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SOME OSCILLATION RESULTS FOR SECOND ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS

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Abstract. This paper is concerned with the oscillatory behavior of second order neutral difference equations. Four oscillation theorems for such equations are established and examples are given to illustrate the results.

1. Inroduction

In this paper we are concerned with the oscillation problem of second order neutral type difference equation of the form

$$\Delta\Big(a_n(\Delta(x_n+p_nx_{\tau(n)}))^{\alpha}\Big)+q_nf(x_{\sigma(n+1)})=0 \tag{1.1}$$

where $n\in\mathbb{N}(n_0)=\{n_0,n_0+1,\dots\}$, n_0 is a positive integer and α is a ratio of odd positive integers. Further, we assume that the following conditions hold.

(H_1) $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are positive real valued sequences with $p_n\geqslant 1$ for all $n\in\mathbb{N}(n_0)$, and $A(n_0)=\sum_{n=n_0}^{\infty}\frac{1}{a_n^{1/\alpha}}<\infty$;

(H_2) $\sigma(n)$ and $\tau(n)$ are strictly increasing sequences of integers on $\mathbb{N}(n_0)$ with $\lim_{n\to\infty}\sigma(n)=\lim_{n\to\infty}\tau(n)=\infty$;

(H_3) $f:\mathbb{R}\to\mathbb{R}$ is continuous and there exists a constant $L>0$ such that $\frac{f(x)}{x^{\alpha}}\geqslant L$ for all $x\neq 0$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined and satisfies equation (1.1) for all $n\in\mathbb{N}(n_0)$. We consider only those solutions $\{x_n\}$ of equation (1.1) which satisfy $\sup\{|x_n|:n\geqslant N\}>0$ for all $N\in\mathbb{N}(n_0)$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and otherwise it is called nonoscillatory.

In [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15], the authors studied the oscillatory behavior of equation of the form (1.1) when $0\leqslant p_n<1$ and

$$\text{either}\quad \sum_{n=n_0}^{\infty}\frac{1}{a_n^{1/\alpha}}=\infty\quad\text{or}\quad\sum_{n=n_0}^{\infty}\frac{1}{a_n^{1/\alpha}}<\infty.$$

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In [13], the authors consider the following difference equation

$$\Delta(a_n(\Delta(x_n + p_n x_{\tau(n)}))^{\alpha}) + q_n x^{\beta}_{\sigma(n)} = 0, \ \ n \in \mathbb{N}(n_0) \tag{1.2}$$

and established oscillation criteria for the equation (1.2) for the case

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} < \infty, \ 0 \leqslant p_n \leqslant p < \infty \ \text{ and } \ \tau \circ \sigma = \sigma \circ \tau.$$

Clearly, the assumptions given in [13] are quite restrictive and now the problem is how to derive new oscillation tests for equation (1.1) without such conditions. Motivated by this observation, in this paper we establish some new oscillation criteria for the equation (1.1). In Section 2, we establish oscillation criteria for the equation (1.1) and in Section 3, we present some examples to illustrate the main results.

2. Oscillation Results

In this section, we establish four new oscillation results for equation (1.1) when

(H_4) $\tau(n) \geqslant n$ and $\sigma(n+1) \leqslant n$;

(H_5) $\sigma(n) \geqslant \tau(n) \geqslant n$;

(H_6) $\sigma(n+1) \leqslant \tau(n) \leqslant n$;

(H_7) $\tau(n) \leqslant n$ and $\sigma(n+1) \geqslant n$.

All the occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all n large enough. Define

$$\begin{aligned}
A(n) &= \sum_{s=n}^{\infty} \frac{1}{a_s^{1/\alpha}}, \quad R(n) = \sum_{s=n_0}^{n-1} \frac{1}{a_s^{1/\alpha}}, \\
B(n) &= \frac{1}{p_{\tau^{-1}(n)}}\Big(1 - \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}}\Big), \\
C(n) &= \frac{1}{p_{\tau^{-1}(n)}}\Big(1 - \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}} \frac{A(\tau^{-1}(\tau^{-1}(n)))}{A(\tau^{-1}(n))}\Big), \\
D(n) &= \begin{cases} \Big(\dfrac{\alpha}{\alpha+1}\Big)^{\alpha+1} \dfrac{A^{\alpha^2-1}(n)}{A^{\alpha^2}(n+1)} \dfrac{1}{a_n^{1/\alpha}} & \text{if } \alpha \geqslant 1 \\[2ex] \Big(\dfrac{\alpha}{\alpha+1}\Big)^{\alpha+1} \dfrac{1}{A(n+1)} \dfrac{1}{a_n^{1/\alpha}} & \text{if } \alpha \leqslant 1, \end{cases} \\
E(n) &= \frac{1}{p_{\tau^{-1}(n)}}\Big(1 - \frac{R(\tau^{-1}(\tau^{-1}(n)))}{p_{\tau^{-1}(\tau^{-1}(n))} R(\tau^{-1}(n))}\Big),
\end{aligned}$$

