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GROUND STATE SOLUTION OF A NONCOOPERATIVE ELLIPTIC SYSTEM

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Abstract. In this paper, we study the existence of a ground state solution, that is, a non trivial solution with least energy, of a noncooperative semilinear elliptic system on a bounded domain. By using the method of the generalized Nehari manifold developed recently by Szulkin and Weth, we prove the existence of a ground state solution when the nonlinearity is subcritical and satisfies a weak superquadratic condition.

1. Introduction

In this paper, we are concerned with the following noncooperative elliptic system

$$(\mathscr{P})\qquad \begin{cases} -\Delta u = F_u(x,u,v), & x\in\Omega,\\ \Delta v = F_v(x,u,v), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega,\end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and F_u designates the partial derivative with respect to u of the nonlinearity $F:\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R}$. The solutions of such systems are steady state of reaction-diffusion systems which arise in many applications such as Chemistry, Biology, Geology, Physics or Ecology. It is well known $(\mathscr{P})$ has variational structure, that is, its solutions can be found as critical points of the following functional

$$\Phi(u,v) := \int_\Omega\Big(\frac{1}{2}|\nabla u|^2-\frac{1}{2}|\nabla v|^2 - F(x,u,v)\Big)$$

defined on $H_0^1(\Omega)\times H_0^1(\Omega)$ $\big($i.e the solutions of the equation $\Phi'(u,v)=0$, where Φ' is the Fréchet derivative of $\Phi\big)$. In this paper, we will be interested in the existence of a ground state solution, that is, a non trivial solution which minimizes the energy functional Φ. Let us recall that ground state solutions play an important role in applications. For instance, in the study of the formation of spacial patterns in various reaction-diffusion systems, the solutions of the system often converge to a ground state of a simplified semilinear elliptic system, as time tends to infinity (see [2]).

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In recent years, the existence of ground state solutions of elliptic equations and systems has been widely study, and many interesting results have been obtained (see for instance [2, 3, 7, 9, 1, 6] and the references therein). In ([7], chapter 3), the authors presented the well known method of the Nehari manifold in a unified way, which can be applied to find ground state solutions of the following elliptic system of cooperative type:

$$\begin{cases} -\Delta u = F_u(x,u,v), \ x \in \Omega, \\ -\Delta v = F_v(x,u,v), \ x \in \Omega, \\ u = v = 0, \ x \in \partial\Omega. \end{cases}$$

However, there appears to be no result in the noncooperative case.

Let us now introduce the precise assumptions on the nonlinearity F under which our problem is studied:

(F_1) $F \in \mathscr{C}^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ and $F(x,0) = 0$ for every x in $\overline{\Omega}$, and $0 \in \mathbb{R}^2$.

(F_2) $|\nabla F(x,u)| \leqslant a\big(1+|u|^{p-1}\big)$, for some $p \in (2, 2^\star)$, $x \in \Omega$, $u = (u_1,u_2) \in \mathbb{R}^2$, where $2^\star := 2N/(N-2)$ if $N \geqslant 3$ and $2^\star := \infty$ if $N = 1,2$.

(F_3) $F(x,u) = \circ(|u|^2)$ as $|u| \to 0$, uniformly in x.

(F_4) $\frac{F(x,u)}{|u|^2} \to \infty$ as $|u| \to \infty$, uniformly in x.

(F_5) $F(x,u) > 0$ and $u \cdot \nabla F(x,u) > 2F(x,u)$, $\forall u \in \mathbb{R}^2 \backslash \{0\}$.

(F_6) $\big(v \cdot \nabla F(x,u)\big)(u \cdot v) \geqslant 0$, $\forall v \in \mathbb{R}^2$.

(F_7) If $|u| = |v|$, then $F(x,u) = F(x,v)$ and $v \cdot \nabla F(x,u) \leqslant u \cdot \nabla F(x,u)$, with strict inequality if in addition $u \neq v$.

(F_8) $|u| \neq |v|$ and $u \cdot v \neq 0 \ \Rightarrow \ v \cdot \nabla F(x,u) \neq u \cdot \nabla F(x,v)$.

