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SUFFICIENT CONDITIONS FOR EXISTENCE OF SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. We provide easily verifiable sufficient conditions for the existence of solutions to nonlinear ordinary differential equations subject to nonlocal boundary conditions. These conditions are based on the solution space of the corresponding linear, homogeneous problem and on the size of the nonlinear perturbation. The results presented here are more general nonlinearities than those in [17] and [26].

1. Introduction

In this paper, we consider boundary value problems of the form

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=f(y(t),y'(t))+\varepsilon G(y,\cdots,y^{(n-1)})(t) \tag{1}$$

subject to

$$\sum_{j=1}^{n} b_{ij}(0)y^{(j-1)}(0)+\sum_{j=1}^{n} b_{ij}(t_1)y^{(j-1)}(t_1)+\cdots+\sum_{j=1}^{n} b_{ij}(t_N)y^{(j-1)}(t_N)=0 \tag{2}$$

for $i=1,2,\cdots,n$, and for $0\leqslant t\leqslant 1$. The points t_k for $k=0,1,\cdots,n$ are fixed and $0=t_0<t_1<\cdots<t_N=1$.

We assume that $f:\mathbb{R}^2\to\mathbb{R}$ is continuous and that the limits $f(\infty,\infty)$, $f(\infty,-\infty)$, $f(-\infty,\infty)$ and $f(-\infty,-\infty)$ exist. The map G is a continuous, nonlinear operator on the space of $C^{(n-1)}$ functions. Some examples for the nonlinear operator G include $G(y,y',\cdots,y^{(n-1)})(t)=g(y(t),y'(t),\cdots,y^{(n-1)}(t))$, where g is a continuous, real-valued mapping; and $G(y,y',\cdots,y^{(n-1)})(t)=\int_0^1 k(t,s)H(y(s),y'(s),\cdots,y^{(n-1)}(s))ds$, which would allow the reader to consider integro-differential equations.

We devote our study to problems where the corresponding linear, homogeneous boundary value problem

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=0 \tag{3}$$

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subject to (2) has a one dimensional solution space. For these problems, we provide sufficient conditions for the existence of solutions to (1),(2). Our conditions are based on the limiting behavior of the real valued function f, the properties of the solution space of the linear homogeneous boundary value problem (3)-(2), and the behavior of the nonlinear map G.

In [26], Rodriguez and Taylor approach a similar problem with less general nonlinearities using the Lyapunov-Schmidt Procedure. Due to the multipoint boundary conditions, this approach required a Lipschitz condition on the nonlinear term. In [17], Rodriguez was able to approach the problem in a more direct manner that allowed the author to eliminate the need for a Lipschitz condition. The results we present here allow us to establish the solvability of boundary value problems that do not fall within the scope of the results previously obtained by Rodriguez [17]. Approaches similar to the one presented in this paper have been successfully used in the analysis of periodic behavior in discrete and continuous dynamical systems [3], [4], [6], [9], [13], boundary value problems for differential and difference equations [1], [7], [8], [12], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], and more general systems [2], [10], [28].

2. Preliminaries

In order to analyze the boundary value problem (1),(2), we formulate it in system form.

The matrix $A(t)$ is defined by

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{bmatrix}.$$

The vector

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is given by $x_1 = y,\ x_2 = y',\ \cdots,\ x_n = y^{(n-1)}$ and the boundary matrices $B_0,\ B_1,\ \cdots,$ B_N are given by

$$B_l = [b_{ij}(l)].$$

Throughout this discussion, we will assume that the augmented $n \times n(N+1)$ matrix $[B_0|B_1|\cdots|B_N]$ has full rank. This condition is to ensure that the boundary conditions are not redundant.

For

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$\mathscr{F}(x)(t):\mathbb{R}^n \to \mathbb{R}^n$ is defined by $\mathscr{F}(x)(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f(x_1(t),x_2(t)) \end{bmatrix}$ and $\mathscr{G}(x)(t):\mathbb{R}^n \to \mathbb{R}^n$ is given by $\mathscr{G}(x)(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ G(x(t)) \end{bmatrix}$.

