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OSCILLATION OF DELAY DYNAMIC EQUATIONS WITH OSCILLATING COEFFICIENTS

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Abstract. In this paper, we study the following delay dynamic equation

$$x^{\Delta}(t)+p(t)x(\tau(t))=0 \text{ for } t\in[t_0,\infty)_{\mathbb{T}},$$

where $t_0\in\mathbb{T}$, $\sup\mathbb{T}=\infty$, $p\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ alternates in sign infinitely many times and $\tau\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ is a strictly increasing unbounded function satisfying $\tau(t)\leqslant t$ for all $t\in[t_0,\infty)_{\mathbb{T}}$. Our results extend recent results for arbitrary time scales.

1. Introduction

In 1988, Hilger introduced the theory of time scales in order to unify continuous and discrete calculus in his Ph. D. thesis (see [9]). This theory received attention by the researchers studying differential and difference equations. For the fundamentals of the time scale theory the readers are referred to the books [3, 4], which summarize and organize much of the time scale calculus.

In the papers [2, 5, 6, 7, 10, 11, 16], the readers may find studies on the oscillation and nonoscillation of the following type of delay dynamic equations

$$x^{\Delta}(t)+p(t)x(\tau(t))=0 \text{ for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{1}$$

where $t_0\in\mathbb{T}$, $\sup\mathbb{T}=\infty$, $p\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^{+})$ and $\tau\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ is a strictly increasing unbounded function satisfying $\tau(t)\leqslant t$ for all $t\in[t_0,\infty)_{\mathbb{T}}$.

If $\mathbb{T}=\mathbb{R}$, then $x^{\Delta}=x'$ (the usual derivative), while if $\mathbb{T}=\mathbb{Z}$, then $x^{\Delta}=\Delta x$ (the usual forward difference). On a time scale, the *forward jump operator* and the *graininess function* is defined as follows

$$\sigma(t):=\inf(t,\infty)_{\mathbb{T}} \quad\text{and}\quad \mu(t):=\sigma(t)-t,$$

where $(t,\infty)_{\mathbb{T}}:=(t,\infty)\cap\mathbb{T}$ and $t\in\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}$ is called *positively regressive* if $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $1+\mu(t)f(t)>0$ for all $t\in\mathbb{T}$, and such a function is represented as $f\in\mathscr{R}^{+}(\mathbb{T},\mathbb{R})$. It is

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well known that if $f \in \mathscr{R}^+([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, then there exists a positive function x satisfying the initial value problem

$$x^{\Delta}(t) = f(t)x(t) \quad \text{with} \quad x(t_0) = 1,$$

where $t_0 \in \mathbb{T}$ and $t \in [t_0,\infty)_{\mathbb{T}}$, and it is called the *exponential function* and denoted by $\mathrm{e}_f(\cdot,t_0)$, another definition for the exponential function is

$$\mathrm{e}_f(t,s) := \exp\left\{\int_s^t \xi_{\mu(\eta)}(f(\eta))\Delta\eta\right\},$$

where $s,t \in [t_0,\infty)_{\mathbb{T}}$ and the *cylinder transformation* is defined by

$$\xi_h(\lambda) := \begin{cases} \lambda, & h = 0 \\ \dfrac{1}{h}\mathrm{Log}(1+h\lambda), & h > 0 \end{cases}$$

for $\lambda \in \mathbb{R}$ and $h \in \mathbb{R}^+$. For useful properties of the exponential function, the readers may refer to [3, Theorem 2.36].

Unlike the previously mentioned papers, in this work, we study (1) with the following primary assumptions:

(H1) $\tau : [t_0,\infty)_{\mathbb{T}} \to \mathbb{T}$ is a strictly increasing unbounded function, which satisfies $\tau(t) \leqslant t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$,

(H2) $p \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is allowed to oscillate,

(H3) $\{\vartheta_k\}_{k\in\mathbb{Z}}, \{\zeta_k\}_{k\in\mathbb{Z}} \subset [t_0,\infty)_{\mathbb{T}}$ are strictly increasing divergent sequences satisfying the following three conditions:

(i) $p(t) \geqslant 0 (\not\equiv 0)$ for all $t \in \bigcup_{i\in\mathbb{N}}[\vartheta_i,\zeta_i]_{\mathbb{T}}$,