Here we write

$$F(x,u) = \circ(|u|^2) \quad \text{as } |u| \to 0$$

to mean that

$$\lim_{|u| \to 0} \frac{F(x,u)}{|u|^2} = 0.$$

Also, $\nabla F(x,u)$ denotes the gradient of F with respect to u and $u \cdot v$ is the usual inner product in $\mathbb{R}^2$. A simple example of a nonlinearity satisfying these conditions is $F(x,u) = f(x)|u|^p$, where $2 < p < 2^\star$ and $f > 0$ is of class $\mathscr{C}^1$ on $\overline{\Omega}$.

The main result of this paper is the following:

THEOREM 1. *Under assumptions* (F_1)-(F_8), $(\mathscr{P})$ *has a ground state solution.*

We point out here that the energy functional Φ associated to $(\mathscr{P})$ is strongly indefinite, in the sense that the negative and positive eigenspaces of its quadratic part are both infinite-dimensional. Therefore, the set

$$\mathscr{N} := \left\{z \in H_0^1(\Omega) \times H_0^1(\Omega) \,\middle|\, z \neq 0 \,\text{and}\, \left\langle \Phi'(z), z \right\rangle = 0\right\}$$

need not be closed (since $\inf_{\mathscr{N}} \Phi$ can be 0), and Theorem 1 cannot be proved by using the usual method of the Nehari manifold (see [7], chapter 3 for a description and some applications of this method). To circumvent the difficulty posed by the strongly indefiniteness of Φ, we will use the method of the generalized Nehari manifold inspired by Pankov [4], and developed recently by Szulkin and Weth [7], which consists in a reduction into two steps.

We organize the paper in the following way: In section 2, the method of the generalized Nehari manifold is briefly presented while in section 3, the existence of a ground state solution is proved.

2. The method of the generalized Nehari manifold

Let X be a Hilbert space with norm $\|\cdot\|$, and an orthogonal decomposition $X = X^+ \oplus X^-$. We denote by S^+ the unit sphere in X^+; that is,

$$S^+ := \left\{u \in X^+ \,\middle|\, \|u\| = 1\right\}.$$

For $u = u^+ + u^- \in X$, where $u^\pm \in X^\pm$, we define

$$X(u) := \mathbb{R}u \oplus X^- \equiv \mathbb{R}u^+ \oplus X^- \text{ and } \widehat{X}(u) := \mathbb{R}^+ u \oplus X^- \equiv \mathbb{R}^+ u^+ \oplus X^-, \tag{2.1}$$

where $\mathbb{R}v := \{\lambda v; \lambda \in \mathbb{R}\}$ and $\mathbb{R}^+ v := \{\lambda v; \lambda \geqslant 0\}$ for $v \in X$.

Let Φ be a $\mathscr{C}^1-$functional defined on X by

$$\Phi(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u).$$

We consider the following situation:

(A_1) $I(0) = 0$, $\frac{1}{2}\left\langle I'(u), u\right\rangle > I(u) > 0$ for all $u \neq 0$, and I is weakly lower semicontinuous.

(A_2) For each $w \in X \backslash X^-$ there exists a unique nontrivial critical point $\widehat{m}(w)$ of $\Phi|_{\widehat{X}(w)}$. Moreover, $\widehat{m}(w)$ is the unique global maximum of $\Phi|_{\widehat{X}(w)}$.

(A_3) There exists $\delta > 0$ such that $\|\widehat{m}(w)^+\| \geqslant \delta$ for all $w \in X \backslash X^-$, and for each compact subset $\mathscr{K} \subset X \backslash X^-$ there exists a constant $C_{\mathscr{K}}$ such that $\|\widehat{m}(w)\| \leqslant C_{\mathscr{K}}$.

We consider the following set introduced by Pankov [4]:

$$\mathscr{M} := \left\{u \in X \backslash X^- : \left\langle \Phi'(u), u \right\rangle = 0 \text{ and } \left\langle \Phi'(u), v \right\rangle = 0 \; \forall v \in X^-\right\}.$$

Following Szulkin and Weth [7], we will call $\mathscr{M}$ the generalized Nehari manifold.

REMARK 1. By (A_1) $\mathscr{M}$ contains all nontrivial critical points of Φ and by (A_2) $\widehat{X}(w)\cap\mathscr{M}=\{\widehat{m}(w)\}$ whenever $w\in X\backslash X^-$.

