand side goes to zero because $v_n\to 0$ strongly in L^2_{loc} thanks to Lemma 3. Our claim (2.7) follows by letting $n\to+\infty$, $R\to+\infty$ and finally $\delta\to 0$. □

As a consequence, our problem fits into the abstract framework. Here is a possible existence result.

THEOREM 1. *Pick $N>0$ so small that the variational problem $E(N)$ has a unique non-degenerate ground state $Q\in H^{\frac12}(\mathbb{R}^3)$; let $\mu\in\mathbb{R}^3$ be the corresponding Lagrange multiplier. Assume moreover that $g\in L^\infty(\mathbb{R}^3)$ vanishes at infinity and does not change sign. Then, for every ε sufficiently small, equation (2.1) has (at least) a solution $u_\varepsilon\in H^{\frac12}(\mathbb{R}^3)$ such that $u_\varepsilon\simeq Q(\cdot-\xi)$, for a suitable choice of $\xi\in\mathbb{R}^3$.*

Proof. Set $\widetilde{Q}=\left(|x|^{-1} * |Q|^2\right)|Q|^2\in L^1(\mathbb{R}^3)$, and recall that $|x|^{-1} * |Q|^2\in L^\infty(\mathbb{R}^3)$ by (2.6). The abstract scheme invites us to looking for stable critical points of the (finite-dimensional) function

$$G(\xi)=\int_{\mathbb{R}^3}g(x)\widetilde{Q}(x-\xi)\,dx,\qquad \xi\in\mathbb{R}^3.$$

Let us decompose $G(\xi) = G_1(R,\xi) + G_\infty(R,\xi)$, where $R > 0$ and

$$G_1(\xi) = \int_{|x|<R} g(x)\widetilde{Q}(x-\xi)\,dx$$
$$G_\infty(\xi) = \int_{|x|\geqslant R} g(x)\widetilde{Q}(x-\xi)\,dx.$$

We can estimate, for some constant $C_\infty > 0$,

$$|G_\infty(R,\xi)| \leqslant \sup_{|x|\geqslant R} |g(x)| \int_{|x-\xi|\geqslant R} \widetilde{Q}(x)\,dx \leqslant C_\infty \sup_{|x|\geqslant R} |g(x)|. \tag{2.8}$$

On the other hand,

$$|G_1(R,\xi)| \leqslant \sup_{|x|<R} |g(x)| \int_{|x-\xi|<R} \widetilde{Q}(x)\,dx. \tag{2.9}$$

Let $\varepsilon > 0$. Fix $R = R(\varepsilon) > 0$ such that $\sup_{|x|\geqslant R} |g(x)| < \varepsilon$. The right-hand side of (2.9) tend to zero as $|\xi| \to +\infty$, and thus

$$\limsup_{|x|\to+\infty} |G(\xi)| \leqslant C_\infty \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we conclude that

$$\lim_{|\xi|\to+\infty} G(\xi) = 0,$$

and that G does not change sign. Therefore G must have a strict local maximum (or minimum) point at some ξ_0. It now suffices to apply Proposition 1. $\square$

By exploiting the exponential decay of Q, we can prove a similar result under different assumptions.

THEOREM 2. *Pick $N > 0$ so small that the variational problem $E(N)$ has a unique non-degenerate ground state $Q \in H^{\frac{1}{2}}(\mathbb{R}^3)$; let $\mu \in \mathbb{R}^3$ be the corresponding Lagrange multiplier. Assume moreover that $g \in L^\theta(\mathbb{R}^3)$ for some $\theta > 1$ and g does not change sign. Then, for every ε sufficiently small, equation (2.1) has (at least) a solution $u_\varepsilon \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that $u_\varepsilon \simeq Q(\cdot - \xi)$, for a suitable choice of $\xi \in \mathbb{R}^3$.*