It is clear that the boundary value problem (1),(2) is equivalent to

$$\dot{x}(t) = A(t)x(t) + \mathscr{F}(x(t)) + \varepsilon\mathscr{G}(x(t)), 0 \leqslant t \leqslant 1 \tag{4}$$

subject to

$$\sum_{k=0}^{N} B_k x(t_k) = 0. \tag{5}$$

The solution space to the corresponding linear problem

$$\dot{x}(t) = A(t)x(t) \tag{6}$$

subject to boundary conditions (5) will play a crucial role in solving (4),(5).

Throughout the paper we will assume that $f:\mathbb{R}^2 \to \mathbb{R}$ is continuous and that it has finite limits at (∞,∞), $(\infty,-\infty)$, $(-\infty,\infty)$, and $(-\infty,-\infty)$. We write

$$f(\infty,\infty) = \lim_{(s,t)\to(\infty,\infty)} f(s,t),$$

$$f(\infty,-\infty) = \lim_{(s,t)\to(\infty,-\infty)} f(s,t),$$

$$f(-\infty,\infty) = \lim_{(s,t)\to(-\infty,\infty)} f(s,t),$$

and

$$f(-\infty,-\infty) = \lim_{(s,t)\to(-\infty,-\infty)} f(s,t).$$

For any integer $p \geqslant 1$ the space $(\mathscr{C}([0,1],\mathbb{R}^p)$ will denote

$$\{\phi : [0,1] \to \mathbb{R}^p : \phi \text{ is continuous}\}.$$

The norm used on this space is the sup norm; this is, $\|\phi\|_\infty = \sup\{|\phi(t)| : 0 \leqslant t \leqslant 1\}$ where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^p$. For $n \geqslant 2$, let $\mathscr{C}^{(n-1)}([0,1],\mathbb{R}^p)$ denote the collection of functions $\phi : [0,1] \to \mathbb{R}^p$ such that ϕ has $n-1$ continuous derivatives. We define $\|\phi\| = \sup_{0\leqslant k\leqslant n-1}\{\sup\{|\phi^{(k)}(t)| : 0 \leqslant t \leqslant 1\}\}$. The map $G : \mathscr{C}^{(n-1)}([0,1],\mathbb{R}^n) \to \mathscr{C}^{(n-1)}([0,1],\mathbb{R})$ is continuous.

We will denote the principal matrix solution at $t = 0$ of $\dot{x}(t) = A(t)x(t)$ by $\Gamma(t)$ and we define the matrix D by

$$D = \sum_{k=0}^{N} B_k \Gamma(t_k).$$

We notate V as an arbitrary, fixed subspace of $\mathbb{R}^n$ so that

$$\ker(D) \oplus V = \mathbb{R}^n.$$

It is well-known that x is a solution to (6), (5) if and only if

$$x(t) = \Gamma(t)v$$

where $\Gamma(t)$ is the principal matrix solution of $\dot{x}(t) = A(t)x(t)$ and $v \in \ker(D)$.

In this paper, we consider the case when the dimension of the solution space of (6), (5) is one, or equivalently, $\dim(\ker(D)) = 1$. Since we are considering the case where $\ker(D)$ is one-dimensional, we may assume that v spans $\ker(D)$ and

$$\int_0^1 |\Gamma(t)v|^2 dt = 1.$$

The following construction appears in [23]. We define

$$\Psi(t) = \begin{cases} [(B_1\Gamma(t_1) + B_2\Gamma(t_2) + \cdots + B_N\Gamma(t_N))\Gamma^{-1}(t)]^T c & \text{for } 0 \leqslant t \leqslant t_1, \\ [(B_2\Gamma(t_2) + \cdots + B_N\Gamma(t_N))\Gamma^{-1}(t)]^T c & \text{for } t_1 < t \leqslant t_2, \\ \vdots & \vdots \\ [B_N\Gamma(t_N)\Gamma^{-1}(t)]^T c & \text{for } t_{N-1} < t \leqslant t_N, \end{cases}$$

where $c \in \ker(D^T)$ so that

$$\int_0^1 |\Psi(t)|^2 dt = 1.$$

Except for minor details, the following proof is contained in [17]. We omit the details.