In the following we consider the mappings:

$$\widehat{m}:X\backslash X^-\to\mathscr{M},\, w\mapsto\widehat{m}(w) \text{ and } m:=\widehat{m}|_{S^+}.$$

The following results are due to A. Szulkin and T. Weth $\big($[7], Chapter 4$\big)$. For reader's convenience we provide the proofs here.

PROPOSITION 1. *If* (A_1), (A_2) *and* (A_3) *are satisfied, then*

(a) $\widehat{m}$ *is continuous,*

(b) m *is a homeomorphism between* S^+ *and* $\mathscr{M}$.

Proof. (a) Let $(w_n)\subset X\backslash X^-$ such that $w_n\to w\notin X^-$. We want to show that $\widehat{m}(w_n)\to\widehat{m}(w)$. Since $\widehat{m}(w_n)=\widehat{m}(w_n^+/\|w_n^+\|)$, we may assume without loss of generality that $w_n\in S^+$. Therefore, it suffices to show that $\widehat{m}(w_n)\to\widehat{m}(w)$ after passing to a subsequence. Write $\widehat{m}(w_n)=s_nw_n+v_n$, with $s_n\geqslant 0$ and $v_n\in X^-$. By (A_3), the sequence $(\widehat{m}(w_n))$ is bounded. So taking a subsequence, we have $s_n\to s$ and $v_n\rightharpoonup v$. Setting $\widehat{m}(w)=sw+v$, it follows from (A_2) that

$$\Phi(\widehat{m}(w_n))\geqslant\Phi(s_nw_n+v)\to\Phi(sw+v)=\Phi(\widehat{m}(w))$$

and hence, using the weak lower semicontinuity of the norm and I,

$$\begin{aligned}\Phi(\widehat{m}(w))&\leqslant\lim_{n\to\infty}\Phi(\widehat{m}(w_n))=\lim_{n\to\infty}\Big(\frac{1}{2}s_n^2-\frac{1}{2}\|v_n\|^2-I(\widehat{m}(w_n))\Big)\\&\leqslant\frac{1}{2}s^2-\frac{1}{2}\|v\|^2-I(sw+v)\leqslant\Phi(\widehat{m}(w)).\end{aligned}$$

Hence the inequalities above must be equalities. It follows that (v_n) is strongly convergent, and so $v_n\to v$. Hence $\widehat{m}(w_n)=s_nw_n+v_n\to sw+v=\widehat{m}(w)$.

(b) It is easy to see that m is a bijection whose inverse m^{-1} is given by

$$m^{-1}(u)=\frac{u^+}{\|u^+\|},\quad\forall u\in\mathscr{M}.$$

Since m^{-1} is clearly continuous, we then deduce from (a) that m is a homeomorphism between S^+ and $\mathscr{M}$.

Let

$$\widehat{\Psi}:X^+\backslash\{0\}\to\mathbb{R},\,\widehat{\Psi}(w):=\Phi(\widehat{m}(w)) \text{ and } \Psi:=\widehat{\Psi}|_{S^+}.$$

PROPOSITION 2. *Under assumptions* (A_1), (A_2) *and* (A_3), $\widehat{\Psi}$ *is of class* $\mathscr{C}^1$ *and*

$$\big\langle\widehat{\Psi}'(w),z\big\rangle=\frac{\|\widehat{m}(w)^+\|}{\|w\|}\big\langle\widehat{\Phi}'(w),z\big\rangle,\text{ for all } w,z\in X^+,\ w\neq 0.$$

Proof. Let $w \in X^+ \setminus \{0\}$, $z \in X^+$ and put $\widehat{m}(w) = s_w w + v_w$, $v_w \in X^-$. Using the maximality property of $\widehat{m}(w)$ given by (A_2) and the mean value theorem, we obtain

$$\begin{aligned}\widehat{\Psi}(w+tz) - \widehat{\Psi}(w) &= \Phi(s_{w+tz}(w+tz) + v_{w+tz}) - \Phi(s_w w + v_w) \\ &\leqslant \Phi(s_{w+tz})(w+tz) + v_{w+tz}) - \Phi(s_{w+tz})w + v_{w+tz}) \\ &= \big\langle \Phi'(s_{w+tz}w + v_{w+tz} + \tau_t s_{w+tz}tz), s_{w+tz}tz \big\rangle,\end{aligned}$$

