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SINGLE POINT BLOW–UP SOLUTIONS TO THE HEAT EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study finite blow-up solutions of the heat equation with nonlinear boundary conditions. We provide a sufficient condition for the single point blow-up at the origin and a precise spacial singularity of the blow-up profile.

1. Introduction

We study positive solutions of the heat equation with nonlinear boundary conditions:

$$\begin{cases} \partial_t u = \Delta u, & (x,t) \in \mathbb{R}^n_+ \times (0,T), \\ \partial_\nu u = u^q, & (x,t) \in \partial\mathbb{R}^n_+ \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n_+, \end{cases} \tag{P}$$

where $u_0 \geqslant 0$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}$, $\partial_\nu = -\partial/\partial x_n$ and $1 < q < n/(n-2)$ if $n \geqslant 3$. We are concerned with finite time blow-up solutions and their asymptotic behavior. We call a solution $u(x,t)$ blow-up in a finite time, if there exists $T > 0$ such that

$$\limsup_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^n_+)} = \infty.$$

Moreover for a finite blow-up solution $u(x,t)$, if a limit

$$U(x) = \lim_{t\to T} u(x,t) \in [0,\infty]$$

exists for any $x \in \overline{\mathbb{R}^n_+}$, we call $U(x)$ a blow-up profile of $u(x,t)$. The blow-up profile of positive solutions of the semilinear heat equation:

$$\partial_t u = \Delta u + u^p \qquad \text{in } \mathbb{R}^n \times (0,T) \tag{F}$$

for $p \in (1,(n+2)/(n-2))$ is well studied ([2]–[4], [7], [11], [14]–[16], [20]–[23]). For a one dimensional case, the blow-up profile of solutions of (F) is completely classified by Herrero-Velázquez [14]–[16]. Let $n = 1$ and $u(x,t)$ be a positive solution of (F) which blows up at the origin. Then they proved that the blow-up profile of $u(x,t)$

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satisfies one of the following two cases:

(i) there exists $c_p > 0$ such that $\lim_{x \to 0} \left(\frac{x^2}{|\log x|} \right)^{1/(p-1)} U(x) = c_p$,

(ii) there exist $c > 0$ and an even number $m \geqslant 4$ such that $\lim_{x \to 0} x^{m/(p-1)} U(x) = c$.

The blow-up profile of solutions of (F) for a higher dimensional case was also studied ([23]), however the situation is different from that of one dimensional case since the blow-up set is not always isolated (see [10]).

Recently, the author in [13] studied the blow-up profile of solutions of (P). To study the asymptotic behavior of blow-up solutions, we introduce a rescaled function:

$$\varphi(y,s) = (T-t)^{1/(p-1)} u((T-t)^{1/2} x, t), \qquad s = -\log(T-t),$$

where $T > 0$ is a blow-up time. This rescaled function $\varphi(y,s)$ solves ($s_T = -\log(T - t)$)

$$\begin{cases} \partial_s \varphi = \Delta \varphi - \dfrac{y}{2} \cdot \nabla \varphi - \dfrac{\varphi}{2(q-1)}, & (y,s) \in \mathbb{R}^n_+ \times (s_T, \infty), \\ \partial_\nu \varphi = \varphi^q, & (y,s) \in \partial \mathbb{R}^n_+ \times (s_T, \infty). \end{cases}$$

Chlebík-Fila in [5], [6] (see also [19]) proved that $\varphi(y,s)$ is uniformly bounded on $\mathbb{R}^n_+ \times (s_T, \infty)$ and

$$\lim_{s \to \infty} \varphi(y,s) = \varphi_0(y_n)$$

uniformly any compact set on $\overline{\mathbb{R}^n_+}$, where $\varphi_0(y_n)$ is the unique positive bounded solution of

$$\begin{cases} \varphi_0'' + \dfrac{\xi}{2} \varphi_0' - \dfrac{\varphi_0}{2(q-1)} = 0, & \xi \in \mathbb{R}_+, \\ \partial_\nu \varphi_0 = \varphi_0^q, & \xi = 0. \end{cases}$$

From this fact, the original solution $u(x,t)$ asymptotically behaves like

$$u(x,t) \sim (T-t)^{1/2(q-1)} \varphi_0((T-t)^{-1/2} x_n), \qquad t \sim T.$$

By virtue of the asymptotic formula of $\varphi_0(\xi)$: $\varphi_0(\xi) \sim c_q \xi^{-1/(q-1)}$ (see p. 202 [9]), this asymptotic formula gives the spacial singularity of the blow-up profile $U(x)$, that is $U(x) \sim c_q x_n^{-1/(q-1)}$. However this formula of the blow-up profile has no meaning on the boundary $\partial \mathbb{R}^n_+$ since $x_n = 0$ on $\partial \mathbb{R}^n_+$. In order to investigate the blow-up profile on the boundary $\partial \mathbb{R}^n_+$, we need more precise asymptotic behavior of

$$v(y,s) = \varphi(y,s) - \varphi_0(y). \tag{1.1}$$

Then $v(y,s)$ satisfies

$$\begin{cases} \partial_s v = \Delta v - \dfrac{y}{2} \cdot \nabla v - \dfrac{v}{2(q-1)}, & (y,s) \in \mathbb{R}^n_+ \times (s_T, \infty), \\ \partial_\nu v = q \varphi_0^{q-1} v + O(v^2), & (y,s) \in \partial \mathbb{R}^n_+ \times (s_T, \infty). \end{cases}$$

A corresponding eigenvalue problem is given by

$$\begin{cases} -\left(\Delta e - \frac{y}{2}\cdot \nabla e - \frac{1}{2(q-1)} e\right) = \lambda e & \text{in } \mathbb{R}^n_+, \\ \partial_\nu e = q\varphi_0^{q-1} e & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.2}$$

Let $e_i(y) \in L^2_\rho(\mathbb{R}^n_+)$ be the i-th eigenfunction of (1.2), where $L^2_\rho(\mathbb{R}^n_+)$ is a weighted L^2-space defined by

$$L^2_\rho(\mathbb{R}^n_+) = \left\{ v \in L^2_{\mathrm{loc}}(\mathbb{R}^n_+);\ \|v\|^2_{L^2_\rho(\mathbb{R}^n_+)} := \int_{\mathbb{R}^n_+} |v(y)|^2 e^{-|y|^2/4} dy < \infty \right\}.$$

Then $v(y,s)$ behaves as one of the following two cases (see [13]):

(I) there exists $c \neq 0$ depending only on n,q such that $\|v(s) - cs^{-1}e_2\|_{L^2_\rho(\mathbb{R}^n_+)} = o(s^{-1})$,

(II) there exist $c > 0$ and $\gamma > 0$ such that $\|v(s)\|_{L^2_\rho(\mathbb{R}^n_+)} \leqslant ce^{-\gamma s}$.

Moreover we proved in [13] that if the case (I) occurs, then the blow-up profile for x_n-axial symmetric solutions with $x' \cdot \nabla' u_0 \leqslant 0$ is given by

$$U(x',0) \sim c\left(\frac{|\log|x'||}{|x'|^2}\right)^{1/2(q-1)} \qquad \text{on } \partial\mathbb{R}^n_+.$$

However we did not clarify when the case (I) occurs. Our goal of this paper is to provide sufficient conditions on the initial data for the case (I). To state our result, we define

$$BC^1(\overline{\mathbb{R}^n_+}) = \{u \in C^1(\overline{\mathbb{R}^n_+}); u, |\nabla u| \in L^\infty(\mathbb{R}^n_+)\}.$$

THEOREM 1. *Let $u_0(x) \in BC^1(\overline{\mathbb{R}^n_+})$ be x_n-axial symmetric and satisfy $x' \cdot \nabla' u_0 \leqslant 0$, $\partial_n u_0 \leqslant 0$. If a solution $u(x,t)$ blows up in a finite time, then $u(x,t)$ blows up only on the origin and behaves as* (I).

REMARK 1. As for the case (F), if the initial data is radially symmetric and monotone decreasing, then these properties are preserved for $t > 0$. Therefore the solution has a unique local maximum point at the origin for $t > 0$ and no local minimum points for $t > 0$. From the view point of this geometry of the solution, it is easily proved that the rescaled solution $\varphi(y,s)$ satisfies the asymptotic formula (i). However this kind of observation can not be applicable to solutions of (P), since solutions treated here are not radially symmetric but x_n-axial symmetric.

The rest of paper is organized as follows. In the Section 2, we show a single point blow-up of solutions of (P) by using methods given in [12]. In section 3, we provide a sufficient condition for the case (I). In particular, we mainly study the singularity of the blow-up profile.

Throughout this paper, for simplicity we set

$$m = 1/2(q-1), \qquad B = \varphi_0(0).$$

2. Single point blow-up

2.1. Blow-up profile along $x_n = |x|\cos\theta$

In this subsection, we provide a blow-up profile along $x_n = |x|\cos\theta$ for fixed $\theta \in [0,\pi/2)$. At the formal level, the blow-up profile $U(x)$ is given by $U(x) \sim c_q x_n^{-1/(q-1)}$ as is stated in Introduction. Here we provide a rigorous proof of this formula.

PROPOSITION 1. *Let $u(x,t)$ be a finite time blow-up solution at $x = x_0$. Then there exists $c_q > 0$ such that*

$$u(x+x_0,T) = c_q(1+o(1))(\cos\theta)^{-2m}|x|^{-2m}$$

along $x_n = |x|\cos\theta$ for any fixed $\theta \in [0,\pi/2)$.