(ii) $p(t) \leqslant 0$ for all $t \in [t_0,\infty)_{\mathbb{T}} \setminus \bigcup_{i\in\mathbb{N}}(\vartheta_i,\zeta_i)_{\mathbb{T}}$,

(iii) $\sup I_1 = \infty$, where

$$I_k := \bigcup_{i=k}^{\infty}[\tau^{-2}(\vartheta_i),\zeta_i]_{\mathbb{T}} \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad \tau^{-2} := \tau^{-1} \circ \tau^{-1}.$$

The three properties in (H3) imply that $\zeta_k \in [\vartheta_k,\vartheta_{k+1}]_{\mathbb{T}}$ and $\vartheta_k \in [\zeta_k,\zeta_{k+1}]_{\mathbb{T}}$ hold for all $k \in \mathbb{N}$. From now on, we always suppose without furthermore mentioning that (H1)–(H3) hold.

For delay differential equations with oscillating coefficients, the readers are referred to the paper [13], and for delay difference equations with oscillating coefficients, the readers are refereed to the papers [12, 14, 15].

Set $t_{-1} := \tau(t_0)$. As is customary, by a *solution* of (1), we mean a real valued rd-continuous function, which is defined on $[t_{-1},\infty)_{\mathbb{T}}$ and has a rd-continuous delta-derivative on $[t_0,\infty)_{\mathbb{T}}$ and identically satisfies (1). A solution is called *oscillatory* if it alternates in sign infinitely many times; otherwise, it is called *nonoscillatory*.

2. Main Results

In this section, we give our main results.

THEOREM 1. *Assume that*

$$\limsup_{k\to\infty} \sup_{t\in I_k} \int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta > 1, \tag{2}$$

then every solution of (1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1), which can be assumed to be eventually positive without loss of generality since (1) is linear. Let $x(t), x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$ for some fixed $t_1 \in [t_0,\infty)_{\mathbb{T}}$, and pick $k_1 \in \mathbb{N}$ satisfying

$$\bigcup_{i=k_1}^{\infty} [\vartheta_i, \zeta_i]_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}.$$

From (1), we get

$$x^{\Delta}(t) = -p(t)x(\tau(t)) \leqslant 0 \quad \text{for all } t \in \bigcup_{i=k_1}^{\infty} [\vartheta_i, \zeta_i]_{\mathbb{T}}, \tag{3}$$

which indicates that x is nonincreasing on $[\vartheta_k, \zeta_k]_{\mathbb{T}}$ for all $k \in [k_1,\infty)_{\mathbb{Z}}$. For any $k \in [k_1,\infty)_{\mathbb{Z}}$, let $t \in I_{k_1}$, then we learn that x is nonincreasing on $[\tau^2(t), \tau(t)]_{\mathbb{T}}$ since $[\tau^2(t), \tau(t)]_{\mathbb{T}} \subset [\tau^{-2}(\vartheta_k), \zeta_k]_{\mathbb{T}} \subset [\vartheta_k, \zeta_k]_{\mathbb{T}}$ and $p \geqslant 0$ on $[\tau(t), t]_{\mathbb{T}} \subset [\tau^{-2}(\vartheta_k), \zeta_k]_{\mathbb{T}} \subset [\vartheta_k, \zeta_k]_{\mathbb{T}}$. Therefore, integrating (1) from $\tau(t)$ to $\sigma(t)$, where $t \in I_{k_1}$ for some $k \in [k_1,\infty)_{\mathbb{Z}}$, we get

$$\begin{aligned} 0 =& x(\sigma(t)) - x(\tau(t)) + \int_{\tau(t)}^{\sigma(t)} p(\eta)x(\tau(\eta))\Delta\eta \\ \overset{(3)}{\geqslant}& x(\sigma(t)) + \left(\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta - 1\right)x(\tau(t)), \end{aligned}$$

which proves

$$\int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta < 1 \quad \text{for all } t \in I_{k_1}. \tag{4}$$

Clearly, (4) contradicts (2). The proof is therefore completed. □

Next, we give another theorem.