where $|t|$ is small enough and $\tau_t \in (0,1)$. Similarly,

$$\begin{aligned}\widehat{\Psi}(w+tz) - \widehat{\Psi}(w) &\geqslant \Phi(s_w(w+tz) + v_w) - \Phi(s_w w + v_w) \\ &= \big\langle \Phi'(s_w w + v_w + \eta_t s_w tz), s_w tz \big\rangle,\end{aligned}$$

where $\eta_t \in (0,1)$. Since the mappings $w \mapsto s_w$ and $w \mapsto v_w$ are continuous according to Proposition 1, we see by combining these two inequalities that

$$\begin{aligned}\big\langle \widehat{\Psi}'(w), z \big\rangle &= \lim_{t \to 0} \frac{\widehat{\Psi}(w+tz) - \widehat{\Psi}(w)}{t} \\ &= s_w \big\langle \Phi'(s_w w + v_w), z \big\rangle = \frac{\|\widehat{m}(w)^+\|}{\|w\|} \big\langle \Phi'(\widehat{m}(w)), z \big\rangle.\end{aligned}$$

Hence the Gâteaux derivative of $\widehat{\Psi}$ is bounded linear in z and continuous in w. It follows from Proposition 1.3 in [8] that $\widehat{\Psi}$ is of class $\mathscr{C}^1$.

Before giving a consequence of the previous propositions, which is the main result of this section, we recall some definitions.

DEFINITION 1. Let $\varphi \in \mathscr{C}^1(X, \mathbb{R})$.

1. A sequence $(u_n) \subset X$ is a Palais-Smale sequence (resp. a Palais-Smale sequence at level $c \in \mathbb{R}$) for φ if $(\varphi(u_n))$ is bounded (resp. $\varphi(u_n) \to c$) and $\varphi'(u_n) \to 0$ as $n \to \infty$.
2. We say that φ satisfies the Palais-Smale condition (resp. the Palais-Smale condition at level c) if every Palais-Smale sequence (resp. every Palais-Smale sequence at level c) has a convergent subsequence.

COROLLARY 1. *Assume that (A_1), (A_2) and (A_3) are satisfied. Then:*

(a) $\Psi \in \mathscr{C}^1(S^+, \mathbb{R})$ *and*

$$\big\langle \Psi'(w), z \big\rangle = \|m(w)^+\| \big\langle \Phi'(m(w)), z \big\rangle \text{ for all } z \in T_w(S),$$

where $T_w(S)$ is the tangent space of S at w.

(b) If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for Φ. If $(u_n) \subset \mathscr{M}$ is a bounded Palais-Smale sequence for Φ, then $(m^{-1}(w_n))$ is a Palais-Smale sequence for Ψ.

(c) w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of Φ. Moreover, the corresponding critical values coincide and $inf_{S^+}\Psi = inf_{\mathscr{M}}\Phi$.

Proof. (a) This is a direct consequence of Proposition 2, since $m(w) = \widehat{m}(w)$ for $w \in S^+$.

(b) Let $(w) \subset S^+$ and let $u = m(w) \in \mathscr{M}$. We have an orthogonal decomposition

$$X = X(w) \oplus T_w(S^+) = X(u) \oplus T_w(S^+).$$

Using (a) we have

$$\|\Psi'(w)\| = \sup_{\substack{z\in T_w(S^+)\\ \|z\|=1}} \langle \Psi'(w), z\rangle = \sup_{\substack{z\in T_w(S^+)\\ \|z\|=1}} \|u^+\|\langle \Phi'(m(w)), z\rangle = \|u^+\|\|\Phi'(u)\|, \quad (2.2)$$

where the last equality holds because $\langle \Phi'(u), v\rangle = 0$ for all $v \in X(w)$, and $T_w(S^+)$ is orthogonal to $X(u)$. By (A_3), there is $\delta > 0$ such that $\|u^+\| \geqslant \delta$. It is then easy to conclude.

(c) By (2.2), $\Psi'(w) = 0$ if and only if $\Phi'(m(w)) = 0$. The other part is clear.