Proof. Without loss of generality, we can assume that $x_0 = 0$. It is known (cf. p. 173 in [8]) that $u(x,t)$ is expressed by

$$\begin{aligned} u(x,t) = & \int_{\mathbb{R}^n_+} G_N(x,\xi,t)u_0(\xi)d\xi + \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G_N(x,\xi',t-\tau)u(\xi',0,\tau)^q d\xi' \\ =: & \, J_1 + J_2, \end{aligned}$$

where $G_N(x,\xi,t)$ is given by

$$G_N(x,\xi,t) = \frac{c}{t^{1/2}}\mathcal{G}(x',\xi',t)\left(e^{-(x_n-\xi_n)^2/4t} + e^{-(x_n+\xi_n)^2/4t}\right),$$

$$\mathcal{G}(x',\xi',t) = t^{-(n-1)/2}e^{-|x'-\xi'|^2/4t}.$$

By changing variables: $\xi' - x' = \sqrt{t-\tau}\cdot z'$, we see that

$$\begin{aligned} J_2 &= \int_0^t (t-\tau)^{-1/2}e^{-x_n^2/4(t-\tau)}d\tau \int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4}u(\sqrt{t-\tau}\cdot z' + x',0,\tau)^q dz' \\ &= \int_0^t \frac{e^{-x_n^2/4(t-\tau)}d\tau}{(t-\tau)^{1/2}(T-\tau)^{mq}} \int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4}\left\{(T-\tau)^m u(\sqrt{t-\tau}\cdot z' + x',0,\tau)\right\}^q dz'. \end{aligned}$$

Since $T > t$, we note that

$$\begin{aligned} (T-\tau)^{-mq}e^{-x_n^2/4(t-\tau)} &= (T-\tau)^{-mq}\left(\frac{x_n^2}{4(t-\tau)}\right)^{-mq}\left(\frac{x_n^2}{4(t-\tau)}\right)^{mq}e^{-x_n^2/4(t-\tau)} \\ &\leqslant c x_n^{-2mq}\left(\frac{t-\tau}{T-\tau}\right)^{mq}e^{-x_n^2/8(t-\tau)} \leqslant c x_n^{-2mq}e^{-x_n^2/8(t-\tau)}. \end{aligned}$$

Therefore, since $(T-t)^m\|u(t)\|_{L^\infty(\mathbb{R}^n_+)}$ is uniformly bounded for $t \in (0,T)$ (see Section 3 in [5]), by the Lebesgue dominant convergence lemma, we obtain for $x_n > 0$

$$\lim_{t\to T} J_2 = \int_0^T \frac{e^{-x_n^2/4(T-\tau)}d\tau}{(T-\tau)^{mq+1/2}} \int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4}\left\{(T-\tau)^m u(\sqrt{T-\tau}\cdot z' + x',0,\tau)\right\}^q dz'$$

$$= \int_0^T \frac{e^{-x_n^2/4(T-\tau)}d\tau}{(T-\tau)^{mq+1/2}} \int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4} \varphi\left(z' + \frac{x'}{\sqrt{T-\tau}}, 0, -\log(T-\tau)\right)^q dz',$$

where $\varphi(y,s) = (T-\tau)^m u(\sqrt{T-\tau}\cdot y, s)$ and $s = -\log(T-\tau)$. We divide this integration into three parts:

$$\begin{aligned}\lim_{t\to T} J_2 = & \left(\int_0^{T-R^2|x'|^2} + \int_{T-R^2|x'|^2}^{T-R^{-2}|x'|^2} + \int_{T-R^{-2}|x'|^2}^{T}\right) \frac{e^{-x_n^2/4(T-\tau)}d\tau}{(T-\tau)^{mq+1/2}} \\ & \times \int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4} \varphi\left(z' + \frac{x'}{\sqrt{T-\tau}}, 0, -\log(T-\tau)\right)^q dz' \\ =: & K_1 + K_2 + K_3.\end{aligned}$$

Now we fix $\theta \in [0, \pi/2)$ and

$$x_n = |x|\cos\theta, \qquad |x'| = |x|\sin\theta.$$

Since $\varphi(y,s)$ is uniformly bounded on $\mathbb{R}^n_+ \times (s_T, \infty)$, by $mq - 1/2 = m$, we verify that

$$K_1(\theta) \leqslant c\int_0^{T-R^2x_n^2} (T-\tau)^{-(mq+1/2)} d\tau \leqslant cR^{-2m}|x_n|^{-2m} = cR^{-2m}(\cos\theta)^{-2m}|x|^{-2m}.$$

Since $g(\xi) = \xi e^{-\xi} \leqslant ce^{-\xi/2}$ for $\xi \in \mathbb{R}_+$, K_3 is estimated by

$$\begin{aligned}K_3(\theta) &\leqslant c\int_{T-R^{-2}x_n^2}^{T} \frac{e^{-x_n^2/4(T-\tau)}}{(T-\tau)^{mq+1/2}} d\tau \leqslant cx_n^{-2(mq+1/2)} \int_{T-R^{-2}|x'|^2}^{T} e^{-x_n^2/8(T-\tau)} d\tau \\ &\leqslant cR^{-2}x_n^{-2m} = cR^{-2}(\cos\theta)^{-2m}|x|^{-2m}.\end{aligned}$$

Finally we calculate K_2. By changing variables: $T-\tau = x_n^2/\mu$, we see that

$$\begin{aligned}K_2(\theta) = & \; x_n^{-2m} \int_{R^{-2}}^{R^2} \mu^{mq-3/2} e^{-\mu/4} d\mu \\ & \times \int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4} \varphi\left(z' + \sqrt{\mu}\left(\frac{x'}{|x'|}\right)\sin\theta, 0, -\log\left(\frac{x_n^2}{\mu}\right)\right)^q dz' \\ =: & \; A(R, x_n) x_n^{-2m} = A(R, x_n)(\cos\theta)^{-2m}|x|^{-2m}.\end{aligned}$$

Then, since $\varphi(y', 0, s) \to \varphi_0(0) = B$ in $L^2_\rho(\partial\mathbb{R}^n_+)$, it holds that

$$\lim_{x_n\to 0} A(R, x_n) = B^q \left(\int_{R^{-2}}^{R^2} \mu^{mq-3/2} e^{-\mu/4} d\mu\right) \left(\int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4} dz'\right).$$

Moreover, since $mq - 3/2 = -1 + m$, we see that

$$\lim_{R\to\infty}\left(\lim_{x_n\to 0} A(R, x_n)\right) = B^q \left(\int_0^\infty \mu^{mq-3/2} e^{-\mu/4} d\mu\right) \left(\int_{\mathbb{R}^{n-1}} e^{-|z'|^2/4} dz'\right) =: A_\infty.$$

Therefor we obtain

$$\lim_{x_n\to 0}|x|^{2m}J_2=\lim_{x_n\to 0}|x|^{2m}(K_1(\theta)+K_2(\theta)+K_3(\theta))=A_\infty(\cos\theta)^{-2m}$$

for any fixed $\theta\in[0,\pi/2)$. As a consequence, it follows that

$$u(x,T)=A_\infty(1+o(1))(\cos\theta)^{-2m}|x|^{-2m}$$

along $x_n=|x|\cos\theta$ for any fixed $\theta\in[0,\pi/2)$, which completes the proof. □

2.2. Single point blow-up

In this subsection, we provide a sufficient condition on initial data for a single point blow-up. For the case where Ω is the unit ball and $n=2$, a single point blow-up was proved in [18], however our argument is different from that of [18]. Let $u(x,t)$ be a x_n-axial symmetric function. Then $u(x,t)$ is written by

$$u(x,t)=U(r,z,t)\qquad(r=|x'|,\ z=x_n).$$

For simplicity of notations, we set

$$J=(0,\infty)\times(0,\infty),\qquad I=(0,\infty)\times\{0\},\qquad J_r=(0,r)\times(0,r).$$

Equation (P) is rewritten in the coordinate (r,z,t) by

$$\begin{cases}U_t=U_{rr}+\dfrac{n-2}{r}U_r+U_{zz}, & (r,z)\in J,\ t\in(0,T),\\ \partial_\nu U=U^q, & (r,z)\in I,\ t\in(0,T),\\ U(r,z,0)=U_0(r,z):=u_0(x), & (r,z)\in J.\end{cases}\tag{R}$$

In this subsection, we impose the following monotonicity conditions on initial data:

$$\partial_z U_0\leqslant 0,\qquad \partial_r U_0\leqslant 0.\tag{2.1}$$

To apply a technique given in [12], we prepare several lemmas.

LEMMA 1. *Let $u_0(x)\in BC^1(\overline{\mathbb{R}^n_+})$ be x_n-axial symmetric and satisfy $\partial_r U_0\leqslant 0$. Then it follows that $\partial_r U\leqslant 0$ for $t\in(0,T)$.*

Proof. Differentiating (R) with respect to r and set $\chi(r,z,t)=\partial_r U(r,z,t)$, then we obtain

$$\begin{cases}\chi_t=\chi_{rr}+\dfrac{n-2}{r}\chi_r+\chi_{zz}-\dfrac{n-2}{r^2}\chi, & (r,z)\in J,\ t\in(0,T),\\ \partial_\nu\chi=qU^{q-1}\chi, & (r,z)\in I,\ t\in(0,T).\end{cases}$$

Now we set $\bar{\chi}(r,z,t) = e^{-\sqrt{1+r^2+z^2}}\chi(r,z,t)$. Then it is verified that

$$\begin{cases} \bar{\chi}_t = \bar{\chi}_{rr} + \dfrac{n-2}{r}\bar{\chi}_r + \bar{\chi}_{zz} - \dfrac{n-2}{r^2}\bar{\chi} + G, & (r,z) \in J,\ t \in (0,T), \\ \partial_\nu \bar{\chi} = qU^{q-1}\bar{\chi}, & (r,z) \in I,\ t \in (0,T). \end{cases}$$

where G is given by

$$G = a_1(r,z)\bar{\chi} + a_2(r,z)\bar{\chi}_r + a_3(r,z)\bar{\chi}_z$$

with some bounded coefficients $a_i(r,z)$ $(i = 1,2,3)$. Since $u_0 \in BC^1(\overline{\mathbb{R}^n_+})$, we note that $\bar{\chi} \in C([0,T);H)$, where

$$H = \{U \in L^1_{\mathrm{loc}}(J);\ \|U\|_H < \infty\} \quad \text{with} \quad \|U\|_H^2 = \int_0^\infty dz \int_0^\infty U(r,z)^2 r^{n-2} dr.$$