Proof. Once more, G does not change sign, and $G(\xi) = G_1(R,\xi) + G_\infty(R,\xi)$. Now,

$$\begin{aligned}|G_\infty(R,\xi)| &\leqslant \left(\int_{|x|\geqslant R} |g(x)|^\theta\,dx\right)^{\frac{1}{\theta}} \left(\int_{|x-\xi|\geqslant R} \left|\widetilde{Q}(x)\right|^{\theta'}\,dx\right)^{\frac{1}{\theta'}} \\ &\leqslant \|\widetilde{Q}\|_{L^{\theta'}} \left(\int_{|x|\geqslant R} |g(x)|^\theta\,dx\right)^{\frac{1}{\theta}}.\end{aligned}$$

Similarly,

$$|G_1(R,\xi)| \leqslant \left(\int_{|x|<R} |g(x)|^\theta \, dx\right)^{\frac{1}{\theta}} \left(\int_{|x-\xi|<R} \left|\widetilde{Q}(x)\right|^{\theta'}\right)^{\frac{1}{\theta'}}$$
$$\leqslant \|g\|_{L^\theta} \left(\int_{|x-\xi|<R} \left|\widetilde{Q}(x)\right|^{\theta'}\right)^{\frac{1}{\theta'}}.$$

In these estimates, $1/\theta + 1/\theta' = 1$, and $\widetilde{Q} \in L^{\theta'}(\mathbb{R}^3)$ thanks to the exponential decay. Letting $R \to +\infty$ and then $|\xi| \to +\infty$, we conclude as before that $\lim_{|\xi|\to+\infty} G(\xi) = 0$, and consequently G attains either a strict local maximum or a strict local minimum (or both). □

REMARK 2. The assumption that g has constant sign ensures that G is non-constant. We can also allow sign-changing perturbations g, for example under the assumption that $\int_{\mathbb{R}^3} g(x)\widetilde{Q}(x)\, dx = G(0) \neq 0$.

REMARK 3. The reader will realize that we can deal with perturbed equations other than (2.1). For example we could prove a similar existence result for

$$\sqrt{-\Delta + m^2}\, u + \mu u = \left(|x|^{-1} * |u|^2\right) u + \varepsilon g(x) |u|^{p-1} u,$$

provided that the integral $\int_{\mathbb{R}^3} g(x)|u(x)|^{p+1}\, dx$ is finite for every $u \in H$.

REMARK 4. As far as we know, our existence results are new when the perturbation term g has no *a-priori* symmetry.

3. A model with the fractional laplacian

Equations governed by fractional powers of the Laplace operator Δ arise in several physical models, and we refer to [13] for some discussion. In this section we show that some existence results can be easily proved also for some of these problems.

Let us consider the problem

$$(-\Delta)^s u + u = (1 + \varepsilon h(x)) |u|^p u \quad \text{in } \mathbb{R}, \tag{3.1}$$

where $0 < s < 1$ and

$$0 < p < p^\dagger = \begin{cases} \frac{4s}{1-2s} & \text{in } 0 < s < 1/2 \\ +\infty & \text{if } 1/2 \leqslant s < 1. \end{cases}$$

The function h is a *bounded* potential and $\varepsilon > 0$ is a "small" parameter. Our aim is to find solution of (3.1) as $\varepsilon \to 0$.

Equations of this form have been widely investigated in the last decades when $s = 1$, i.e. when the differential operator coincides with the standard laplacian. For fractional operators, to the best of our knowledge, the literature is still growing. With

$\varepsilon = 1$ and in dimension $N \geqslant 2$, a similar problem is studied in [11] and in [10] under suitable assumptions on h. See also references therein. A comparable problem is studied in [20] by means of the Concentration-Compactness Alternative in the half-space, but still the coefficients of the equation are constant.

REMARK 5. Problems like (3.1) with a non-costant term h are still rare in the literature. We point out the recent paper [19] for some related results without a perturbative structure.

We wish to spend a few words on the operator $(-\Delta)^s$ appearing on the left-hand side of (3.1). This operator is called *fractional laplacian* (of order s) and there are several *almost equivalent* definitions. A first approach is to regard this operator via Fourier analysis: for every test function φ,

$$(-\Delta)^s \varphi(\xi) = \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(\varphi)(\xi)\right),$$

where $\mathscr{F}$ stands for the Fourier transform. Hence $(-\Delta^s)$ is pseudo-differential operator with symbol $|\xi|^{2s}$.