3. Proof of the main result

Let $X := H_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the norm

$$\|(a,b)\| = \left(\|\nabla a\|^2_{L^2(\Omega)} + \|\nabla b\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}},$$

which by the Poincaré inequality is equivalent to its usual norm. Define

$$X^+ := H_0^1(\Omega) \times \{0\} \text{ and } X^- := \{0\} \times H_0^1(\Omega).$$

Then for $u = u^+ + u^- \in X$, we have

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u), \quad (3.1)$$

where $I(u) := \int_\Omega F(x,u)dx.$

We recall that for $u \in X$,

$$X(u) := \mathbb{R}u \oplus X^- \equiv \mathbb{R}u^+ \oplus X^- \text{ and } \widehat{X}(u) := \mathbb{R}^+u \oplus X^- \equiv \mathbb{R}^+u^+ \oplus X^-. \quad (3.2)$$

By a standard argument we have:

LEMMA 1. *Under (F_1)-(F_2), $\Phi \in \mathscr{C}^1(X,\mathbb{R})$ and*

$$\langle \Phi'(u), v\rangle = \int_\Omega \Big(\nabla u^+ \cdot \nabla v^+ - \nabla u^- \cdot \nabla v^- - v \cdot \nabla F(x,u)\Big). \quad (3.3)$$

Before giving the proof of the main theorem, we need some preliminary results.

LEMMA 2. *Assume* (F_1) *and* (F_5). *Then* (A_1) *is satisfied.*

Proof. Clearly by (F_1) and (F_5) we have $I(0)=0$ and

$$\frac{1}{2}\langle I'(u),u\rangle > I(u) > 0,\ \ \forall u \neq 0.$$

Let $(u_n)\subset X$ and $c\in\mathbb{R}$ such that $u_n \rightharpoonup u$ and $I(u_n)\leqslant c$. By Rellich-Kondrachov theorem $u_n\to u$ in $L^2(\Omega)\times L^2(\Omega)$, and taking a subsequence if necessary we have $u_n(x)\to u(x)$ a.e on Ω. Since F is continuous, we conclude by applying Fatou's Lemma that I is weakly lower semicontinuous.

LEMMA 3. *Under* (F_1), (F_3)-(F_8), (A_2) *is satisfied.*

Proof. **(1)** We first show that $\widehat{X}(w)\cap\mathscr{M}\neq\emptyset$ for any $w\in X\backslash X^-$. Let $w\in X\backslash X^-$. Then $\Phi\leqslant 0$ on $\widehat{X}(w)\backslash B_R$ for R large enough, where $B_R:=\{u\in X\mid \|u\|\leqslant R\}$. In fact, if this is not true then there exists a sequence $(u_n)\subset\widehat{X}(w)$ such that $\|u_n\|\to\infty$ and $\Phi(u_n)>0$. Up to a subsequence we have $v_n=u_n/\|u_n\|\rightharpoonup v$. By (3.1) we have

$$0<\frac{\Phi(u_n)}{\|u_n\|^2}=\frac{1}{2}\|v_n^+\|^2-\frac{1}{2}\|v_n^-\|^2-\int_\Omega\frac{F(x,\|u_n\|v_n)}{\left|v_n\|u_n\|\right|^2}|v_n|^2.$$

If $v\neq 0$ we deduce, by using Fatou's Lemma and (F_4), that $0\leqslant-\infty$; a contradiction. Consequently $v=0$. Since $\widehat{X}(w)=\widehat{X}(w^+/\|w^+\|)$, we may assume that $w\in S^+$. Now since $I(u_n)\geqslant 0$ and $1=\|v_n^+\|^2+\|v_n^-\|^2$, then necessarily $v_n^+=s_nw\nrightarrow 0$. Hence there is $r>0$ such that $\|v_n^+\|=\|s_nw\|>r\ \forall n$. So $\|v_n^+\|=s_n$ is bounded and bounded away from 0. But then, up to a subsequence, $v_n^+\to sw,\ s>0$, which contradicts the fact that $v_n\rightharpoonup 0$.