Then, since

$$\bar{\chi}(r,z,0) = e^{-\sqrt{1+r^2+z^2}}\partial_r U_0(r,z) \leqslant 0 \ \text{ for } (r,z) \in J,$$

a comparison argument shows that $\bar{\chi} \leqslant 0$ in $J \times (0,T)$, which completes the proof. □

LEMMA 2. *Let u_0 be as in Lemma 1. Then for any $t_0 > 0$ there exists $\varepsilon_0 > 0$ such that*

$$-U_r(r,z,t) \geqslant \varepsilon_0 r, \qquad (r,z) \in J_1,\ t \in (t_0,T).$$

Proof. We set $v(r,z,t) = -U_r(r,z,t)/r$. Then $v(r,z,t)$ satisfies

$$\begin{cases} v_t = v_{rr} + \dfrac{n}{r}v_r + v_{zz}, & (r,z) \in J,\ t \in (0,T), \\ \partial_\nu v = qu^{q-1}v, & (r,z) \in I,\ t \in (0,T). \end{cases}$$

Since $U_{rr}(r,z,t)$ is uniformly bounded on $J \times (\delta, T-\delta)$ for any $\delta \in (0,T)$, $v(r,z,t)$ is uniformly bounded on $J \times (\delta, T-\delta)$. Hence a parabolic regularity theory shows that $v(r,z,t)$ is a classical solution. Since $v \geqslant 0$, by a strong maximum principle, there exists $\varepsilon_0 > 0$ such that

$$v(r,z,t_0/2) \geqslant \varepsilon_0 \qquad \text{in } J_2.$$

Let $\tilde{v}(r,z,t)$ be the solution of

$$\begin{cases} v_t = v_{rr} + \dfrac{n}{r}v_r + v_{zz}, & (r,z) \in J_2,\ t \in (0,\infty), \\ \partial_\nu v = 0, & (r,z) \in (0,2) \times \{0\},\ t \in (0,\infty), \\ v = 0, & (r,z) \in \{2\} \times (0,2) \cup (0,2) \times \{2\},\ t \in (0,\infty), \\ v(r,z,0) = \varepsilon_0 \chi_{J_1}, & (r,z) \in J_2, \end{cases}$$

where $\chi_{J_1}(r,z) = 1$ if $(r,z) \in J_1$ and $\chi_{J_1} = 0$ if $(r,z) \notin J_1$. From a strong maximum principle, there exists $\varepsilon_1 > 0$ such that

$$\tilde{v}(r,z,t) \geqslant \varepsilon_1, \qquad (r,z) \in J_1,\ t \in (t_0/2, T).$$

Then by a comparison argument, we see that

$$v(r,z,t) \geqslant \tilde{v}(r,z,t-t_0/2), \qquad (r,z) \in J_2,\ t \in (t_0/2,T).$$

Therefore the proof is completed. $\square$

Here for simplicity, we set

$$B_R = \{x \in \mathbb{R}^n_+; |x| < R\}, \qquad D_R = \{x \in \partial\mathbb{R}^n_+; |x| < R\}, \qquad S_R = \{x \in \mathbb{R}^n_+; |x| = R\}.$$

LEMMA 3. *Let u_0 be x_n-axial symmetric and satisfy (2.1). Then for $t_1 \in (0,T)$ there exist $c_1, c_2 > 0$ such that*

$$-\partial_n u(x,t) \geqslant c_1 u(x,t)^q, \qquad (x,t) \in B_{1/2} \times (t_1,T).$$

Moreover it holds that

$$u(x,t) \leqslant c_2 x_n^{-1/(q-1)}, \qquad (x,t) \in B_{1/2} \times (t_1,T).$$

Proof. Let $\varphi_1(x) > 0$ and $\mu_1 > 0$ be the first eigenfunction with $\|\phi_1\|_{L^\infty(B_1)} = 1$ and the first eigenvalue of

$$-\Delta\phi = \mu\phi \quad \text{in } B_1, \qquad \partial_\nu\phi = 0 \quad \text{on } D_1, \qquad \phi = 0 \quad \text{on } S_1.$$

We set

$$w_a(x,t) = -a\,\partial_n u(x,t) \ \ (a > 0), \qquad g(x,t) = e^{-\mu_1 t}\phi_1(x)u(x,t)^q.$$

Then it is verified that

$$\partial_t w_a - \Delta w_a = 0$$

and

$$\begin{aligned}\partial_t g - \Delta g = &\left(-\mu_1\phi_1 u^q + qu^{q-1}\phi_1 u_t\right)e^{-\mu_1 t}\\ &\qquad - \left((\Delta\phi_1)u^q + 2\nabla\phi_1\cdot\nabla u^q + \phi_1(\Delta u^q)\right)e^{-\mu_1 t}\\ = &-\left(2\nabla\phi_1\cdot\nabla u^q + q(q-1)\phi_1 u^{q-2}|\nabla u|^2\right)e^{-\mu_1 t}.\end{aligned}$$

A standard comparison argument with (2.1) implies that $\partial_n u \leqslant 0$. Moreover, by Lemma 1, we note that $U_r \leqslant 0$. Therefore we obtain

$$\partial_t g - \Delta g \leqslant 0.$$

By a boundary condition, we note that $w_a = au^q$ on $D_1 \times (0,T)$. Hence it is verified that for $a \geqslant 1$

$$g \leqslant u^q \leqslant au^q = w_a, \qquad (x,t) \in D_1 \times (0,T).$$

Since $g = 0$ on $S_1 \times (0,T)$, it is clear that

$$g \leqslant w_a, \qquad (x,t) \in S_1 \times (0,T).$$

Moreover by a strong maximum principle, we see that $w_a(x,t_1) > \varepsilon_0$ in B_1 for some $\varepsilon_0 > 0$. Hence we take $a > 1$ large enough, then we get

$$g(x,t_1) \leqslant w_a(x,t_1), \qquad x \in B_1.$$

Therefore, by a comparison argument, we obtain $g(x,t) \leqslant w_a(x,t)$ in $B_1 \times (t_1,T)$, which implies that $-\partial_n u(x,t) \geqslant cu(x,t)^q$ for $(x,t) \in B_{1/2} \times (t_1,T)$. Finally integrating both sides, we obtain $u(x,t) \leqslant cx_n^{-1/(q-1)}$ for $(x,t) \in B_{1/2} \times (t_1,T)$, which completes the proof. □

PROPOSITION 2. *Let u_0 be as in Lemma 3. If the solution $u(x,t)$ blows up in a finite time, then the solution blows up only on the origin.*

Proof. Suppose that the blow-up set is larger than $\mathscr{B} := \{(r,0) \in \bar{J}; 0 \leqslant r \leqslant r_0\}$ for some $r_0 > 0$. By using the same idea as in the proof of Theorem 2.4 in [12], we compare $-U_r$ and U^q. We set

$$V_a(r,z,t) = -aU_r(r,z,t), \qquad W(r,z,t) = d(r)U(r,z,t)^q,$$

where $d(r) > 0$ is chosen later. Then $V_a(r,z,t)$ is a solution of

$$\begin{cases} \partial_t V_a = \partial_r^2 V_a + \dfrac{n-2}{r}\partial_r V_a + \partial_z^2 V_a - \dfrac{n-2}{r^2} V_a, & (r,z) \in J,\ t \in (0,T), \\ \partial_\nu V_a = qU^{q-1} V_a, & (r,z) \in I,\ t \in (0,T). \end{cases}$$

Moreover we find that

$$\begin{aligned} LW &:= \partial_t W - \partial_r^2 W - \frac{n-2}{r}\partial_r W - \partial_z^2 W + \frac{n-2}{r^2} W \\ &= -\left(d'' + \frac{n-2}{r}d' - \frac{n-2}{r^2}d\right) U^q - qd'U_rU^{q-1} - q(q-1)d\left(U_r^2 + U_z^2\right)U^{q-2}. \end{aligned}$$

From Lemma 3, there exists $c_0 > 0$ such that

$$q(q-1)U_z^2 U^{q-2} \geqslant c_0 U^{3q-2}, \qquad (r,z) \in J_{1/2},\ t \in (t_1,T).$$

We set

$$\mathscr{U}_a = \{(r,z,t) \in J \times (0,T); V_a(r,z,t) < W(r,z,t)\}.$$

Then we see that for $(r,z,t) \in \mathscr{U}_a$

$$-U_rU^{q-1} = \left(\frac{V_a}{a}\right) U^{q-1} < \left(\frac{W}{a}\right) U^{q-1} = \left(\frac{d}{a}\right) U^{2q-1}.$$

Hence we obtain for $(r,z,t) \in \mathscr{U}_a$, $(r,z) \in J_{1/2}$ and $t \in (t_1,T)$

$$LW \leqslant -\left(d'' + \frac{n-2}{r}d' - \frac{n-2}{r^2}d\right) U^q + \frac{q}{a}|d'|dU^{2q-1} - c_0 dU^{3q-2}.$$

Let $r_1 = \min\{r_0/2, 1/2\}$ and $d(r)$, μ be the first eigenfunction and the first eigenvalue of

$$\begin{cases} -\left(d'' + \dfrac{n-2}{r}d'\right) + \dfrac{n-2}{r^2}d = \mu d, & r \in (0,r_1), \\ d = 0, & r \in \{0,r_1\}. \end{cases}$$

Since the proof for the case $n = 2$ is easier than that of the case $n \geqslant 3$, here we give a proof only for the case $n \geqslant 3$. Then $d(r)$ and μ are explicitly expressed by

$$d(r) = (\sqrt{\mu}r)^{-(n-3)/2} J_{(n-1)/2}(\sqrt{\mu}r), \qquad \mu = Z_1^2/r_1^2,$$

where $J_\nu(r)$ is the ν-th Bessel function and $Z_1 > 0$ is the first zero of $J_{(n-1)/2}(r)$. Then we obtain for $(r,z,t) \in \mathscr{U}_a$ and $a \geqslant 1$

$$LW \leqslant \left(\mu_1 U^{-2(q-1)} + q|d'|U^{-(q-1)} - c_0\right) dU^{3q-2}, \quad r \in (0,r_1),\ z \in (0,1/2). \tag{2.2}$$