Equivalently, we may define[3]

$$(-\Delta^s)\varphi(x) = C_s \,\mathrm{P.V.} \int_{\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{|x-y|^{1+2s}}dy,$$

where

$$C_s = \left(\int_{\mathbb{R}} \frac{1-\cos x}{|x|^{1+2s}}dx\right)^{-1}.$$

Either definition makes it clear that $(-\Delta^s)$ is a *non-local* operator: unlike the standard laplacian ($s = 1$), the value of $(-\Delta^s)\varphi$ depends on the values of φ in the whole $\mathbb{R}$. In particular, compactly supported functions won't have, in general, compactly supported fractional laplacians. This is a serious obstruction to the use of standard techniques of nonlinear differential equations such as localization and cut-offs.

Recently, Caffarelli and Silvestre proved in [5] a very interesting *local realization* of the fractional laplacian via a Dirichlet-to-Neumann operator. Roughly speaking, we can add one more variable and solve the linear problem in $\mathbb{R}^2_+ = \{(x,y) \mid x \in \mathbb{R},\ y > 0\}$

$$\begin{cases} -\operatorname{div}\left(y^{1-2s}\nabla u\right) = 0 & \text{in } \mathbb{R}^2_+ \\ u(x,0) = \varphi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Then

$$(-\Delta^s\varphi)(x) = -b_s \lim_{y\to 0+} y^{1-2s}\frac{\partial u}{\partial y}$$

for a suitable constant b_s. This approach is very useful for solving differential equations, as we can work again in a local setting. Once more, it is clear from either definition that problems like (3.1) are non-compact, due to the action of the group of translations.

[3] P.V. denotes here the *principal value* of the integral.

In the sequel, the rôle of $(-\Delta)^s$ will be somehow hidden in the *unperturbed problem* (see below for the definition), and we will use the definition via Fourier analysis only for definiteness.

A function space that can be used quite naturally to study equation (3.1) is the fractional Sobolev space[4]

$$H^s(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} |\xi|^{2s} |\hat{u}|^2 \, d\xi < \infty \right\}$$

endowed with the norm

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \int_{\mathbb{R}} |\xi|^{2s} |\hat{u}|^2 \, d\xi.$$

We recall an embedding property for this space. For more information about fractional Sobolev spaces, we recommend the recent survey [9].

LEMMA 3. *Assume $0 < s < 1/2$, and let $2 \leqslant q \leqslant 2_s^\star = 2/(1-2s)$. Then $H^s(\mathbb{R})$ is continuously embedded into $L^q(\mathbb{R})$. Moreover, this embedding is locally compact provided that $2 \leqslant q < 2_s^\star$.*

If $s = 1/2$, then $H^s(\mathbb{R})$ is continuously embedded into $L^p(\mathbb{R})$, for every $p \in [2, +\infty)$. Finally, if $1/2 < s < 1$, then $H^s(\mathbb{R})$ is continuously embedded in $C^{0,\alpha}(\mathbb{R})$, with $\alpha = (2s-1)/2$.

REMARK 6. Even in dimension one, functions in $H^s(\mathbb{R})$ need not be continuous. Actually, Sobolev's critical exponent for H^s is $2/(1-2s)$: continuity is granted only when $1/2 < s < 1$.

We assume that $h \in C(\mathbb{R})$ is bounded. Then we can easily prove that (weak) solutions to (3.1) correspond to critical points of the functional

$$f_\varepsilon(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p+2}\int_{\mathbb{R}} |u|^{p+2} - \frac{\varepsilon}{p+2}\int_{\mathbb{R}} h(x)|u(x)|^{p+2} \, dx.$$

Setting

$$f_0(u) = \frac{1}{2}\|u\|_{H^s}^2 - \frac{1}{p+2}\int_{\mathbb{R}} |u|^{p+2},$$

we can write

$$f_\varepsilon(u) = f_0(u) + \varepsilon G(u), \tag{3.2}$$

where

$$G(u) = -\frac{1}{p+2}\int_{\mathbb{R}} h(x)|u(x)|^{p+2} \, dx.$$

When $\varepsilon = 0$, critical points of the functional $f_0 \colon H^s(\mathbb{R}) \to \mathbb{R}$ correspond to (weak) solutions to the equation

$$(-\Delta^s)u + u = |u|^p u \quad \text{in } \mathbb{R}, \tag{3.3}$$

which we call *the unperturbed problem*. This equation was deeply studied in [13]. We recall here the main results.