By (F_3), $\Phi(sw)=s^2/2+\circ(s^2)$ as $s\to 0$. Hence

$$0<\sup_{\widehat{X}(w)}\Phi<\infty.$$

Since Φ is weakly upper semicontinuous on $\widehat{X}(w)$ and $\Phi\leqslant 0$ on $\widehat{X}(w)\cap X^-$, the supremum is attained at some point u_0 such that $u_0^+\neq 0$. So u_0 is a nontrivial critical point of $\Phi|_{\widehat{X}(w)}$ and hence $u_0\in\mathscr{M}$.

(2) Now we show that if $u\in\mathscr{M}$, then u is the unique global maximum of $\Phi|_{\widehat{X}(u)}$. Let $u\in\mathscr{M}$ and $u+w\in\widehat{X}(u)$ with $w\neq 0$. By definition of $\widehat{X}(u)$ we have

$$u+w=(1+s)u+v,\ s\geqslant -1 \text{ and } v\in X^-.$$

By using the fact that

$$s(\frac{s}{2}+1)u+(1+s)v\in X(u)$$

we obtain

$$\Phi(u+w)-\Phi(u)=-\frac{1}{2}\|v\|^2+\int_\Omega\Big[\big(s(\frac{s}{2}+1)u+(1+s)v\big)\cdot\nabla F(x,u)+F(x,u)-F(x,u+w)\Big].$$

We define g on $[-1,\infty[$ by

$$g(s):=\big(s(\frac{s}{2}+1)u+(1+s)v\big)\cdot\nabla F(x,u)+F(x,u)-F(x,u+w).$$

Since $u\neq 0$, then in view of (F_5) we have $g(-1)<0$. On the other hand we deduce from (F_4) and (F_5) that $g(s)\to-\infty$ as $s\to\infty$. Assume that g attains its maximum at a point $s\in[-1,\infty[$, then

$$g'(s)=\big((1+s)u+v\big)\cdot\nabla F(x,u)-u\cdot\nabla F(x,(1+s)u+v)=0. \tag{3.4}$$

Setting $z=u+w=(1+s)u+v$, one can easily verify that

$$g(s)=-\big(\frac{s^2}{2}+s+1\big)u\cdot\nabla F(x,u)+(1+s)z\cdot\nabla F(x,u)+F(x,u)-F(x,z).$$

It is then clear that if $u\cdot z\leqslant 0$, then (F_6) implies $g(s)<0$. Suppose that $u\cdot z>0$, then in view of (3.4), (F_8) implies $|u|=|z|$ and by (F_7) we have

$$F(x,u)=F(x,z)\quad\text{and}\quad z\cdot\nabla F(x,u)<u\cdot\nabla F(x,u)$$

whenever $w\neq 0$. This implies that

$$g(s)<-\frac{s^2}{2}u\cdot\nabla F(x,u)\leqslant 0.$$

Hence $\Phi(u+w)<\Phi(u)$.

LEMMA 4. *Assume* (F_2)-(F_8). *Then* (A_3) *is satisfied.*

Proof. Clearly (F_3) implies $I'(u)=\circ(\|u\|)$ as $|u|\to 0$, which together with (A_1) imply that

$$\forall\varepsilon>0,\exists\alpha>0\,|\,\forall u\in X^+,|u|<\alpha\Rightarrow I(u)<\frac{1}{2}\langle I'(u),u\rangle\leqslant\|I'(u)\|\|u\|\leqslant\frac{\varepsilon}{2}\|u\|^2.$$

Hence we can find $\rho,\eta>0$ such that $\Phi(w)\geqslant\eta$ for any $w\in\{u\in X^+\,|\,\|u\|=\rho\}$. By (A_2), $\Phi(\widehat{m})\geqslant\eta$ for any $w\in X\backslash X^-$. Since $I\geqslant 0$, we deduce from (3.1) that $\|\widehat{m}(w)^+\|\geqslant\sqrt{2\eta}$ for any $w\in X\backslash X^-$.