Since $(r_1,0) \in \mathscr{B}$ is a blow-up point, from Proposition 1, it follows that

$$U(r_1,z,T) = c_q(1+o(1))z^{-2m}.$$

Since $J_\nu(r) = cr^\nu + o(r^\nu)$ and $\partial_r J_\nu(r) = c\nu r^{\nu-1} + o(r^{\nu-1})$, there exist $c_1, c_2 > 0$ such that

$$d(r) = c_1 r + o(r), \qquad d'(r) = c_2 + o(1). \tag{2.3}$$

Hence there exists $z_1 > 0$ such that

$$\mu_1 U(r_1,z_1,T)^{-2(q-1)} + q|d'|_\infty U(r_1,z_1,T)^{-(q-1)} \leqslant c_0/4.$$

From Lemma 3, we note that $(r_1,z_1) \in J$ is not a blow-up point. Hence by a parabolic regularity theory, $u(r_1,z_1,t)$ is continuous on $(0,T]$. As a consequence, there exists $t_1 \in (0,T)$ such that

$$\mu_1 U(r_1,z_1,t)^{-2(q-1)} + q|d'|_\infty U(r_1,z_1,t)^{-(q-1)} \leqslant c_0/2, \quad t \in (t_1,T).$$

Since $U_z, U_r \leqslant 0$, it follows that

$$\mu_1 U(r,z,t)^{-2(q-1)} + q|d'|_\infty U(r,z,t)^{-(q-1)} \leqslant c_0/2,\ \ r(0,r_1),\ z \in (0,z_1),\ t \in (t_1,T).$$

This implies that for $(r,z,t) \in \mathscr{U}_a$ and $a \geqslant 1$

$$LW \leqslant 0, \qquad r \in (0,r_1),\ z \in (0,z_1),\ t \in (t_1,T).$$

Now we set

$$J_1' = (0,r_1) \times (0,z_1), \qquad Q_1 = J_1' \times (t_1,T).$$

By (2.3), we note that $W(r,z,t) \sim rU(r,z,t)^q$. Therefore, from Lemma 2, there exists $a_1 > 1$ such that

$$V_{a_1}(r,z,t_1) \geqslant W(r,z,t_1), \qquad (r,z) \in J_1'. \tag{2.4}$$

Moreover, by Proposition 2, we find that $V_{a_1}(r,z_1,t)$ and $W(r,z_1,t)$ are continuous functions on $\{(r,t); r\in[0,r_1], t\in[0,T]\}$. Therefore there exists $a_2 \geqslant a_1$ such that

$$V_{a_2}(r,z_1,t) \geqslant W(r,z_1,t), \qquad r\in(0,r_1),\ t\in(t_1,T). \tag{2.5}$$

Furthermore, since $W(r_1,z,t)=0$, it is clear that

$$V_{a_2}(r_1,z,t) \geqslant W(r_1,z,t), \qquad z\in(0,z_1),\ t\in(t_1,T). \tag{2.6}$$

Since $L(W-V_{a_2})\leqslant 0$ in $Q_1\cap\mathscr{U}_{a_2}$, multiplying both sides by $(W-V_{a_2})_+ r^{n-2}$ and integrating over J_1', we obtain by (2.5) and (2.6)

$$\partial_t\|(W-V_{a_2})_+\|^2_{L^2(J_1')} \leqslant c\|(W-V_{a_2})_+\|^2_{L^2(J_1')}.$$

Hence applying the Gronwall inequality with (2.4), we obtain $(W-V_{a_2})_+\equiv 0$ in Q_1, which implies that

$$d(r)U(r,z,t)^q \leqslant -a_2U_r(r,z,t) \qquad \text{in } Q_1.$$

By (2.3), we obtain

$$U(r,0,t)\leqslant cr^{-2/(q-1)} = cr^{-4m}, \qquad r\in(0,r_1),\ t\in(t_1,T),$$

which contradicts definition of $\mathscr{B}$. Hence a single point blow-up is assured. □

3. Sufficient condition for the case (I)

This section is a main part of this paper. Before going to the proof of Theorem 1, we recall several facts studied in [13]. Let $u(x,t)$ be a finite time blow-up solution of (P) and $v(y,s)$ be defined by (1.1). Then $v(y,s)$ satisfies

$$\begin{cases} \partial_s v = \Delta v - \dfrac{y}{2}\cdot\nabla v - mv, & (y,s)\in\mathbb{R}^n_+\times(s_T,\infty),\\ \partial_\nu v = qB^{q-1}v + f(v), & (y,s)\in\partial\mathbb{R}^n_+\times(s_T,\infty), \end{cases} \tag{3.1}$$

where we recall that $B=\varphi_0(0)$ and $f(v)$ is given by

$$f(v) = (v+\varphi_0)^q - \varphi_0^q - q\varphi_0^{q-1}v.$$

We define wight functions:

$$\rho(y) = e^{-|y|^2/4}, \qquad \bar\rho(y') = e^{-|y'|^2/4} \qquad \text{for } y=(y',y_n).$$

From this definition, it is clear that $\rho|_{\partial\mathbb{R}^n_+} = \bar\rho$. Moreover we define functional spaces:

$$L^p_\rho(\mathbb{R}^n_+) = \left\{ v\in L^p_{\mathrm{loc}}(\mathbb{R}^n_+);\ \int_{\mathbb{R}^n_+}|v(y)|^p\rho(y)dy<\infty \right\},$$

$$H^k_\rho(\mathbb{R}^n_+) = \left\{ v\in L^2_\rho(\mathbb{R}^n_+); D^\alpha v\in L^2_\rho(\mathbb{R}^n_+) \text{ for any } \alpha=(\alpha_1,\cdots,\alpha_n) \text{ satisfying } |\alpha|\leqslant k \right\},$$

$$L^p_\rho(\partial\mathbb{R}^n_+) = \left\{ v\in L^p_{\mathrm{loc}}(\partial\mathbb{R}^n_+);\ \int_{\partial\mathbb{R}^n_+}|v(y')|^p\bar\rho(y')dy'<\infty \right\}.$$

The norm is given by

$$\|v\|^p_{L^p_\rho(\mathbb{R}^n_+)} = \int_{\mathbb{R}^n_+} |v(y)|^p \rho(y)dy, \qquad \|v\|^2_{H^k_\rho(\mathbb{R}^n_+)} = \sum_{|\alpha|\leqslant k} \|D^\alpha v\|^2_{L^2_\rho(\mathbb{R}^n_+)},$$
$$\|v\|^p_{L^p_\rho(\partial\mathbb{R}^n_+)} = \int_{\partial\mathbb{R}^n_+} |v(y')|^p \bar{\rho}(y')dy'$$

and the inner product on $L^2_\rho(\mathbb{R}^n_+)$ is naturally defined by

$$(v_1, v_2)_\rho = \int_{\mathbb{R}^n_+} v_1(y)v_2(y)\rho(y)dy.$$

For simplicity, the norm of $L^2_\rho(\mathbb{R}^n_+)$ is denoted by $\|\cdot\|_\rho = \|\cdot\|_{L^2_\rho(\mathbb{R}^n_+)}$. Moreover we define a functional space whose element is y_n-axial symmetric function.

$$L^2_{\mathrm{sym},\rho}(\mathbb{R}^n_+) = \{v \in L^2_\rho(\mathbb{R}^n_+); v(y) = v(|y'|, y_n)\}.$$

To study the asymptotic behavior of $v(y,s)$, we introduce a linear operator A related to (3.1).

$$Av = \Delta v - \frac{y}{2}\cdot\nabla v - mv,$$
$$D(A) = \{v \in H^2(\mathbb{R}^n_+); \partial_\nu v = qB^{q-1}v \text{ on } \partial\mathbb{R}^n_+\}.$$

Since the operator $A\colon D(A) \to L^2_\rho(\mathbb{R}^n_+)$ is self-adjoint and has a compact inverse from $L^2_\rho(\mathbb{R}^n_+)$ to $L^2_\rho(\mathbb{R}^n_+)$ (see Appendix [13]), $L^2_{\mathrm{sym},\rho}(\mathbb{R}^n_+)$ is spanned by y_n-axial symmetric eigenfunctions of

$$\begin{cases} -\left(\Delta - \frac{y}{2}\cdot\nabla - m\right)e = \lambda e & \text{in } \mathbb{R}^n_+, \\ \partial_\nu e = qB^{q-1}e & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{3.2}$$

Let $K_i(r)$ and σ_i be the i-th eigenfunction with

$$\int_{\mathbb{R}^{n-1}} K_i(\xi')^2 e^{-|\xi'|^2/4}d\xi' = 1$$

and the i-th eigenvalue of

$$-\left(K'' + \frac{n-2}{r}K' - \frac{r}{2}K'\right) = \sigma K, \quad r \in (0,\infty).$$

Then it is known that $\sigma_i = i-1$ and $K_i(r)$ is the $2(i-1)$-th polynomial. Let $I_j(\xi)$ and κ_j be the j-th eigenfunction with

$$\int_0^\infty I_j(\xi)e^{-\xi^2/4}d\xi = 1$$

and the j-th eigenvalue of

$$-\left(I'' - \frac{\xi}{2}I'\right) = \kappa I \quad \text{in } \mathbb{R}_+, \qquad \partial_\nu I = qB^{q-1}I \quad \text{on } \{0\}.$$

It is known that $\kappa_1 = -(m+1)$ and $\kappa_2 > 0$ (see Lemma 2.2 [13]). Then eigenfunctions and their eigenvalues of (3.2) are completely given by

$$e_{ij}(|y'|,y_n) = K_i(|y'|)I_j(y_n), \qquad \lambda_{ij} = (i-1)+\kappa_j+m \qquad (i,j\in\mathbb{N}). \tag{3.3}$$

3.1. Dynamical system

As is stated in Introduction, the author studied the asymptotic behavior of $v(y,s)$ and obtained the following result in [13].