where τ^{-1} is the inverse function of τ,

$$Q_n = q_n B^{\alpha}(\sigma(n)), \quad \overline{Q}_n = q_n C^{\alpha}(\sigma(n)).$$

Define

$$z_n = x_n + p_n x_{\tau(n)}. \tag{2.1}$$

We begin with the following lemma.

LEMMA 1. *Let z_n be defined by* (2.1) *with $z_n > 0$, $\Delta z_n > 0$ and $\Delta(a_n(\Delta z_n)^{\alpha}) \leqslant 0$ for all $n \in \mathbb{N}(n_0)$. Then for $\tau(n) \geqslant n$ and $B(n) > 0$, we have*

$$x_n \geqslant B(n) z_{\tau^{-1}(n)}, \quad n \in \mathbb{N}(n_0).$$

Proof. From (2.1), we have

$$z_n = x_n + p_n x_{\tau(n)}$$

or

$$x_n = \frac{1}{p_{\tau^{-1}(n)}}\Big(z_{\tau^{-1}(n)} - x_{\tau^{-1}(n)}\Big).$$

Since $\{z_n\}$ is nondecreasing and $x_{\tau(n)} < \frac{1}{p_n} z_n$, we obtain

$$x_n \geqslant B(n) z_{\tau^{-1}(n)}.$$

This completes the proof.

LEMMA 2. *Let z_n be defined by* (2.1) *with $z_n > 0$, $\Delta z_n < 0$ and $\Delta(a_n(\Delta z_n)^{\alpha}) \leqslant 0$ for all $n \in \mathbb{N}(n_0)$. Then for $A(n_0) < \infty$, $\tau(n) \geqslant n$ and $C(n) > 0$, we have*

$$x_n \geqslant C(n) z_{\tau^{-1}(n)}, \quad n \in \mathbb{N}(n_0).$$

Proof. From $\Delta(a_n(\Delta z_n)^{\alpha}) \leqslant 0$ for all $n \in \mathbb{N}(n_0)$, we have

$$\Delta z_s \leqslant \frac{a_n^{1/\alpha} \Delta z_n}{a_s^{1/\alpha}} \quad \text{for} \quad s \geqslant n \geqslant n_0.$$

Summing the last inequality from ℓ to n and then letting $\ell \to \infty$, we obtain

$$0 \leqslant z_n + a_n^{1/\alpha} A(n) \Delta z_n$$

or

$$\Delta\Big(\frac{z_n}{A(n)}\Big) \geqslant 0 \quad \text{for} \quad n \in \mathbb{N}(n_0).$$

From (2.1), we have

$$x_n = \frac{1}{p_{\tau^{-1}(n)}}\Big(z_{\tau^{-1}(n)} - x_{\tau^{-1}(n)}\Big)$$

$$\begin{aligned} &\geqslant \frac{1}{p_{\tau^{-1}(n)}}\Big(z_{\tau^{-1}(n)} - \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}}\frac{A(\tau^{-1}(\tau^{-1}(n)))}{A(\tau^{-1}(n))}z_{\tau^{-1}(\tau^{-1}(n))}\Big) \\ &\geqslant C(n)z_{\tau^{-1}(n)}. \end{aligned}$$

This completes the proof.