THEOREM 2. *Assume that*

$$\liminf_{k\to\infty} \inf_{t\in I_k} \inf_{\substack{\lambda>0 \\ -\lambda p\in\mathscr{R}^+([\tau(t),t)_{\mathbb{T}},\mathbb{R})}} \left\{ \frac{1}{\lambda \mathrm{e}_{-\lambda p}(t,\tau(t))} \right\} > 1, \tag{5}$$

then every solution of (1) *is oscillatory.*

Proof. Suppose that x is a nonoscillatory solution of (1), and without loss of generality let x be an eventually positive solution. Say $x(t), x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$ for some fixed $t_1 \in [t_0, \infty)_{\mathbb{T}}$, and let $k_1 \in \mathbb{N}$ satisfy $\bigcup_{i=k_1}^{\infty}[\vartheta_i, \zeta_i]_{\mathbb{T}} \subset [t_1, \infty)_{\mathbb{T}}$. From (3), we learn that x is nonincreasing on $[\vartheta_k, \zeta_k]_{\mathbb{T}}$ for all $k \in [k_1, \infty)_{\mathbb{T}}$. Set

$$y(t) := \frac{x(\tau(t))}{x(t)} \quad \text{for } t \in J_{k_1}, \tag{6}$$

where

$$J_k := \bigcup_{i=k}^{\infty} [\tau^{-1}(\vartheta_i), \zeta_i]_{\mathbb{T}} \quad \text{for } k \in \mathbb{N}$$

Then, from (1) and (6), we get

$$x^{\Delta}(t) + p(t)y(t)x(t) = 0 \quad \text{for all } t \in J_{k_1}. \tag{7}$$

Integrating (7) from t to $\sigma(t)$, where $t \in J_{k_1}$, we see that

$$0 = x(\sigma(t)) - x(t) + \mu(t)p(t)y(t)x(t) > -x(t)\big[1 - \mu(t)y(t)p(t)\big], \tag{8}$$

which proves $-yp \in \mathscr{R}^+(J_{k_1}, \mathbb{R})$. From (7), we see that

$$x(t) = \mathrm{e}_{-yp}(t, t_1)x(t_1) \quad \text{for all } t \in J_{k_1}. \tag{9}$$

Substituting (9) into (6), we obtain

$$y(t) = \frac{1}{\mathrm{e}_{-yp}(t, \tau(t))} \quad \text{for all } t \in J_{k_1}. \tag{10}$$

Now, set

$$z(t) := \inf\big\{y(\eta) : \eta \in [\tau(t), t)_{\mathbb{T}}\big\} \quad \text{for } t \in I_{k_2}, \tag{11}$$

where $t_2 \in [\tau^{-1}(t_1), \infty)_{\mathbb{T}}$ and $I_{k_2} \subset [t_2, \infty)_{\mathbb{T}}$. It is obvious that $-z(t)p \in \mathscr{R}^+([\tau(t), t)_{\mathbb{T}}, \mathbb{R})$ for all $t \in I_{k_2}$. We deduce that

$$y(t) \overset{(10),(11)}{\geqslant} \frac{1}{\mathrm{e}_{-z(t)p}(t, \tau(t))} = \frac{1}{z(t)\mathrm{e}_{-z(t)p}(t, \tau(t))} z(t) \tag{12}$$

holds for all $t \in I_{k_2}$. Now, we prove

$$\liminf_{k \to \infty} \inf_{t \in I_k} y(t) = \infty. \tag{13}$$

Note that

$$\liminf_{k \to \infty} \inf_{t \in I_k} y(t) \geqslant \liminf_{k \to \infty} \inf_{t \in J_k} y(t) \geqslant 1 \tag{14}$$

holds because of the nondecreasing nature of x on $[\vartheta_k, \zeta_k]_{\mathbb{T}}$ for all $k \in [k_1, \infty)_{\mathbb{N}}$ and the definition of y in (6). Clearly, if

$$\liminf_{k \to \infty} \inf_{t \in I_k} y(t) < \infty \tag{15}$$

holds, then we obtain

$$
\begin{aligned}
\liminf_{k\to\infty\, t\in I_k} y(t) &\overset{(12)}{\geqslant} \liminf_{k\to\infty\, t\in I_k}\left(\frac{1}{z(t)\mathrm{e}_{-z(t)p}(t,\tau(t))}z(t)\right)\\
&\geqslant \left(\liminf_{k\to\infty\, t\in I_k}\frac{1}{z(t)\mathrm{e}_{-z(t)p}(t,\tau(t))}\right)\left(\liminf_{k\to\infty\, t\in I_k} z(t)\right)\\
&\overset{(11)}{=} \left(\liminf_{k\to\infty\, t\in I_k}\frac{1}{z(t)\mathrm{e}_{-z(t)p}(t,\tau(t))}\right)\left(\liminf_{k\to\infty\, t\in I_k} y(t)\right), \qquad (16)
\end{aligned}
$$