PROPOSITION 3. (Proposition 3.1, Proposition 3.3 [13]) *Let u_0 be x_n-axial symmetric. Then $v(y,s)$ satisfies one of two cases*:

(I) *there exists $c_0 > 0$ such that $\|v(s)+c_0 s^{-1}e_{21}\|_\rho = o(s^{-1})$,*

(II) *$\|v(s)\|_\rho$ decays to zero exponentially.*

In this subsection we consider the case (II) and derive a precise decay rate of $\|v(s)\|_\rho$.

PROPOSITION 4. *Let u_0 be x_n-axial symmetric and satisfy $x'\cdot\nabla' u_0 \leqslant 0$. If the case (II) in Proposition 3 occurs, then one of two cases holds.*

(i) *$\|v(s)\|_\rho \leqslant c_\gamma e^{-\gamma s}$ for any $\gamma > 0$,*

(ii) *there exist $(i_1,j_1),(i_2,j_2) \in \mathbb{N}^2\setminus\{(1,1),(2,1)\}$ such that*

$$v(s) = \sum_{\lambda_{kl}<\lambda_{i_2 j_2}} a_{kl}(s)e_{kl} + h(s) \qquad \text{in } L^2_\rho(\mathbb{R}^n_+) \tag{3.4}$$

where

$$|a_{kl}(s)-\alpha_{kl}e^{-\lambda_{kl}}| \leqslant c_{kl,\varepsilon}e^{-2(\lambda_{i_1 j_1}-\varepsilon)s}, \qquad \|h(s)\|_\rho \leqslant c_\varepsilon e^{-2(\lambda_{i_1 j_1}-\varepsilon)s}$$

with $\alpha_{i_1 j_1} \neq 0$ and $\alpha_{kl} = 0$ if $\lambda_{kl} < \lambda_{i_1 j_1}$. Moreover expansion (3.4) holds in $C^2_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$.

Proof. We assume the case (II) in Proposition 3. Then there exists $\gamma > 0$ such that

$$\|v(s)\|_\rho \leqslant ce^{-\gamma s}. \tag{3.5}$$

We set

$$v_{ij}(y,s) = a_{ij}(s)e_{ij}(y), \qquad a_{ij}(s) = (v(s),e_{ij})_\rho$$

and

$$h_{ij}(y,s) = v(y,s) - \sum_{\lambda_{kl}<\lambda_{ij}} v_{kl}(y,s).$$

Form (3.1), we verify that

$$\begin{gathered}\dot a_{ij} = -\lambda_{ij}a_{ij} + \int_{\partial\mathbb{R}^n_+} f(v)e_{ij}\bar\rho\, dy', \\ \frac{1}{2}\partial_s\|h_{ij}\|^2_\rho = -m\|h_{ij}\|^2_\rho - \|\nabla h_{ij}\|^2_\rho + qB^{q-1}\|h_{ij}\|^2_{L^2_\rho(\partial\mathbb{R}^n_+)} + \int_{\partial\mathbb{R}^n_+} f(v)h_{ij}\bar\rho\, dy'.\end{gathered} \tag{3.6}$$

To estimate the last term of (3.6), here we provide a pointwise estimate of $v(y,s)$. By Lemma 1, we note that $x'\cdot\nabla' u(x,t)\leqslant 0$ for $t\in(0,T)$. Therefore, since $v(y,s)=\varphi(y,s)-\varphi_0(y_n)$, we see that $v(y,s)\leqslant v(0,s)$ for $y\in\partial\mathbb{R}^n_+$, $s\in(s_T,\infty)$. By a parabolic regularity theory with (3.5), we get $v(0,s)\leqslant ce^{-\gamma s}$ for $s\in(s_T,\infty)$. Therefore we obtain

$$v(y,s)\leqslant ce^{-\gamma s},\qquad (y,s)\in\partial\mathbb{R}^n_+\times(s_T,\infty).$$

Next we derive a lower estimate of $v(y,s)$. Put $b(y_n)=I_1(y_n)/I_1(0)$ and $w(y,s)=v(y,s)/b(y_n)$. Then $w(y,s)$ satisfies

$$\begin{cases}\partial_s w=\Delta w-\dfrac{y}{2}\cdot\nabla w+\left(\dfrac{2b'}{b}\right)\partial_n w+w, & (y,s)\in\mathbb{R}^n_+\times(s_T,\infty),\\ \partial_\nu w=f(w), & (y,s)\in\partial\mathbb{R}^n_+\times(s_T,\infty).\end{cases}$$

Let $S(s)w_0$ be a solution of

$$\begin{cases}\partial_s w=\Delta w-\dfrac{y}{2}\cdot\nabla w+\left(\dfrac{2b'}{b}\right)\partial_n w, & (y,s)\in\mathbb{R}^n_+\times(0,\infty),\\ \partial_\nu w=0, & (y,s)\in\partial\mathbb{R}^n_+\times(0,\infty),\\ w(y,0)=w_0(y), & y\in\mathbb{R}^n_+.\end{cases}$$

Since $f(v)\geqslant 0$, by a representation formula (28) in [13], we obtain

$$w(s)\geqslant e^{s-s_0}S(s-s_0)w(s_0).$$

By Lemma 2.9 in [13], the right-hand side is estimated by

$$e^{s-s_0}S(s-s_0)|w(s_0)|\leqslant ce^{s-s_0}\exp\left(\frac{e^{-(s-s_0)}|y|^2}{4(1+e^{-(s-s_0)})}\right)\|w(s_0)b(y_n)\|_\rho\qquad\text{on }\partial\mathbb{R}^n_+.$$

Therefore we obtain

$$e^{s-s_0}S(s-s_0)|w(s_0)|\leqslant ce^{s-s_0-\gamma s_0}\exp\left(\frac{e^{-(s-s_0)}|y|^2}{4(1+e^{-(s-s_0)})}\right)\qquad\text{on }\partial\mathbb{R}^n_+.$$

We fix $\varepsilon\in(0,\gamma/4)$ such that $2(\gamma-\varepsilon)\notin\{\lambda_{ij}\}_{(i,j)\in\mathbb{N}^2}$ and choose $s_0\in(0,s)$ such that

$$\begin{aligned}s_0=\left(\frac{1+\gamma-\varepsilon}{1+\gamma}\right)s &\iff s-s_0=\left(\frac{\varepsilon}{1+\gamma}\right)s\\ &\iff (s-s_0)-\gamma s_0=-(\gamma-\varepsilon)s.\end{aligned}$$

Hence it follows that

$$w(y,s)\geqslant -e^{s-s_0}S(s-s_0)|w(s_0)|\geqslant -ce^{-(\gamma-\varepsilon)s}\exp\left(\frac{e^{-\varepsilon s/(1+\gamma)}|y|^2}{4(1+e^{-(s-s_0)})}\right)\qquad\text{on }\partial\mathbb{R}^n_+.$$

Here we note that $v(y,s)$ is uniformly bounded and $|f(v)| \leqslant cv^2$. Therefore, since $v(y,s) = w(y,s)$ on $\partial\mathbb{R}^n_+$, we see that

$$\int_{\partial\mathbb{R}^n_+} f(v)e_{ij}\bar{\rho}\,dy' = \int_{|y'|\leqslant e^{\varepsilon s/2(1+\gamma)}} f(w)e_{ij}\bar{\rho}\,dy' + \int_{|y'|>e^{\varepsilon s/2(1+\gamma)}} f(w)e_{ij}\bar{\rho}\,dy'$$
$$\leqslant ce^{-2(\gamma-\varepsilon)s}\int_{\partial\mathbb{R}^n_+} |e_{ij}|\bar{\rho}\,dy' + c\int_{|y'|>e^{\varepsilon s/2(1+\gamma)}} |e_{ij}|\bar{\rho}\,dy'.$$

Since $e_{ij}(y') = K_i(|y'|)I_j(0)$ on $\partial\mathbb{R}^n_+$ and K_i is the $2(i-1)$-th polynomial, there exists $c_{ij} > 0$ such that $|e_{ij}(y')|\bar{\rho}(y') \leqslant c_{ij}e^{-|y'|^2/8}$. Hence we get

$$\int_{\partial\mathbb{R}^n_+} f(v)e_{ij}\bar{\rho}\,dy' \leqslant c_{ij,\varepsilon}e^{-2(\gamma-\varepsilon)s}.$$

Therefore, by (3.6), there exists $\alpha_{ij} \in \mathbb{R}$ such that

$$\left|a_{ij}(s) - \alpha_{ij}e^{-\lambda_{ij}s}\right| \leqslant c_{ij,\varepsilon}e^{-2(\gamma-\varepsilon)s}, \qquad (i,j) \in \mathbb{N}^2. \tag{3.7}$$

Next we provide a estimate of h_{ij}. Since $|f(v)h_{ij}| \leqslant \delta h_{ij}^2 + \delta^{-1}f(v)^2$, form (3.6), we verify that

$$\frac{1}{2}\partial_s\|h_{ij}\|_\rho^2 \leqslant -\|\nabla h_{ij}\|_\rho^2 - m\|h_{ij}\|_\rho^2$$
$$+ (qB^{q-1} + \delta)\|h_{ij}\|^2_{L^2_{\bar{\rho}}(\partial\mathbb{R}^n_+)} + \delta^{-1}\int_{\partial\mathbb{R}^n_+} f(v)^2\bar{\rho}\,dy'.$$

Let Π_{ij} be a subspace defined by

$$\Pi_{ij} = \{e \in H^1_\rho(\mathbb{R}^n_+); (e, e_{kl})_\rho = 0 \text{ for any } (k,l) \in \mathbb{N}^2 \text{ such that } \lambda_{kl} < \lambda_{ij}\}.$$