Now let $\mathscr{K}$ be a compact subset of $X\backslash X^-$. We want to show that there exists a constant $C_{\mathscr{K}}$ such that $\|\widehat{m}(w)\|\leqslant C_{\mathscr{K}}$, $\forall w\in\mathscr{K}$. Since $\widehat{m}(w)=\widehat{m}(w^+/\|w^+\|)$ $\forall w\in X\backslash X^-$, we may assume that $\mathscr{K}\subset S^+$. Suppose by contradiction that there exists a sequence $(w_n)\subset\mathscr{K}$ such that $\|\widehat{m}(w_n)\|\to\infty$. Since $\widehat{m}(w_n)\in\widehat{X}(w_n)$, we have $\widehat{m}(w_n)=\lambda_n w_n+v_n$, with $\lambda_n\geqslant 0$ and $v_n\in X^-$. Since $\Phi(\widehat{m}(w_n))>0$, $\|w_n\|=1$

and $I \geqslant 0$, we deduce from (3.1) that $\lambda_n \geqslant \|v_n\|$. Hence $\lambda_n \to \infty$, which implies $|\lambda_n w_n + v_n| \to \infty$ as $n \to \infty$. By (3.1) we have

$$\begin{aligned} 0 < \frac{\Phi(\widehat{m}(w_n))}{\lambda_n^2} &= \frac{1}{2} - \frac{1}{2}\frac{\|v_n\|^2}{\lambda_n^2} - \int_\Omega \frac{F(x,\lambda_n w_n + v_n)}{\lambda_n^2} \\ &= \frac{1}{2} - \frac{1}{2}\frac{\|v_n\|^2}{\lambda_n^2} - \int_\Omega \frac{F(x,\lambda_n w_n + v_n)}{|\lambda_n w_n + v_n|^2}\frac{|\lambda_n w_n + v_n|^2}{\lambda_n^2} \\ &\leqslant \frac{1}{2} - \int_\Omega \frac{F(x,\lambda_n w_n + v_n)}{|\lambda_n w_n + v_n|^2}|w_n|^2. \end{aligned} \tag{$\star$}$$

Since $\mathscr{K}$ is compact we have, by taking a subsequence if necessary that $w_n \to w \in S^+$ and $w_n \to w$ a.e on Ω. Clearly $w \neq 0$. Then by using (F_4) and Fatou's Lemma, we deduce from $(\star)$ that $0 \leqslant -\infty$; a contradiction.

We need the following result:

LEMMA 5. *Let* $1 \leqslant q, r < \infty$ *and* $G \in \mathscr{C}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$ *such that*

$$|G(x,a,b)| \leqslant c\big(1 + |a|^{\frac{q}{r}} + |b|^{\frac{q}{r}}\big).$$

Then for all $a,b \in L^q(\Omega)$, $G(\cdot,a,b) \in L^r(\Omega)$ *and the operator* $A : L^q(\Omega) \times L^q(\Omega) \to L^r(\Omega)$, $(a,b) \mapsto G(x,a,b)$ *is continuous.*

The proof of Lemma 5 follows the lines of the proof of Theorem $A.2$ in [8] and is omitted here.

LEMMA 6. *Assume* (F_1)-(F_8). *Then* Φ *satisfies the Palais-Smale condition on* $\mathscr{M}$.

Proof. Let $(u_n) \subset \mathscr{M}$ be a sequence such that $\Phi(u_n) \leqslant d$ for some $d > 0$ and $\Phi'(u_n) \to 0$. We want to show that (u_n) has a convergent subsequence.

Let us first show that (u_n) is bounded.
If (u_n) is not bounded, then up to a subsequence we have $\|u_n\| \to \infty$. Define $v_n := u_n/\|u_n\|$. We easily deduce from (3.1) that

$$\begin{aligned} 0 < \frac{\Phi(u_n)}{\|u_n\|^2} &= \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \frac{I(u_n)}{\|u_n\|^2} \\ &= \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_\Omega |v_n|^2 \frac{F(x, v_n\|u_n\|)}{\big|v_n\|u_n\|\big|^2}. \end{aligned} \tag{$\star\star$}$$

Since (v_n) is bounded we have, by taking a subsequence if necessary, $v_n \rightharpoonup v$. If $v \neq 0$, then by using one more time (F_4) and Fatou's Lemma we obtain from $(\star\star)$ the contradiction $0 \leqslant -\infty$. Hence $v = 0$. Since $\Phi(u_n) > 0$ and $I(u_n) > 0$, (3.1) implies $\|v_n^+\| \geqslant \|v_n^-\|$. Hence we cannot have $v_n^+ \to 0$ (since $\|v_n\| = 1$). There then exists