LEMMA 3. *Let z_n be defined by* (2.1) *with* $z_n > 0$, $\Delta z_n > 0$ *and* $\Delta(a_n(\Delta z_n)^\alpha) \leqslant 0$ *for all* $n \in \mathbb{N}(n_0)$. *Then for* $\tau(n) \leqslant n$ *and* $E(n) > 0$, *we have*

$$x_n \geqslant E(n)z_{\tau^{-1}(n)}, \quad n \in \mathbb{N}(n_0).$$

Proof. From the proof of Lemma 1, we have

$$x_n = \frac{1}{p_{\tau^{-1}(n)}}\Big(z_{\tau^{-1}(n)} - x_{\tau^{-1}(n)}\Big).$$

On the otherhand

$$z_n = z_{n_0} + \sum_{s=n_0}^{n-1} \frac{\Big(a_s(\Delta z_s)^\alpha\Big)^{1/\alpha}}{a_s^{1/\alpha}} \geqslant \Big(a_n^{1/\alpha}\sum_{s=n_0}^{n-1}\frac{1}{a_s^{1/\alpha}}\Big)\Delta z_n.$$

Hence

$$\Delta\Big(\frac{z_n}{R(n)}\Big) = \frac{R(n)\Delta z_n - \frac{z_n}{a_n^{1/\alpha}}}{R(n)R(n+1)} \leqslant 0.$$

Thus $\frac{z_n}{R(n)}$ is nonincreasing. Further

$$\begin{aligned} x_{\tau^{-1}(n)} &\leqslant \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}} z_{\tau^{-1}(\tau^{-1}(n))} \\ &\leqslant \frac{R(\tau^{-1}(\tau^{-1}(n)))}{p_{\tau^{-1}(\tau^{-1}(n))}}\frac{z_{\tau^{-1}(n)}}{R(\tau^{-1}(n))}. \end{aligned}$$

Therefore,

$$x_n \geqslant \frac{1}{p_{\tau^{-1}(n)}}\Big(1 - \frac{R(\tau^{-1}(\tau^{-1}(n)))}{p_{\tau^{-1}(\tau^{-1}(n))}R(\tau^{-1}(n))}\Big)z_{\tau^{-1}(n)},$$

or

$$x_n \geqslant E(n)z_{\tau^{-1}(n)}.$$

This completes the proof.

First we establish oscillation criteria for equation (1.1) when $\tau(n) \geqslant n$.

THEOREM 1. *Assume conditions* (H_1)-(H_4) *hold. If there exists a positive non-deccreasing function* $\{\rho_n\}$ *such that*

$$\sum_{n=n_0}^{\infty}\Big[L\rho_n Q_n - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^\alpha}a_{\tau^{-1}(\sigma(n))}\Big] = \infty \tag{2.2}$$

and

$$\sum_{n=n_0}^{\infty}\left[LA^{\alpha}(n+1)\overline{Q}_n - D(n)\right] = \infty \tag{2.3}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_{\sigma(n)} > 0$ for all $n \geqslant n_1 \in \mathbb{N}(n_0)$. Then it follows from equation (1.1) that

$$\Delta\Big(a_n(\Delta z_n)^{\alpha}\Big) \leqslant -Lq_n x^{\alpha}_{\sigma(n+1)} \leqslant 0, \quad n \in \mathbb{N}(n_1). \tag{2.4}$$

Hence $\{a_n(\Delta z_n)^{\alpha}\}$ has one sign for all $n \geqslant n_1$. If $\Delta z_n > 0$ for all $n \geqslant n_1$, then from (2.1), we have from Lemma 1, that

$$x_n \geqslant B(n) z_{\tau^{-1}(n)}, \quad n \geqslant n_1. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\Delta(a_n(\Delta z_n)^{\alpha}) + Lq_n B^{\alpha}(\sigma(n+1)) z^{\alpha}_{\tau^{-1}(\sigma(n+1))} \leqslant 0, \quad n \in \mathbb{N}(n_1). \tag{2.6}$$

Define

$$u_n = \rho_n \frac{a_n(\Delta z_n)^{\alpha}}{z^{\alpha}_{\tau^{-1}(\sigma(n))}}, \quad n \geqslant n_1$$

then

$$\Delta u_n \leqslant -\rho_n L q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\rho_n}{\rho_{n+1}} u_{n+1} \frac{\Delta z^{\alpha}_{\tau^{-1}(\sigma(n))}}{z^{\alpha}_{\tau^{-1}(\sigma(n))}}, \; n \geqslant n_1. \tag{2.7}$$