Then it is known that

$$\lambda_{ij} = \inf_{e\in\Pi_{ij}}\left(\frac{\|\nabla e\|_\rho^2 + m\|e\|_\rho^2 - qB^{q-1}\|e\|_{L^2_{\bar{\rho}}(\partial\mathbb{R}^n_+)}}{\|e\|_\rho^2}\right).$$

We set

$$\lambda_{ij}(\delta) = \inf_{e\in\Pi_{ij}}\left(\frac{\left(\|\nabla e\|_\rho^2 + m\|e\|_\rho^2\right) - (qB^{q-1} + \delta)\|e\|_{L^2_{\bar{\rho}}(\partial\mathbb{R}^n_+)}}{\|e\|_\rho^2}\right).$$

From definition of $\lambda_{ij}(\delta)$, we get

$$\frac{1}{2}\partial_s\|h_{ij}\|_\rho^2 \leqslant -\lambda_{ij}(\delta)\|h_{ij}\|_\rho^2 + \delta^{-1}\int_{\partial\mathbb{R}^n_+} f(v)^2\bar{\rho}\,dy'.$$

By the same way as above, it holds that

$$\int_{\partial\mathbb{R}^n_+} f(v)^2\bar{\rho}\,dy' \leqslant c_\varepsilon e^{-4(\gamma-\varepsilon)s}.$$

Hence we obtain

$$\|h_{ij}\|_\rho \leqslant c_{ij,\varepsilon}\left(e^{-2(\gamma-\varepsilon)s}+e^{-\lambda_{ij}(\delta)s}\right). \tag{3.8}$$

First we assume that $\alpha_{ij}=0$ for any $(i,j)\in\mathbb{N}^2$. Then by (3.7) and (3.8), if $\|v\|_\rho \leqslant ce^{-\gamma s}$, it holds that $\|v\|_\rho \leqslant ce^{-2(\gamma-\varepsilon)s}$. Therefore by induction, for any $\gamma>0$ there exists $c_\gamma>0$ such that $\|v\|_\rho < c_\gamma e^{-\gamma s}$, which implies the case (i). Next we consider the case $\alpha_{i_1j_1}\neq 0$ and $\alpha_{kl}=0$ if $\lambda_{kl}<\lambda_{i_1j_1}$. Then we choose $(i_2,j_2)\in\mathbb{N}^2$ such that $\lambda_{i_2j_2}>2\lambda_{i_1j_1}$. Since $\lim_{\delta\to 0}\lambda_{ij}(\delta)=\lambda_{ij}$, there exists $\delta>0$ such that $\lambda_{i_2j_2}(\delta)>2\lambda_{i_1j_1}$. Therefore by (3.7) and (3.8), we obtain a conclusion. □

3.2. Spacial singularities

First we list lower eigenvalues of (3.2) below (see (3.3) and Appendix).

$$\begin{aligned}
&\lambda_{11}=-1,\ \lambda_{12}>m+1/2,\ \lambda_{13}>m+1,\\
&\lambda_{21}=0,\quad \lambda_{22}>m+3/2,\ \lambda_{23}>m+2,\\
&\lambda_{31}=1,\quad \lambda_{32}>m+5/2,\ \lambda_{33}>m+3,\\
&\lambda_{41}=2,\quad \lambda_{42}>m+7/2,\ \lambda_{43}>m+4.
\end{aligned}$$

In this subsection, $v(y,s)$ stands for a y_n-axial symmetric function defined by (1.1). Since $v(y,s)$ is y_n-axial symmetric, $v(y,s)$ is expressed by $v(y,s)=V(r,z,s)$ $(r=|y'|$, $z=y_n)$. Since there is no confusion, we denote $V(r,z,s)$ by $v(r,z,s)$.

LEMMA 4. *Let u_0 be as in Theorem 1. Assume that the case* (II) *in Proposition 3 and the case* (ii) *in Proposition 4 occurs. Then expansion* (3.4) *holds with $i_1\in\{1,2\}$, $j_1\geqslant 2$.*

Proof. From Lemma 1, it follows that

$$\partial_r v(r,0,t)\leqslant 0,\qquad r\in(0,\infty).$$

By assumption, it holds that

$$\partial_r v(r,0,s)=\alpha_{i_1j_1}e^{-\lambda s}\partial_r e_{i_1j_1}(r,0)+o(e^{-\lambda s}),\qquad r\in(0,r_0)$$

for any fixed $r_0\in(0,\infty)$. If $i_1\notin\{1,2\}$, then by the shape of $K_i(r)$, there exists $r_1>0$ such that $\partial_r K_1(r_1)>0$. However, since $\partial_r e_{i_1j_1}(r,0)=\partial_r K_{i_1}(r)I_{j_1}(0)$, this contradicts $\partial_r v\leqslant 0$. Therefore the proof is completed. □

LEMMA 5. *Assume the same condition as in Lemma 4. If expansion* (3.4) *holds with $i_1=1$, $j_1=2$ and put $a_{12}(s)=(v(s),e_{12})_\rho$, then there exists $\gamma>1$ such that*

$$v(s)=a_{12}(s)e_{12}+O(e^{-\gamma s})\qquad \text{in } L^2_\rho(\mathbb{R}^n_+).$$

Proof. For the case $\lambda_{12} > 1$, from Proposition 4, this lemma is trivial. Now we assume that $\lambda_{12} \leqslant 1$. Since $\lambda_{12} > 1/2$, we can fix $\varepsilon > 0$ such that $2(\lambda_{12} - \varepsilon) > 1$. Put $h(s) = v(s) - \sum_{\lambda_{kl} < \lambda_{i_2 j_2}} (v(s), e_{kl}) e_{kl}$. Then by Proposition 4, we get

$$\begin{aligned}\|v(s) - a_{12}(s)e_{12}\|_\rho^2 &= |a_{11}(s)|^2 + |a_{21}(s)|^2 + \sum_{\lambda_{12} < \lambda_{kl} < \lambda_{i_2 j_2}} |a_{kl}(s)|^2 + \|h(s)\|_\rho \\ &\leqslant \Big(2 \sum_{\lambda_{12} < \lambda_{kl} < \lambda_{i_2 j_2}} \alpha_{kl}^2 e^{-2\lambda_{kl} s} + c e^{-4(\lambda_{12} - \varepsilon)s}\Big).\end{aligned}$$

From a list of eigenvalues, we see that $\lambda_{kl} > 1$ if $(k,l) \notin \{(1,1),(1,2),(2,1),(3,1)\}$. Therefore it is sufficient to show that $\alpha_{31} = 0$. Suppose that $\alpha_{31} \neq 0$. Then, since $a_{31}(s) = \alpha_{31} e^{-s} + O(e^{-\gamma s})$ for some $\gamma > 1$, it holds that

$$v(s) = a_{12}(s)e_{12} + \alpha_{31} e^{-s} e_{31} + O(e^{-\gamma s}) \qquad \text{in } L_\rho^2(\mathbb{R}_+^n) \cap C_{\mathrm{loc}}^2(\overline{\mathbb{R}_+^n})$$

for some $\gamma > 1$. Differentiating with respect to r, we get from $\partial_r e_{12} = 0$

$$\begin{aligned}\partial_r v(r,0,s) &= a_{12}(s)\partial_r e_{12}(r,0) + \alpha_{31} e^{-s} \partial_r e_{31}(r,0) + O\left(e^{-\gamma s}\right) \\ &= \alpha_{31} e^{-s} \partial_r e_{31}(r,0) + O\left(e^{-\gamma s}\right).\end{aligned}$$

Hence, by the shape of $e_{31}(r,0)$, there exist $r_1 > 0$ and $s = s_1$ such that

$$\partial_r v(r_1, 0, s_1) > 0,$$

which contradicts $\partial_r v \leqslant 0$. Therefore $\alpha_{31} = 0$ is assured, which completes the proof. □

LEMMA 6. *If $v(s)$ satisfies*

$$\|v(s)\|_\rho \leqslant c e^{-\gamma s}$$

for some $\gamma > 1$, then the blow-up profile satisfies

$$\lim_{r \to 0} r^{4m} u(r,0,T) = +\infty.$$

Proof. Let $b(y_n)$ and $S(s)$ be as in the proof of Proposition of 4. Since $f(v) \geqslant 0$, a representation formula (28) in [13] gives

$$v(s) \geqslant e^{s-s_0} S(s-s_0)\left(\frac{v(s_0)}{b}\right) \qquad \text{on } \partial\mathbb{R}_+^n.$$

Applying Lemma 2.9 in [13], we obtain

$$\begin{aligned}\left|S(s-s_0)\left(\frac{v(s_0)}{b}\right)\right| &\leqslant c \exp\left(\frac{e^{-(s-s_0)}|y|^2}{4(1+e^{-(s-s_0)})}\right)\|v(s_0)\|_\rho \\ &\leqslant c \exp\left(\frac{e^{-(s-s_0)}|y|^2}{4(1+e^{-(s-s_0)})}\right) e^{-\gamma s_0} \qquad \text{on } \partial\mathbb{R}_+^n.\end{aligned}$$

We choose $s_0 \in (0,s)$ such that

$$s_0 = \left(\frac{1+\varepsilon}{1+\gamma}\right) s \iff s - s_0 = \left(\frac{\gamma-\varepsilon}{1+\gamma}\right) s$$
$$\iff (s-s_0) - \gamma s_0 = -\varepsilon s.$$

Then it holds that

$$v(r,0,s) \geqslant -ce^{s-s_0}e^{-\gamma s_0} = -ce^{-\varepsilon s}, \qquad r \leqslant e^{(\gamma-\varepsilon)s/2(1+\gamma)}.$$

By (1.1), this implies that

$$\varphi(r,0,s) \geqslant B - ce^{-\varepsilon s}, \qquad r \leqslant e^{(\gamma-\varepsilon)s/2(1+\gamma)}. \tag{3.9}$$