[4]We have set $\hat{u} = \mathscr{F}(u)$, as usual.

THEOREM 3. (Frank and Lenzmann) *Let $0 < s < 1$ and $0 < p < p^{\dagger}$. Then the following holds.*

(i) ***Existence:*** *There exists a solution $Q \in H^s(\mathbb{R})$ of equation (3.3) such that $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$. Moreover, the function $Q \in H^s(\mathbb{R})$ is a minimizer for*

$$J^{s,p}(u) = \frac{\left(\int (-\Delta)^{\frac{s}{2}} u|^2\right)^{\frac{p}{4s}} \left(\int |u|^2\right)^{\frac{p}{4s}(2s-1)+1}}{\int |u|^{p+2}}.$$

(ii) ***Symmetry and Monotonicity:*** *If $Q \in H^s(\mathbb{R})$ with $Q \geqslant 0$ and $Q \not\equiv 0$ solves (3.3), then there exists $x_0 \in \mathbb{R}$ such that $Q(\cdot - x_0)$ is even, positive and strictly decreasing in $|x - x_0|$.*

(iii) ***Regularity and Decay:*** *If $Q \in H^s(\mathbb{R})$ solves (3.3), then $Q \in H^{2s+1}(\mathbb{R})$. Moreover, we have the decay estimate*

$$|Q(x)| + |xQ'(x)| \leqslant \frac{C}{1 + |x|^{2s+1}}$$

for all $x \in \mathbb{R}$ and some constant $C > 0$.

REMARK 7. Unlike the familiar case $s = 1$, ground state solutions Q do *not* decay exponentially fast at infinity.

THEOREM 4. (Frank and Lenzmann) *Let $0 < s < 1$ and $0 < p < p^{\dagger}$. Suppose that $Q \in H^s(\mathbb{R})$ is a positive solution of (3.3) and consider the linearized operator*

$$L_+ = (-\Delta)^s + I - (p+1)Q^p$$

acting on $L^2(\mathbb{R})$. Then the following condition holds: If $Q \in H^s(\mathbb{R})$ is a local minimizer for $J^{s,p}$, then $L+$ is non degenerate, i.e. its kernel satisfies

$$\ker L+ = \operatorname{span}\{Q'\}.$$

In particular, any ground state solution $Q = Q(|x|)$ of equation (3.3) has a non degenerate linearized operator L_+.

THEOREM 5. (Frank and Lenzmann) *Let $0 < s < 1$ and $0 < p < p^{\dagger}$. Then the ground state solution $Q = Q(|x|) > 0$ for equation (3.3) is unique.*

We now introduce our manifold

$$Z = \{Q(\cdot - \theta) \mid \theta \in \mathbb{R}\},$$

where Q is the unique, radially symmetric, positive ground state solution of (3.3). Each element of Z is a critical point of f_0; moreover, since Q decays at infinity, it is standard to check that $D^2 f_0(z)$ is a compact perturbation of the identity, for every $z \in Z$, and assumption (h_2) is thus matched.

By the same token as in the previous section, we can prove the next existence result.

THEOREM 6. *Assume $h \in L^\infty(\mathbb{R})$ has constant sign and $\lim_{|x|\to+\infty} h(x) = 0$. Suppose that $0 < s < 1$ and $0 < p < p^\dagger$. Then, for every ε sufficiently small, equation (3.1) has a non-trivial solution $u_\varepsilon \simeq Q(\cdot - \bar{\theta})$, for some $\bar{\theta} \in \mathbb{R}$.*

Proof. We need to find stable critical points of the function

$$G(\theta) = \int_{\mathbb{R}} h(x)|Q(x-\theta)|^{p+2}\,dx.$$

As in the proof of Theorem 1, $G(\theta) \to 0$ as $\theta \to \pm\infty$, and $\operatorname{sign} G = \operatorname{sign} h$. Hence G has either a strict minimum or a strict maximum point (or both), and we conclude. □

Also in this case we can modify the assumptions on h and replace them by some integrability condition. However, the solution Q no longer decays exponentially fast at infinity, and we must be more precise.