By Mean Value Theorem

$$\Delta z^{\alpha}_{\tau^{-1}(\sigma(n))} \leqslant \begin{cases} \alpha z^{\alpha-1}_{\tau^{-1}(\sigma(n+1))} \Delta z_{\tau^{-1}(\sigma(n))} & \text{if} \quad \alpha \geqslant 1 \\[2ex] \alpha z^{\alpha-1}_{\tau^{-1}(\sigma(n))} \Delta z_{\tau^{-1}(\sigma(n))} & \text{if} \quad \alpha \leqslant 1. \end{cases}$$

Using the last inequality in (2.7) and then using the nonincreasing nature of $\Delta z_{\tau^{-1}(\sigma(n))}$ and the nondecreasing nature of $a^{1/\alpha}_{\tau^{-1}(\sigma(n))} z_{\tau^{-1}(\sigma(n))}$ we obtain

$$\Delta u_n \leqslant -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\alpha\rho_n}{\rho^{1+1/\alpha}_{n+1}} \frac{u^{(\alpha+1)/\alpha}_{n+1}}{a^{1/\alpha}_{\tau^{-1}(\sigma(n))}}, \; n \geqslant n_1. \tag{2.8}$$

By using the inequality

$$AV - BV^{1+1/\alpha} \leqslant \frac{\alpha^{\alpha} A^{\alpha+1}}{(\alpha+1)^{\alpha+1} B^{\alpha}} \quad \text{for} \quad A \geqslant 0, \quad B > 0 \;\text{ and }\; V > 0,$$

we obtain

$$\Delta u_n \leqslant -L\rho_n q_n B^{\alpha}(\sigma(n+1)) + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha}} a_{\tau^{-1}(\sigma(n))}, \quad n \geqslant n_1.$$

Summing the last inequality from n_1 to n and then letting $n \to \infty$, we obtain

$$\sum_{n=n_1}^{\infty} \left[L\rho_n q_n B^{\alpha}(\sigma(n+1)) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha}} a_{\tau^{-1}(\sigma(n))} \right] \leqslant u_{n_1} < \infty$$

which contradicts the assumption (2.2). Next, consider the case $\Delta z_n < 0$ for all $n \in \mathbb{N}(n_1)$. Define

$$w_n = \frac{a_n(\Delta z_n)^{\alpha}}{z_n^{\alpha}}, \quad n \geqslant n_1. \tag{2.9}$$

Then $w_n < 0$ for $n \geqslant n_1$. Since $\Delta(a_n(\Delta z_n)^{\alpha}) \leqslant 0$, we have

$$a_s^{1/\alpha}\Delta z_s \leqslant a_n^{1/\alpha}\Delta z_n, \quad s \geqslant n \geqslant n_1.$$

Dividing the last inequality by $a_s^{1/\alpha}$ and then summing the resulting inequality from n to ℓ, we obtain

$$z_{\ell+1} \leqslant z_n + a_n^{1/\alpha}\Delta z_n \sum_{s=n}^{\ell} \frac{1}{a_s^{1/\alpha}}, \quad \ell \geqslant n \geqslant n_1. \tag{2.10}$$

Letting $\ell \to \infty$ in (2.10), we obtain

$$\frac{a_n^{1/\alpha}\Delta z_n A(n)}{z_n} \geqslant -1, \quad n \geqslant n_1. \tag{2.11}$$

From (2.9) and (2.11), we obtain

$$-1 \leqslant w_n A^{\alpha}(n) \leqslant 0, \quad n \geqslant n_1. \tag{2.12}$$

Also from (2.11), we have

$$\frac{\Delta z_n}{z_n} \geqslant -\frac{1}{A(n)a_n^{1/\alpha}}, \quad n \geqslant n_1. \tag{2.13}$$

From Lemma 2, we have

$$x_n \geqslant C(n) z_{\tau^{-1}(n)}, \quad n \geqslant n_1. \tag{2.14}$$

From (2.4) and (2.14), we obtain

$$\Delta\Big(a_n(\Delta z_n)^{\alpha}\Big) + Lq_n C^{\alpha}(\sigma(n+1)) z^{\alpha}_{\tau^{-1}(\sigma(n+1))} \leqslant 0, \quad n \geqslant n_1. \tag{2.15}$$

From (2.9) and (2.15) we have

$$\Delta w_n \leqslant -L\overline{Q}_n - w_n \frac{\Delta z_n^{\alpha}}{z_{n+1}^{\alpha}}$$

$$\leqslant -L\overline{Q}_n - \frac{\alpha}{a_n^{1/\alpha}} w_n^{1+1/\alpha}, \quad n \geqslant n_1. \tag{2.16}$$