We set

$$2g_\varepsilon(s) = e^{(\gamma-\varepsilon)s/2(1+\gamma)}.$$

Following [13] (originally [15]), we introduce ($r,z > 0$, $t \in (0,1)$)

$$U_s(r,z,t) = e^{-ms}u(e^{-s/2}r + e^{-s/2}g_\varepsilon(s), z, T + (t-1)e^{-s}),$$

where $s > s_T$ is a parameter. We consider a rescaled function $w_s(r,z,\tau)$ defined by ($y \in \mathbb{R}^n_+$, $\tau \in \mathbb{R}_+$)

$$\begin{aligned} w_s(r,z,\tau) &= e^{-m\tau}U_s\left(e^{-\tau/2}r, e^{-\tau/2}z, 1-e^{-\tau}\right) \\ &= e^{-m(\tau+s)}u\left(e^{-(\tau+s)/2}r + e^{-s/2}g_\varepsilon(s), e^{-(\tau+s)/2}z, T - e^{-(\tau+s)}\right) \\ &= \varphi\left(r + e^{\tau/2}g_\varepsilon(s), z, \tau+s\right). \end{aligned}$$

From (3.9), there exists $s_1 > 0$ such that for $s \geqslant s_1$

$$w_s(r,0,0) = \varphi(r + g_\varepsilon(s), 0, s) \geqslant B/2, \qquad r \in (0,1).$$

From Lemma 2.1 in [13], we recall that $|\nabla\varphi|$ is uniformly bounded. Hence there exists $z_1 > 0$ such that for $s \geqslant s_1$

$$w_s(r,z,0) \geqslant B/4, \qquad r \in (0,1,), \ z \in (0,z_1).$$

Since $w_s(r,z,\tau) \geqslant 0$, by the way as in the proof of Proposition 5.1 (lower bound) in [13], there exist $c_0 > 0$ and $\tau_1 \in (0,\infty)$ such that for $s \geqslant s_1$ and $\tau \in (\tau_1,\infty)$

$$w_s(0,0,\tau) \geqslant c_0 e^{-m\tau}.$$

Hence it follows that for $s \geqslant s_1$ and $\tau \in (\tau_1,\infty)$

$$u\left(e^{-s/2}g_\varepsilon(s), 0, T - e^{-(\tau+s)}\right) \geqslant c_0 e^{ms}.$$

Taking $\tau \to \infty$, we obtain for $s \geqslant s_1$

$$u\left(e^{-s/2}g_\varepsilon(s), 0, T\right) \geqslant c_0 e^{ms}.$$

Let $\tilde{s}(b)$ be an inverse function defined by

$$2b = 2e^{-\tilde{s}/2}g_\varepsilon(\tilde{s}) = e^{-(1+\varepsilon)\tilde{s}/2(1+\gamma)}.$$

Thus there exists $b_1 > 0$ such that for $b \in (0, b_1)$ and $t \in (1 - e^{-\tau_1}, 1)$

$$u(b, 0, T) \geqslant c_0' b^{-2(1+\gamma)m/(1+\varepsilon)}.$$

Since $2(1+\gamma) > 4$ if $\gamma > 1$, the proof is completed. □

LEMMA 7. *Let u_0 and $v(s)$ be as in Lemma 5. Then the blow-up profile satisfies*

$$\lim_{r \to 0} r^{4m} u(r, 0, T) = \infty.$$

Proof. From Lemma 5, we note that $v(s) - a_{12}(s)e_{12} = O(e^{-\gamma s})$ in $L^2_\rho(\mathbb{R}^n_+)$ for some $\gamma > 1$. For the case $\lambda_{12} > 1$, this lemma is reduced to Lemma 6. Therefore we assume that $\lambda_{12} \leqslant 1$. We repeat arguments given in the proof of Lemma 6. A representation formula shows that

$$\left(\frac{v(s)}{b}\right) \geqslant e^{s-s_0} S(s-s_0)\left(\frac{v(s_0)}{b}\right)$$
$$= e^{s-s_0} S(s-s_0)\left(\frac{a_{12}(s_0)e_{12}}{b}\right) + e^{s-s_0} S(s-s_0)\left(\frac{v(s_0) - a_{12}(s_0)e_{12}}{b}\right).$$

Since $S(s-s_0)(e_{12}/b) = e^{-(\lambda_{12}+1)(s-s_0)}(e_{12}/b)$ (see the proof of Lemma 4.3 [13]), we obtain

$$\left(\frac{v(s)}{b}\right) \geqslant a_{12}(s_0)e^{-\lambda_{12}(s-s_0)}\left(\frac{e_{12}}{b}\right) + e^{s-s_0} S(s-s_0)\left(\frac{v(s_0) - a_{12}(s_0)e_{12}}{b}\right).$$

By the same argument as in the proof of Lemma 6, we choose s_0 such that $(1+\gamma)s_0 = (1+\varepsilon)s$, then we see that

$$\left| e^{s-s_0} S(s-s_0)\left(\frac{v(s_0) - a_{12}(s_0)e_{12}}{b}\right)\right| \leqslant ce^{-\varepsilon s}, \qquad r \leqslant e^{(\gamma-\varepsilon)s/2(1+\gamma)},\ z = 0.$$

Therefore, since $a_{12}(s_0) = (\alpha_{12} + o(1))e^{-\lambda_{12}s_0}$ and $b = 1$ on $\partial\mathbb{R}^n_+$, it holds that

$$v(s) \geqslant (\alpha_{12} + o(1))e^{-\lambda_{12}s}e_{12} - ce^{-\varepsilon s}, \qquad r \leqslant e^{(\gamma-\varepsilon)s/2(1+\gamma)},\ z = 0.$$

Here we note that $e_{12}(y) = K_1(|y'|)I_2(0)$ on $\partial\mathbb{R}^n_+$ and $K_1(r)$ is a positive constant function. Therefore the rest of proof follows from that of Lemma 6, which completes the proof. □

Proof. (*Proof of Theorem 1*) We prove by contradiction. Suppose that the case (II) in Proposition 3 occurs. Hence, from Proposition 4 and Lemma 4–5, there are two possibilities:

$$\text{(i) } \|v(s)\|_\rho \leqslant e^{-\gamma s} \ (\gamma > 1), \qquad \text{(ii) } v(s) \sim \alpha_{12} e^{-\lambda_{12} s} \ (\alpha_{12} \neq 0).$$

For the both cases, from Lemma 6–7, it follows that

$$\lim_{r\to 0} r^{4m} u(r,0,T) = \infty. \tag{3.10}$$

Now we use the same technique given in Proposition 2 to derive a contradiction. We set

$$V_a(r,z,t) = -aU_r(r,z,t), \qquad W(r,z,t) = d(r;r_1)U(r,z,t)^q,$$

$$d(r;r_1) = (\sqrt{\mu}r)^{-(n-3)/2} J_{(n-1)/2}(\sqrt{\mu}r), \qquad \mu(r_1) = Z_1^2/r_1^2,$$

where $J_\nu(r)$ is the ν-th Bessel function and $Z_1 > 0$ is the first zero of $J_{(n-1)/2}(r)$. Moreover we set

$$\mathscr{U}_a = \{(r,z,t) \in J \times (0,T); V_a(r,z,t) < W(r,z,t)\}.$$

Repeating arguments in Proposition 2, we obtain (2.2), that is

$$LW \leqslant \left(\mu(r_1)U^{-2(q-1)} + q|\partial_r d(r;r_1)|U^{-(q-1)} - c_0\right) dU^{3q-2}$$

for $(r,z,t) \in \mathscr{U}_a$, $r \in (0,r_1)$, $z \in (0,1/2)$ and $a \geqslant 1$. Here we note that

$$r^{-(n-1)/2}|J_{(n-1)/2}(r)| + r^{-(n-3)/2}|\partial_r J_{(n-1)/2}(r)| \leqslant c, \qquad r \in (0,z_1).$$

Hence we see that

$$\begin{aligned} |\partial_r d(r;r_1)| &\leqslant \left(\frac{n-3}{2}\right)\sqrt{\mu}(\sqrt{\mu r})^{-(n-1)/2}|J_{(n-1)/2}(\sqrt{\mu}r)| \\ &\qquad + \sqrt{\mu}(\sqrt{\mu r})^{-(n-3)/2}|J'_{(n-1)/2}(\sqrt{\mu}r)| \\ &\leqslant c\sqrt{\mu}, \qquad r \in (0,r_1). \end{aligned}$$

Therefore, it follows that for $(r,z,t) \in \mathscr{U}_a$, $a \geqslant 1$, $r \in (0,r_1)$, and $z \in (0,1/2)$,

$$LW \leqslant \left(\left(\frac{Z_1}{r_1}\right)^2 U^{-2(q-1)} + c\left(\frac{Z_1}{r_1}\right)U^{-(q-1)} - c_0\right) dU^{3q-2}.$$

From (3.10), it holds that

$$\lim_{r_1\to 0} r_1^{-1} U(r_1,0,T)^{-(q-1)} \leqslant \lim_{r_1\to 0} r_1^{-2} U(r_1,0,T)^{-(q-1)} = 0.$$

Hence there exists $r_1^* > 0$ such that

$$\left(\frac{Z_1}{r_1^*}\right)^2 U(r_1^*,0,T)^{-2(q-1)} + c\left(\frac{Z_1}{r_1^*}\right)U(r_1^*,0,T)^{-(q-1)} \leqslant c_0/2.$$

From Proposition 2, $u(x,t)$ blows up only on the origin. Hence by the continuity, there exist $z_1 > 0$ and $t_1 > 0$ such that

$$\left(\frac{Z_1}{r_1^*}\right)^2 u(r_1^*,z,t)^{-2(q-1)} + c\left(\frac{Z_1}{r_1^*}\right) u(r_1^*,z,t)^{-(q-1)} \leqslant c_0, \qquad z \in (0,z_1),\ t \in (t_1,T).$$