which yields

$$
\begin{aligned}
1 &\overset{(14),(16)}{\geqslant} \liminf_{k\to\infty\, t\in I_k}\frac{1}{z(t)\mathrm{e}_{-z(t)p}(t,\tau(t))}\\
&\geqslant \liminf_{k\to\infty\, t\in I_k} \inf_{\substack{\lambda>0\\ -\lambda p\in\mathscr{R}^+([\tau(t),t)_{\mathbb{T}},\mathbb{R})}}\left\{\frac{1}{\lambda\mathrm{e}_{-\lambda p}(t,\tau(t))}\right\}. \qquad (17)
\end{aligned}
$$

This indicates that (13) holds since (17) is contradicts (5).

To complete the proof, it suffices to prove that (15) holds. Consider (5) and let $t\in I_{k_2}$, where $k_2\in[k_1,\infty)_{\mathbb{Z}}$ satisfies $I_{k_2}\subset[t_2,\infty)_{\mathbb{T}}$ and

$$\frac{1}{\mathrm{e}_{-p}(t,\tau(t))}\geqslant\alpha \qquad (18)$$

for some fixed real $\alpha>1$ since $z(t)\geqslant 1$ holds by the nonincreasing nature of x and this implies $-p\in\mathscr{R}^+([\tau(t),t)_{\mathbb{T}},\mathbb{R})$. From (18) and [5, Lemma 2], for all $t\in I_{k_2}$, we get

$$\int_{\tau(t)}^{t} p(\eta)\Delta\eta\geqslant 2\beta, \quad \text{where } \beta:=\frac{1}{2}\left(1-\frac{1}{\alpha}\right). \qquad (19)$$

For each $k\in[k_2,\infty)_{\mathbb{Z}}$, let $t\in[\tau^{-2}(\vartheta_k),\zeta_k]_{\mathbb{T}}$, and define the function $\Psi_k:[\tau(t),\sigma(t))_{\mathbb{T}}\to\mathbb{R}^+$ by

$$\Psi_k(s):=\int_{\tau(t)}^{s} p(\eta)\Delta\eta-\beta. \qquad (20)$$

Therefore, there exists $\varsigma_k\in[\tau(t),\sigma(t))_{\mathbb{T}}$ such that $\Psi_k^{\sigma}(\varsigma_k)\geqslant 0$ and $\Psi_k(\varsigma_k)\leqslant 0$ hold for all $k\in[k_2,\infty)_{\mathbb{Z}}$ and all $t\in[\tau^{-2}(\vartheta_k),\zeta_k]_{\mathbb{T}}$. Then, we see that

$$\int_{\tau(t)}^{\sigma(\varsigma_k)} p(\eta)\Delta\eta\geqslant\beta \qquad (21)$$

and

$$\int_{\varsigma_k}^{\sigma(t)} p(\eta)\Delta\eta \overset{(20)}{=} \int_{\tau(t)}^{\sigma(t)} p(\eta)\Delta\eta-\big(\Psi_k(\varsigma_k)+\beta\big) \overset{(19)}{\geqslant} 2\beta-\big(\Psi_k(\varsigma_k)+\beta\big)\geqslant\beta \qquad (22)$$

hold for all $t \in I_{k_2}$. Moreover, for each $k \in [k_2, \infty)_{\mathbb{N}}$ and all $t \in [\tau^{-2}(\vartheta_k), \zeta_k]_{\mathbb{T}}$, we see that x is nonincreasing on $[\tau^2(t), \tau(t)]_{\mathbb{T}}$ since

$$[\tau^2(t), \tau(t)]_{\mathbb{T}} \subset [\vartheta_k, \tau(\zeta_k)]_{\mathbb{T}} \subset [\vartheta_k, \zeta_k]_{\mathbb{T}}.$$