PROPOSITION 1. *For a fixed ε, and for each continuous function $x : [0,1] \to \mathbb{R}^n$, there exists a unique $v_x \in V$ such that*

$$Dv_x = -\sum_{k=1}^{N} B_k\Gamma(t_k) \int_0^{t_k} \Gamma^{-1}(s)[(\mathscr{F}(x(s)) + \varepsilon\mathscr{G}(x(s))) - \left(\int_0^1 \Psi^T(u)(\mathscr{F}(x(u)) + \varepsilon\mathscr{G}(x(u)))du\right)\Psi(s)]ds.$$

Furthermore, if $\varepsilon = 0$, there is a constant K such that $|v_x| \leqslant K$ for all $x \in \mathscr{C}([0,1], \mathbb{R}, \|\cdot\|_\infty)$.

3. Fixed Points

Let $\Phi(t) = \Gamma(t)p$, where $p \in \ker(D)$, and

$$w(x(t)) = \Gamma(t)v_x + \Gamma(t)\int_0^t \Gamma^{-1}(s)[(\mathscr{F}(x(s)) + \varepsilon\mathscr{G}(x(s))) - \left(\int_0^1 \Psi^T(u)(\mathscr{F}(x(u)) + \varepsilon\mathscr{G}(x(u)))du\right)\Psi(s)]ds.$$

We will use $\Phi_i(t)$, $\Psi_i(t)$, and $w_i(t)$ to denote the ith entries of $\Phi(t)$, $\Psi(t)$, and $w(t)$, respectively.

We define mappings

$$H_1 : \mathbb{R}\times\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty) \to (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$$
$$H_2 : \mathbb{R}\times\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty) \to \mathbb{R}$$
$$H : \mathbb{R}\times\mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty) \to \mathbb{R}\times(\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$$

by

$$H_1(\varepsilon,\alpha,x) = \Gamma(t)[\alpha p + v_x] + \Gamma(t)\int_0^t \Gamma^{-1}(s)[(\mathscr{F}(x(s)) + \varepsilon\mathscr{G}(x(s))) - \left(\int_0^1 \Psi^T(u)(\mathscr{F}(x(u)) + \varepsilon\mathscr{G}(x(u)))du\right)\Psi(s)]ds,$$

$$H_2(\varepsilon,\alpha,x) = \alpha - \left(\int_0^1 \Psi_n(t)f(\alpha\Phi_1(t) + w_1(x(t), \alpha\Phi_2(t) + w_2(x(t))))dt + \varepsilon\int_0^1 \Psi_n(t)G(\alpha\Phi_1(t) + w_1(x(t)))dt\right),$$

and

$$H(\varepsilon,\alpha,x) = (H_2(\varepsilon,\alpha,x), H_1(\varepsilon,\alpha,x)).$$

The construction of the operators H_1, H_2, and H, as well as the following lemma, stem from a proof found in [17]. This type of result appears either explicitly or implicitly in many papers dealing with resonant boundary value problems [2], [5], [6], [7], [9], [12], [17], [21], [22].

LEMMA 1. *If $(\varepsilon_0, \tilde{\alpha}, \tilde{x})$ is a fixed point of $H(\varepsilon,\alpha,x)$, then $\tilde{x}$ solves the boundary value problem*

$$\dot{x}(t) = A(t)x(t) + \mathscr{F}(x(t)) + \varepsilon_0\mathscr{G}(x(t)), 0 \leqslant t \leqslant 1$$

subject to

$$\sum_{k=0}^{N} B_k x(t_k) = 0.$$

The essential idea behind the proof of the preceding lemma is as follows. We first use a variation of parameters formula to show a fixed point of $H(\varepsilon,\alpha,x)$ satisfies

$$\dot{x}(t) = A(t)x(t) + \mathscr{F}(x(t)) + \varepsilon_0\mathscr{G}(x(t)).$$

We then utilize Proposition 2.1 and the fact that we have a fixed point of $H_2(\varepsilon,\alpha,x)$ to show said fixed point satisfies the boundary conditions. To do this, we compute