Hence by $\partial_z U, \partial_r U \leqslant 0$, we obtain

$$\left(\frac{Z_1}{r_1^*}\right)^2 U(r,z,t)^{-2(q-1)} + c\left(\frac{Z_1}{r_1^*}\right) U(r,z,t)^{-(q-1)} \leqslant c_0$$

for $r \in (0,r_1^*)$, $z \in (0,z_1)$, $t \in (t_1,T)$. This implies that for $(r,z,t) \in \mathscr{U}_a$ and $a \geqslant 1$

$$LW \leqslant 0, \qquad r \in (0,r_1^*),\ z \in (0,z_1),\ t \in (t_1,T).$$

Hence by the same way as in the proof of Proposition 2, we find that there exists $a_1 > 1$ such that

$$-a_1(\partial_r u) \geqslant d(r;r_1^*)u^q, \qquad r \in (0,r_1^*),\ z \in (0,z_1),\ t \in (t_1,T),$$

which implies that

$$u(r,0,t) \leqslant cr^{-4m}, \qquad r \in (0,r_1^*),\ t \in (t_1,T).$$

As a consequence, a blow-up profile satisfies

$$u(r,0,T) \leqslant cr^{-4m}, \qquad r \in (0,r_1^*),$$

which contradicts (3.10). Thus the proof is completed. □

A. Appendix

A.1. Eigenvalue problems

In this section, for simplicity, we denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$. We study the following eigenvalue problem:

$$\begin{cases} -\left(u'' - \dfrac{\xi}{2}u'\right) = \mu u & \text{in } \mathbb{R}_+, \\ \partial_\nu u = Ku & \text{on } \{0\}, \end{cases} \tag{A.1}$$

where $K > 0$. It is known that the first eigenvalue μ_1 is negative and μ_i $(i \geqslant 2)$ is positive. Here we give precise estimate of positive eigenvalues of (A.1). To find two linearly independent fundamental solutions of (A.1), we consider ODE problem without a boundary condition:

$$u'' - \frac{\xi}{2}u' = -\mu u. \tag{A.2}$$

We put $v(\xi)=u(2\sqrt{\xi})$. Then $v(\xi)$ satisfies Kummer's equation:

$$\xi v''+(b-\xi)w'-aw=0, \tag{A.3}$$

where $a=-\mu$ and $b=1/2$. Since $b=1/2\notin\mathbb{N}$, two linearly independent fundamental solutions of (A.3) are given by (see p. 504 [1])

$$v_1(\xi)=M(-\mu,1/2,\xi), \qquad v_2(\xi)=\begin{cases}\xi^{1/2}M(-\mu+1/2,3/2,\xi) & \text{if } 2\mu\in\mathbb{N}_0,\\ U(-\mu,1.2,\xi) & \text{if } 2\mu\notin\mathbb{N}_0,\end{cases}$$

where $M(a,b,\xi)$ (Kummer's function) and $U(a,b,\xi)$ are given by

$$M(a,b,\xi)=1+\sum_{k=1}^{\infty}\frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)}\frac{\xi^k}{k!} \qquad (-b\notin\mathbb{N}_0),$$

$$U(a,b,\xi)=\Big(\frac{\pi}{\sin\pi b}\Big)\Big(\frac{M(a,b,\xi)}{\Gamma(a')\Gamma(b)}-\xi^{1-b}\frac{M(a',b',\xi)}{\Gamma(a)\Gamma(b')}\Big) \qquad (-a,\pm b,-a',-b'\notin\mathbb{N}_0),$$

where $a'=1+a-b$, $b'=2-b$. For the case $a=-n$ $(n\in\mathbb{N})$, $M(a,b,\xi)$ is the n-th polynomial:

$$M(-n,b,\xi)=1+\sum_{k=1}^{n}\frac{-n(-n+1)\cdots(-n+k-1)}{b(b+1)\cdots(b+k-1)}\frac{\xi^k}{k!} \qquad (-b\notin\mathbb{N}_0).$$

Moreover for the case $-a\notin\mathbb{N}_0$, the asymptotic formulas of $M(a,b,\xi)$ and $U(a,b,\xi)$ for large $\xi>0$ are given by (see p. 504 [1])

$$\begin{aligned} M(a,b,\xi)&=\frac{\Gamma(b)}{\Gamma(a)}e^{\xi}\xi^{a-b}(1+O(\xi^{-1})) \qquad (-a,-b\notin\mathbb{N}_0),\\ U(a,b,\xi)&=\xi^{-a}(1+O(\xi^{-1})) \qquad (-a,\pm b,-a',-b'\notin\mathbb{N}_0),\end{aligned} \tag{A.4}$$

where $a'=1+a-b$, $b'=2-b$. Then, since $u(\xi)=v(\xi^2/4)$, the original equation (A.2) has two linearly independent fundamental solutions given by

$$u_1(\xi)=M(-k,1/2,\xi^2/4), \qquad u_2(\xi)=\begin{cases}\frac{\xi}{2}M(-\mu+1/2,3/2,\xi^2/4) & \text{if } 2\mu\in\mathbb{N}_0,\\ U(-\mu,1/2,\xi^2/4) & \text{if } 2\mu\notin\mathbb{N}_0.\end{cases}$$

Then by virtue of asymptotic formula (A.4), we find that a solution $u(x,t)$ of (A.2) in $L^2_\rho(\mathbb{R}_+)$ is given by

$$u_0(\xi)=c\cdot\begin{cases}M(-k,1/2,\xi^2/4) & \text{if } \mu=k\in\mathbb{N}_0,\\ \xi M(-k,3/2,\xi^2/4) & \text{if } \mu-1/2=k\in\mathbb{N}_0,\\ U(-\mu,1/2,\xi^2/4) & \text{if } 2\mu\in\mathbb{N}_0\end{cases}$$

for some constant $c \neq 0$. Since $u_0'(0) = 0$ if $\mu \in \mathbb{N}_0$ and $u_0(0) = 0$ if $\mu - 1/2 \in \mathbb{N}_0$, if u is a solution of (A.1) in $L^2_\rho(\mathbb{R}_+)$, it holds that $2\mu \notin \mathbb{N}_0$. Therefore to find all positive eigenvalue of (A.1), it is sufficient to find all value $\mu > 0$ such that

$$K = -\left(\frac{u_0'(0)}{u_0(0)}\right) = -\frac{1}{2}\lim_{\xi \to 0}\left(\frac{\xi U'(-\mu, 1/2, \xi^2/4)}{U(-\mu, 1/2, \xi)}\right) \qquad (2\mu \notin \mathbb{N}_0). \tag{A.5}$$

Then it is known that asymptotic formulas for $\xi \sim 0$ are given by (see p. 508 [1])

$$U(a,b,\xi) = \frac{\Gamma(1-b)}{\Gamma(a')} + O(|\xi|^{1-b}) \qquad (-a, -a' \notin \mathbb{N}_0, b \in (0,1)),$$
$$U(a,b,\xi) = \frac{\Gamma(b-1)}{\Gamma(a)}\xi^{1-b} + O(1) \qquad (-a, -a' \notin \mathbb{N}_0, b \in (1,2)),$$

where $a' = 1 + a - b$. Since $U'(a,b,\xi) = -aU(a+1, b+1, \xi)$ (see p. 507 [1]), we see that

$$\lim_{\xi \to 0}\left(\frac{\xi U'(-\mu, 1/2, \xi^2/4)}{U(-\mu, 1/2, \xi)}\right) = \frac{2\mu\Gamma(-\mu + 1/2)}{\Gamma(1-\mu)}.$$

Since

$$\Gamma(\mu)\Gamma(1-\mu) = \pi / \sin \pi\mu \quad \text{and} \quad \Gamma(\mu + 1/2)\Gamma(-\mu + 1/2) = \pi / \cos \pi\mu,$$

it holds that

$$\frac{\mu\Gamma(-\mu + 1/2)}{\Gamma(1-\mu)} = \frac{\Gamma(\mu)}{\Gamma(\mu + 1/2)}(\mu \tan \pi\mu).$$

Therefore (A.5) is reduced to

$$K = -\frac{\Gamma(\mu)}{\Gamma(\mu + 1/2)}(\mu \tan \pi\mu) \qquad (2\mu \notin \mathbb{N}_0). \tag{A.6}$$

Since $\Gamma(\xi) > 0$ if $\xi > 0$, the roots of (A.6) are in $\bigcup_{k \in \mathbb{N}}(k - 1/2, k)$. Here we fix $k \in \mathbb{N}$. Now we claim that (A.6) has a unique root in $(k - 1/2, k)$. Put

$$G(\mu) = \Gamma(\mu)/\Gamma(\mu + 1/2) \quad \text{and} \quad f(\mu) = \mu \tan \pi\mu.$$

It is known that $G'(\mu) < 0$ for $\mu > 0$ (see p. 4 [17]). Moreover we see that

$$\begin{aligned} f'(\mu) &= \tan \pi\mu + \frac{\pi\mu}{(\cos \pi\mu)^2} = \frac{1}{(\cos \pi\mu)^2}(\cos \pi\mu \cdot \sin \pi\mu + \pi\mu) \\ &\geqslant \frac{1}{(\cos \pi\mu)^2}(-1 + (k - 1/2)\pi) \\ &> \frac{1}{(\cos \pi\mu)^2}(-1 + \pi/2) \geqslant 0, \qquad \mu > k - 1/2. \end{aligned}$$

Therefore, since $G(\mu) > 0$ and $f(\mu) < 0$ for $\mu \in (k - 1/2, k)$, we obtain

$$\frac{d}{d\mu}(G(\mu)f(\mu)) = G'(\mu)f(\mu) + G(\mu)f'(\mu) > 0, \qquad \mu \in (k - 1/2, k).$$

As a consequence, since

$$\lim_{\mu\to k-1/2+0} G(\mu)f(\mu) = -\infty, \qquad G(k)f(k) = 0,$$

there exists a unique $\mu_k(K) \in (k-1/2,k)$ such that $-K = G(\mu_k)f(\mu_k)$. This proves the claim. Therefore we obtain the following result.

LEMMA 8. *Let* μ_k $(k\in\mathbb{N})$ *be the* k*-th eigenvalue of* (A.1). *Then it follows that* $\mu_1 < 0$ *and* $\mu_k \in (k-3/2,k-1)$ *if* $k \geqslant 2$ $(k\in\mathbb{N})$.

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