Multiplying (2.16) by $A^{\alpha}(n+1)$ and summing from n_1 to $n-1$ and then using the summation by parts formula, we obtain

$$\sum_{s=n_1}^{n-1} LA^{\alpha}(s+1)\overline{Q}_s \leqslant A^{\alpha}(n_1) - A^{\alpha}(n)w_n + \sum_{s=n_1}^{n-1}\left[\Delta A^{\alpha}(s)w_s - \frac{\alpha A^{\alpha}(s+1)}{a_s^{1/\alpha}} w_s^{(\alpha+1)/\alpha}\right]. \tag{2.17}$$

By Mean Value Theorem, we have

$$\Delta A^{\alpha}(s)w_s \leqslant \begin{cases} \frac{\alpha A^{\alpha-1}(s)}{a_s^{1/\alpha}}(-w_s) & \text{if} \quad \alpha \geqslant 1 \\ \frac{\alpha A^{\alpha-1}(s+1)}{a_s^{1/\alpha}}(-w_s) & \text{if} \quad \alpha < 1. \end{cases} \tag{2.18}$$

From (2.17) and (2.18) we obtain for $n \geqslant n_1$,

$$\sum_{s=n_1}^{n-1} LA^{\alpha}(s+1)\overline{Q}_s \leqslant A^{\alpha}(n_1) + 1 + \sum_{s=n_1}^{n-1}\left[\frac{\alpha A^{\alpha-1}(s)}{a_s^{1/\alpha}}(-w_s) - \frac{\alpha A^{\alpha}(s+1)}{a_s^{1/\alpha}}(-w_s)^{(\alpha+1)/\alpha}\right] \tag{2.19}$$

for $\alpha \geqslant 1$, and for $0 < \alpha < 1$, we have for $n \geqslant n_1$,

$$\sum_{s=n_1}^{n-1} LA^{\alpha}(s+1)\overline{Q}_s \leqslant A^{\alpha}(n_1) + 1 + \sum_{s=n_1}^{n-1}\left[\frac{\alpha A^{\alpha-1}(s+1)}{a_s^{1/\alpha}}(-w_s) - \frac{\alpha A^{\alpha}(s+1)}{a_s^{1/\alpha}}(-w_s)^{(\alpha+1)/\alpha}\right]. \tag{2.20}$$

By using the inequality

$$AV - BV^{(\alpha+1)/\alpha} \leqslant \frac{\alpha^{\alpha} A^{\alpha+1}}{(\alpha+1)^{\alpha+1} B^{\alpha}} \text{ for } A \geqslant 0,\ B > 0 \text{ and } V > 0,$$

we obtain from (2.19) and (2.20) that

$$\sum_{s=n_1}^{n-1}\left[LA^{\alpha}(s+1)\overline{Q}_s - D(s)\right] \leqslant A^{\alpha}(n_1) + 1.$$

Letting $n \to \infty$ in the last inequality, we obtain a contradiction with (2.3). This completes the proof.

THEOREM 2. *Assume conditions* (H_1)-(H_3) *and* (H_5) *hold. If there exists a positive nondecreasing function* $\{\rho_n\}$ *such that*

$$\sum_{n=n_0}^{\infty}\left[L\rho_n Q_n - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha}}a_n\right] = \infty \tag{2.21}$$

and

$$\sum_{n=n_0}^{\infty}\left[LA^{\alpha}(n+1)\overline{Q}_n\Big(\frac{A(\tau^{-1}(\sigma(n+1)))}{A(n+1)}\Big)^{\alpha} - D(n)\right] = \infty \tag{2.22}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Proceeding as in the proof of Theorem 1, we have (2.4), and thus there exist two possible cases of the sign of $\{\Delta z_n\}$. If $\Delta z_n > 0$, then by Lemma 1, we obtain

$$\Delta\Big(a_n(\Delta z_n)^{\alpha}\Big) + Lq_nB^{\alpha}(\sigma(n+1))z^{\alpha}_{\tau^{-1}(\sigma(n+1))} \leqslant 0. \tag{2.23}$$