Thus, for each $k \in [k_2, \infty)_{\mathbb{N}}$ and all $t \in [\tau^{-2}(\vartheta_k), \zeta_k]_{\mathbb{T}}$, we can deduce

$$\begin{aligned} x(\varsigma_k) \geqslant x(\varsigma_k) - x^\sigma(t) &= -\int_{\varsigma_k}^{\sigma(t)} x^\Delta(\eta)\Delta\eta \overset{(1)}{=} \int_{\varsigma_k}^{\sigma(t)} p(\eta)x(\tau(\eta))\Delta\eta \\ &\overset{(3)}{\geqslant} x(\tau(t)) \int_{\varsigma_k}^{\sigma(t)} p(\eta)\Delta\eta \geqslant \beta x(\tau(t)) \overset{(22)}{\geqslant} \beta\big(x(\tau(t)) - x^\sigma(\varsigma_k)\big) \\ &= -\beta \int_{\tau(t)}^{\sigma(\varsigma_k)} x^\Delta(\eta)\Delta\eta = \beta \int_{\tau(t)}^{\sigma(\varsigma_k)} p(\eta)x(\tau(\eta))\Delta\eta \\ &\overset{(3)}{\geqslant} \beta x(\tau(\varsigma_k)) \int_{\tau(t)}^{\sigma(\varsigma_k)} p(\eta)\Delta\eta \overset{(21)}{\geqslant} \beta^2 x(\tau(\varsigma_k)), \end{aligned} \tag{23}$$

which indicates that

$$y(\varsigma_k) \overset{(6),(23)}{\leqslant} \frac{1}{\beta^2} \quad \text{for all } t \in I_{k_2}. \tag{24}$$

Clearly, (24) proves (15) but this contradicts (13). This completes the proof. □

3. Applications

In this section, we give some applications to illustrate the applicability of our results.

EXAMPLE 1. Let $\mathbb{T} = \mathbb{Z}$, $\tau(t) = t - 1$, and $p(5k) = -1$, $p(5k+1) = p(5k+2) = 0$ and $p(5k+3) = p(5k+4) = 3/4$ for $k \in \mathbb{N}$ in (1) to obtain the following delay difference equation

$$\Delta x(t) + p(t)x(t-1) = 0 \text{ for } t \in \mathbb{N}. \tag{25}$$

For this equation, letting $\vartheta_k = 5k+1$ and $\zeta_k = 5k+4$ for $k \in \mathbb{N}$, we see that all assumptions of Theorem 1 hold. In fact, we have

$$\lim_{k\to\infty} \sup_{t \in \bigcup_{i=k}^{\infty} [5k+3, 5k+4]_{\mathbb{Z}}} \sum_{i=t-1}^{t} p(i) = \lim_{k\to\infty} \sum_{i=5k+3}^{5k+4} p(i) = \frac{3}{2} > 1.$$

Thus, every solution of (25) oscillates.

EXAMPLE 2. Let $\mathbb{T} = \mathbb{R}$, $\tau(t) = t - \pi/3$ and $p(t) = \sin(t)$ in (1) to obtain the following delay differential equation

$$x'(t) + \sin(t)x(t - \pi/3) = 0 \text{ for } t \in \mathbb{R}^+. \tag{26}$$

For this equation, letting $\vartheta_k = 2k\pi$ and $\zeta_k = 2k\pi + \pi$ for $k \in \mathbb{N}$, we see that all assumptions of Theorem 2 hold since

$$\begin{aligned}
&\lim_{k\to\infty} \inf_{t\in\bigcup_{i=k}^{\infty}[2k\pi+2\pi/3,2k\pi+\pi]_{\mathbb{R}}} \inf_{\lambda\in\mathbb{R}} \left\{\frac{1}{\lambda}\exp\left\{\lambda\int_{t-\pi/3}^{t} p(\eta)\mathrm{d}\eta\right\}\right\} \\
&\quad = \lim_{k\to\infty} \inf_{t\in\bigcup_{i=k}^{\infty}[2k\pi+2\pi/3,2k\pi+\pi]_{\mathbb{R}}} \mathrm{e}\int_{t-\pi/3}^{t} p(\eta)\mathrm{d}\eta \\
&\quad = \lim_{k\to\infty} \mathrm{e}\int_{2k\pi+2\pi/3}^{2k\pi+\pi} p(\eta)\mathrm{d}\eta = \frac{\mathrm{e}}{2} > 1.
\end{aligned}$$

Thus, every solution of (26) oscillates.