THEOREM 7. *Suppose that $0 < s < 1$, $0 < p < p^\dagger$, and let $h \in L^\theta(\mathbb{R})$ with*

$$\theta = \begin{cases} \frac{2}{p+2s(p-2)} & \text{if } 0 < s < \frac{1}{2} \\ \text{any number} & \text{if } \frac{1}{2} \leqslant s < 1. \end{cases}$$

If h does not change sign, then, for every ε sufficiently small, equation (3.1) has a non-trivial solution $u_\varepsilon \simeq Q(\cdot - \bar{\theta})$, for some $\bar{\theta} \in \mathbb{R}$.

Proof. The proof is similar to that of Theorem 2. However, when applying Hölder's inequality, we must be sure that

$$\int_{\mathbb{R}^3} |Q(x)|^{(p+2)\theta'}\,dx < \infty,$$

where $1/\theta + 1/\theta' = 1$. This is true if

$$\theta'(p+2) = \frac{2}{1-2s}, \quad \text{if } 0 < s < \frac{1}{2},$$

which boils down to

$$\theta = \frac{2}{p+2s(p-2)}, \quad \text{if } 0 < s < \frac{1}{2}.$$

The case $1/2 \leqslant s < 1$ is easier. □

REMARK 8. Of course stable critical points of G may also occur under different assumptions on h. Moreover, different stable critical points of G give rise to different solutions of (3.1); if we know that G has two (ore more) stable critical points, then our equation will have two (or more) solutions.

4. Comments and perspectives

- It is easy to use the regularity estimates of [7, 8] and [11] to prove that our solutions have additional regularity.

- The perturbed equations we have treated are only possible models. It is easy to check that, *mutatis mutandis*, we can also deal with other problems like

$$(-\Delta)^s u + (1+\varepsilon V(x))u = |u|^p u,$$

  for instance.

- Although obtained by rather easy considerations, we believe that our existence results are new and cannot be easily recovered by adapting the corresponding methods for the unperturbed problems. We wish to remark that, to the best of our knowledge, for fractional operators there is no precise analysis of loss of compactness in the existing literature. In the recent preprint [18] the author's assumptions allow non-constant potential functions, but they should be radially symmetric. We do not expect our solutions to be necessarily invariant under rotations.

- Unlike the operator $\sqrt{-\Delta+m^2}$, the fractional laplacian $(-\Delta)^s$ scales in a standard way: under a dilation $x \mapsto \varepsilon x$, the fractional laplacian becomes $\varepsilon^{2s}(-\Delta)^s$. Therefore it is tempting to investigate the *singularly perturbed* equation

$$\varepsilon^{2s}(-\Delta)^s + V(x)u = |u|^p u, \tag{4.1}$$

  where $V \colon \mathbb{R} \to \mathbb{R}$ is ax external potential function. When $s=1$, i.e. when the fractional laplacian reduces to the local Laplace operator, equations like (4.1) appear in a lot of papers. Roughly speaking, *single-peak* solutions, i.e. solutions that concentrate at some point as $\varepsilon \to 0$, are generated by "good" critical points of V. These solutions can also be discovered by a suitable modification of the perturbation developed in [1] and generalized in [3]. See also [2] for a survey. However, it seems that the slow decay at infinity of solutions to $(-\Delta)^s u + u = |u|^p u$ is a severe obstruction. We believe that the analysis of singularly perturbed problems for non-local operators should be pursued further.

- Another interesting issue is to extend the results of non-degeneracy for the fractional laplacian to higher dimensions. The main ingredient for our approach to work again is a version of Theorem 4 in general dimension $N > 1$. The proof of [13] is heavily based on results about the number of zeroes of an eigenfunction corresponding to the second eigenvalue for operators like $(-\Delta)^s + V$, where V is a suitable potential (see the comments in [12]). This approach cannot be immediately generalized to any space dimension. Only very recently has this issue been solved in full generality, see [14]. As a consequence, and in the framework of [14], our existence result can be extended, *mutatis mutandis*, to $\mathbb{R}^N$, $N \geqslant 2$.

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