$\alpha > 0$ such that, up to a subsequence, $\|v_n^+\| \geqslant \alpha$ $\forall n$. It is clear that $sv_n^+ \in \widehat{X}(u_n)$ $\forall s > 0$. Then by (A_2) we have

$$d \geqslant \Phi(u_n) \geqslant \Phi(sv_n^+) \geqslant \frac{1}{2}s^2\alpha^2 - I(sv_n^+),\ \forall s > 0.$$

Since $v_n^+ \rightharpoonup 0$, we deduce from the compactness of the embedding $X \hookrightarrow L^p(\Omega)\times L^p(\Omega)$ that $v_n^+ \to 0$ in $L^p(\Omega)\times L^p(\Omega)$. Now since by (F_2) F satisfies the conditions of Lemma 5 (with $q = p$ and $r = 1$), we deduce that $I(sv_n^+) \to 0$. It then follows that $d \geqslant \frac{1}{2}s^2\alpha^2$ $\forall s > 0$. This gives another contradiction if we take s big enough. Hence (u_n) is bounded.

By taking a subsequence if necessary we have $u_n \rightharpoonup u$ in X. It follows from the compactness of the embedding $X \hookrightarrow L^p(\Omega)\times L^p(\Omega)$ that $u_n \to u$ in $L^p(\Omega)\times L^p(\Omega)$. Now we easily obtain from (3.1) and (3.3):

$$\|u_n^\pm - u^\pm\|^2 = \pm\langle \Phi'(u_n) - \Phi'(u), u_n^\pm - u^\pm\rangle \pm \int_\Omega (u_n^\pm - u^\pm)\cdot\big(\nabla F(x,u_n) - \nabla F(x,u)\big).$$

Clearly $\langle \Phi'(u_n) - \Phi'(u), u_n^\pm - u^\pm\rangle \to 0$. By (F_2) the components of ∇F satisfy the conditions of Lemma 5 with $q = p-1$ and $r = \frac{p}{p-1}$, then by using the Hölder inequality and Lemma 5 we obtain

$$\int_\Omega (u_n^\pm - u^\pm)\cdot(\nabla F(x,u_n) - \nabla F(x,u)\big) \to 0.$$

Consequently $u_n \to u$.

We also need the following consequence of the Ekeland variational principle:

LEMMA 7. ([8], Corollary 2.5) *Let E be a Banach space and let $\varphi \in \mathscr{C}^1(E,\mathbb{R})$ be bounded below. If φ satisfies the Palais-Smale condition at level $\theta := \inf_E \varphi$, then there exists $x \in E$ such that $\varphi'(x) = 0$ and $\theta = \varphi(x)$.*

Proof. [Proof of Theorem 1] We already know from Lemmas 2, 3 and 4 that (A_1), (A_2) and (A_3) are satisfied. By Corollary 1-(a) $\Psi \in C^1(S^+,\mathbb{R})$.

Let us show that Ψ satisfies the Palais-Smale condition on S^+.
Let $(w_n) \subset S^+$ be a Palais-Smale sequence for Ψ. By Corollary 1-(b) $(m(w_n))$ is a Palais-Smale sequence for Φ on $\mathscr{M}$. By Lemma 6 we have $m(w_n) \to w$ up to a subsequence. Since m^{-1} is continuous, it follows that $w_n \to m^{-1}(w)$. Hence Ψ satisfies the Palais-Smale condition on S^+. Particularly Ψ satisfies the Palais-Smale condition at level $\theta = \inf_{S^+}\Psi$. By Corollary 1-(c) $\inf_{S^+}\Psi = \inf_{\mathscr{M}}\Phi > 0$ and Ψ is bounded below. By Lemma 7 $\inf_{S^+}\Psi$ is a critical value of Ψ. There then exists $u_0 \in S^+$ such that $\inf_{S^+}\Psi = \Psi(u_0)$ and $\Psi'(u_0) = 0$. It follows from Corollary 1-(c) that $m(u_0)$ is a critical point of Φ and $\Phi(m(u_0)) = \inf_{\mathscr{M}}\Phi$. Hence $m(u_0)$ is a ground state solution for the equation $\Phi'(u) = 0$.

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