EXAMPLE 3. Let $q \in (1,\infty)_{\mathbb{R}}$, $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $\tau(t) = t/q$ and

$$\begin{aligned}
p(q^{4k}) = -\alpha/((q-1)q^{4k}),\ p(q^{4k+1}) = 0, \\
p(q^{4k+2}) = \alpha/((q-1)q^{4k+2}),\ p(q^{4k+3}) = 0 \ \text{ for } k \in \mathbb{N}_0,
\end{aligned}$$

where α is a positive constant, in (1) to obtain the following delay q-difference equation

$$\mathrm{D}_q x(t) + p(t)x(t/q) = 0 \text{ for } t \in q^{\mathbb{N}_0}, \tag{27}$$

where

$$\mathrm{D}_q x(t) := \frac{x(qt) - x(t)}{(q-1)t}.$$

For this equation, letting $\vartheta_k = q^{4k+1}$ and $\zeta_k = q^{4k+3}$ for $k \in \mathbb{N}$, we deduce

$$\begin{aligned}
&\lim_{k\to\infty} \inf_{t\in\bigcup_{i=k}^{\infty}[q^{4k+3},q^{4k+3}]_{q^{\mathbb{Z}}\cup\{0\}}} \inf_{\substack{\lambda>0 \\ 1-(q-1)tp(t/q)/q>0}} \left\{\frac{1}{\lambda(1-\lambda(q-1)tp(t/q)/q)}\right\} \\
&\quad = \lim_{k\to\infty} \inf_{\substack{\lambda>0 \\ 1-\alpha\lambda>0}} \left\{\frac{1}{\lambda(1-\lambda(q-1)q^{4k+3}p(q^{4k+3}/q)/q)}\right\} \\
&\quad = \inf_{\substack{\lambda>0 \\ 1-\alpha\lambda>0}} \left\{\frac{1}{\lambda(1-\alpha\lambda)}\right\} \geqslant \inf_{\lambda>0} \left\{\frac{1}{\lambda(1-\alpha\lambda)}\right\} = 4\alpha
\end{aligned}$$

Therefore, by Theorem 2, every solution of (27) oscillates provided that $\alpha > 1/4$.

4. Final Comments

In § 2, we extended the results in [5, 6, 11, 16] to (1), where p alternates in sign infinitely many times. However, when p is eventually positive, we see that (H3)(iii) does not hold. In such a case, we see that Theorem 1 reduces to [11, Theorem 2.4] and Theorem 2 reduces to [5, Theorem 1], [6, Theorem 2($n_0 = 1$)] and [16, Theorem 1].

It must be mentioned here that the respective conditions

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(\eta)\mathrm{d}\eta>1\quad\text{and}\quad\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(\eta)\mathrm{d}\eta>\frac{1}{\mathrm{e}}$$

of Theorems 1 and 2 for the case $\mathbb{T}=\mathbb{R}$ with $p(t)\geqslant 0$ for $t\in[t_0,\infty)_{\mathbb{R}}$ are known to be critical (see [1, Theorem 2.9]), i.e., for any constant $c\in(1/\mathrm{e},1)$, there exists a nonoscillatory solution of (1) for the case $\mathbb{T}=\mathbb{R}$ with $p(t)\geqslant 0$ for $t\in[t_0,\infty)_{\mathbb{R}}$ such that

$$\sup_{t\geqslant t_0}\int_{\tau(t)}^{t}p(\eta)\mathrm{d}\eta=c.$$

The monotonicity of the delay function in the proof of Theorem 1 is essential since we require the monotonicity of $x\circ\tau$, when the nonoscillatory solution x is monotonic. However, this can be removed by using the technique introduced in the recent paper [8] for differential and difference equations.

In the paper [2], the results in [5, 16] for (1) are extended to following type of delay dynamic equations including several coefficients:

$$x^{\Delta}(t)+\sum_{i=1}^{n}p_i(t)x(\tau_i(t))=0\text{ for }t\in[t_0,\infty)_{\mathbb{T}},\tag{28}$$

where $p_i\in\mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $\tau_i\in\mathrm{C_{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are strictly increasing unbounded functions satisfying $\tau_i(t)\leqslant t$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ and all $i\in[1,n]_{\mathbb{Z}}$. One can easily extend our results in § 2 given for (1) to (28), where p_i are allowed to oscillate for all $i\in[1,n]_{\mathbb{Z}}$, by applying very similar arguments to that in the proofs of our main results. In this case, the results may require the maximal delay ($\tau_{\max}(t):=\max_{i=1,2,\cdots,n}\tau_i(t)$, $t\in[t_0,\infty)_{\mathbb{T}}$) together with a common interval of positivity of all coefficients.

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