Define

$$u_n = \rho_n\frac{a_n(\Delta z_n)^{\alpha}}{z_n^{\alpha}}.$$

Similar as in the proof of Theorem 1 we obtain a contradiction to (2.21).
If $\Delta z_n < 0$, then by using Lemma 2, we obtain

$$x_n \geqslant C(n)z_{\tau^{-1}(n))}. \tag{2.24}$$

Next, define w_n by (2.9) and then by (2.24) we obtain

$$\begin{aligned}\Delta w_n &= \frac{\Delta\Big(a_n(\Delta z_n)^{\alpha}\Big)}{z_{n+1}^{\alpha}} - \frac{a_n(\Delta z_n)^{\alpha}}{z_n^{\alpha}z_{n+1}^{\alpha}}\Delta z_n^{\alpha}\\ &\leqslant -Lq_n\frac{C^{\alpha}(\sigma(n+1))z^{\alpha}_{\tau^{-1}(\sigma(n+1))}}{z_{n+1}^{\alpha}} - \frac{\alpha}{a_n^{1/\alpha}}w_n^{1+1/\alpha}, \quad n \geqslant n_1.\end{aligned}$$

Since $\{\frac{z_n}{A(n)}\}$ is nondecreasing, we have from the last inequality

$$\Delta w_n \leqslant -Lq_nC^{\alpha}(\sigma(n+1))\frac{A^{\alpha}(\tau^{-1}(\sigma(n+1)))}{A^{\alpha}(n+1)} - \frac{\alpha}{a_n^{1/\alpha}}w_n^{1+1/\alpha}, \quad n \geqslant n_1.$$

The rest of the proof is similar to that of Theorem 1 and so is omitted. The proof is now complete.

Now we shall establish some oscillation results for equation (1.1) for the case $\tau(n) \leqslant n$.

THEOREM 3. *Assume conditions* (H_1)-(H_3) *and* (H_6) *hold. If there exists a positive nondecreasing function* $\{\rho_n\}$ *such that*

$$\sum_{n=n_0}^{\infty}\Big[L\rho_n q_n E^{\alpha}(\sigma(n+1)) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha}}a_{\tau^{-1}(\sigma(n))}\Big] = \infty \tag{2.25}$$

and

$$\sum_{n=n_0}^{\infty}\Big[Lq_n A^{\alpha}(n+1)B^{\alpha}(\sigma(n+1)) - D(n)\Big] = \infty \tag{2.26}$$

hold, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_n > 0$, $x_{\tau(n)} > 0$, and $x_{\sigma(n+1)} > 0$ for all $n \geqslant n_1 \in \mathbb{N}(n_0)$. From equation (1.1), we have (2.4). Hence there are two possible cases for the sign of $\{\Delta z_n\}$. If $\Delta z_n > 0$ for all $n \geqslant n_1$, then by Lemma 3, we obtain

$$x_n \geqslant E(n) z_{\tau^{-1}(n)}. \tag{2.27}$$

From (2.4) and (2.27), we have

$$\Delta\Big(a_n(\Delta z_n)^{\alpha}\Big) + Lq_n E^{\alpha}(\sigma(n+1)) z^{\alpha}_{\tau^{-1}(\sigma(n+1))} \leqslant 0, \quad n \geqslant n_1.$$

Define

$$v_n = \frac{\rho_n a_n (\Delta z_n)^{\alpha}}{z^{\alpha}_{\tau^{-1}(\sigma(n))}}.$$

Similar to the proof of Theorem 1, we obtain a contradiction to (2.25).
If $\Delta z_n < 0$, then from the definition of z_n, we have

$$x_n \geqslant \frac{z_{\tau^{-1}(n)}}{p_{\tau^{-1}(n)}} - \frac{z_{\tau^{-1}(\tau^{-1}(n))}}{p_{\tau^{-1}(n)} p_{\tau^{-1}(\tau^{-1}(n))}} \geqslant \frac{1}{p_{\tau^{-1}(n)}}\Big(1 - \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}}\Big) z_{\tau^{-1}(n)}.$$

The remainder of the proof is similar to that of Theorem 1 and hence is omitted. The proof is now complete.

THEOREM 4. *Assume conditions* (H_1)-(H_3) *and* (H_7) *hold. If there exists a positive nondecreasing function* $\{\rho_n\}$ *such that*

$$\sum_{n=n_0}^{\infty}\Big[L\rho_n q_n E^{\alpha}(\sigma(n+1)) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha}}a_n\Big] = \infty \tag{2.28}$$

and

$$\sum_{n=n_0}^{\infty}\Big[Lq_n A^{\alpha}(n+1)C^{\alpha}(\sigma(n+1))\Big(\frac{A(\tau^{-1}(\sigma(n+1)))}{A(n)}\Big)^{\alpha} - D(n)\Big] = \infty \tag{2.29}$$

hold, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x_n > 0$, $x_{\tau(n)} > 0$, and $x_{\sigma(n)} > 0$ for all $n \geqslant n_1 \in \mathbb{N}(n_0)$. From equation (1.1), we have

$$\Delta\Big(a_n(\Delta z_n)^\alpha\Big) \leqslant -Lq_n x^\alpha(\tau^{-1}(\sigma(n+1))) \leqslant 0, \quad n \in \mathbb{N}(n_1). \tag{2.30}$$

Therefore, there are two possible cases for the sign of $\{\Delta z_n\}$. If $\Delta z_n > 0$ for all $n \geqslant n_1$, then from Lemma 3, we have

$$x^\alpha_{\sigma(n+1)} \geqslant E^\alpha(\sigma(n+1)) z^\alpha_{\tau^{-1}(\sigma(n+1))}. \tag{2.31}$$

Define

$$v_n = \frac{\rho_n a_n (\Delta z_n)^\alpha}{z_n^\alpha}.$$

Proceeding as in the proof of Theorem 1, we obtain a contradiction to (2.28).
If $\Delta z_n < 0$ for $n \geqslant n_1$, then as in the proof of Theorem 3, we obtain

$$x_n \geqslant B(n) z_{\tau^{-1}(n)}.$$

On the other hand, by the proof of Lemma 1, we see that $\frac{z_n}{A(n)}$ is nondecreasing. Thus, we have

$$\frac{z_{\tau^{-1}(\sigma(n+1))}}{z_n} \geqslant \frac{A(\tau^{-1}(\sigma(n+1)))}{A(n)}.$$

The rest of the proof is similar to that of Theorem 1 and hence is omitted. The proof is now complete.

3. Examples

In this section, we present some examples to illustrate our main results.

EXAMPLE 1. Consider the difference equation

$$\Delta\Big(2^n \Delta(x_n + 8x_{n+2})\Big) + 54(2^n) x_{n^2-1} = 0, n \geqslant 1. \tag{3.1}$$

Here $\alpha = 1$, $L = 1$, $\tau(n) = n+2$, $\sigma(n+1) = n^2 - 1$,

$$A(n) = \frac{1}{2^{n-1}}, \; Q_n = 189(2^{n-5}), \; \overline{Q}_n = 27(2^{n-3}) \quad \text{and} \quad D(n) = \frac{1}{4}.$$

By taking $\rho_n = 1$, we see that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (3.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (3.1).

EXAMPLE 2. Consider the difference equation

$$\Delta\Big(2^n\Delta(x_n+8x_{n-2})\Big)+54(2^n)x_{n^2-1}=0, n\geqslant 1. \tag{3.2}$$

Here $\alpha=1$, $L=1$, $\tau(n)=n-2$, $\sigma(n+1)=n^2-1$,

$$E(n)=\frac{1}{8}\Big(1-\frac{1}{32}\frac{2^{n+3}-1}{2^{n+1}-1}\Big) \quad \text{and} \quad A(n)=\frac{1}{2^{n-1}}.$$

By taking $\rho_n=1$, we see that all conditions of Theorem 2.4 are satisfied and hence every solution of equation (3.2) is oscillatory. In fact $\{x_n\}=\{(-1)^n\}$ is one such solution of equation (3.2).

We conclude this paper with the following remark.

REMARK 1. The results obtained here improve some of the existing results in the literature. Also the theorems obtained in [13] cannot be applied to equations (3.1) and (3.2) since $\tau\circ\sigma\neq\sigma\circ\tau$.
Further the results obtained in this paper can be extended to

$$\Delta(a_n|\Delta z_n|^{\alpha-1}\Delta z_n)+q_n|x_{\sigma(n)}|^{\alpha-1}x_{\sigma(n)}=0$$

where $\alpha>0$, without any difficulty, and hence the details are left to the reader.

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