$$\begin{aligned}\sum_{k=0}^{N} B_k\tilde{x}(t_k) &= \sum_{k=0}^{N} B_k\Gamma(t_k)[\tilde{\alpha}p+v_{\tilde{x}}] \\ &\quad + \sum_{k=0}^{N} B_k\Gamma(t_k)\int_0^{t_k}\Gamma^{-1}(s)(\mathscr{F}(\tilde{x}(s))+\varepsilon_0\mathscr{G}(\tilde{x}(s)))ds \\ &= Dv_{\tilde{x}} + \sum_{k=1}^{N} B_k\Gamma(t_k)\int_0^{t_k}\Gamma^{-1}(s)(\mathscr{F}(\tilde{x}(s))+\varepsilon_0\mathscr{G}(\tilde{x}(s)))ds\end{aligned}$$

since $p\in\ker(D)$ and $t_0=0$. Remembering that

$$\begin{aligned}Dv_{\tilde{x}} = -\sum_{k=1}^{N} B_k\Gamma(t_k)\int_0^{t_k}\Gamma^{-1}(s)[(\mathscr{F}(\tilde{x}(s))+\varepsilon_0\mathscr{G}(\tilde{x}(s))) \\ -\left(\int_0^1\Psi^T(u)(\mathscr{F}(\tilde{x}(u))+\varepsilon_0\mathscr{G}(\tilde{x}(u)))du\right)\Psi(s)]ds\end{aligned}$$

and

$$\int_0^1\Psi^T(u)(\mathscr{F}(\tilde{x}(u))+\varepsilon_0\mathscr{G}(\tilde{x}(u)))du=0,$$

we have that

$$\sum_{k=0}^{N} B_k\tilde{x}(t_k) = Dv_{\tilde{x}} + \sum_{k=1}^{N} B_k\Gamma(t_k)\int_0^{t_k}\Gamma^{-1}(s)(\mathscr{F}(\tilde{x}(s))+\varepsilon_0\mathscr{G}(\tilde{x}(s)))ds = 0. \quad \square$$

We will define $A_1=\{t\in[0,1]:\Phi_1(t)>0\}$ and $A_2=\{t\in[0,1]:\Phi_2(t)>0\}$, and we use the notation J_1 and J_2 as

$$\begin{aligned}J_1 = f(\infty,\infty)\int_{A_1\cap A_2}\Psi_n(t)dt + f(\infty,-\infty)\int_{A_1\cap([0,1]/A_2)}\Psi_n(t)dt \\ + f(-\infty,\infty)\int_{([0,1]/A_1)\cap A_2}\Psi_n(t)dt + f(-\infty,-\infty)\int_{([0,1]/A_1)\cap([0,1]/A_2)}\Psi_n(t)dt\end{aligned}$$

and

$$\begin{aligned}J_2 = f(-\infty,-\infty)\int_{A_1\cap A_2}\Psi_n(t)dt + f(-\infty,\infty)\int_{A_1\cap([0,1]/A_2)}\Psi_n(t)dt \\ + f(\infty,-\infty)\int_{([0,1]/A_1)\cap A_2}\Psi_n(t)dt + f(\infty,\infty)\int_{([0,1]/A_1)\cap([0,1]/A_2)}\Psi_n(t)dt.\end{aligned}$$

The result of the following lemma appears in [17], but we include a proof for the benefit of the reader.

LEMMA 2. *Suppose that*

(i) $\dim(\sum_{k=0}^{N} B_k\Gamma(t_k)) = 1$*;*

(ii) $f : \mathbb{R}^2 \to \mathbb{R}$ *is continuous;*

(iii) $f(\infty,\infty)$, $f(\infty,-\infty)$, $f(-\infty,\infty)$, *and* $f(-\infty,-\infty)$ *exist;*

(iv) $J_1J_2 < 0$;

(v) $\varepsilon = 0$.

Then $H(0,\alpha,x)$ *has a fixed point.*

Proof.

Since $\{t \in [0,1] : \Phi_1(t) = 0\}$ and $\{t \in [0,1] : \Phi_2(t) = 0\}$ have Lebesgue measure zero, it follows that

$$\begin{aligned}\int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt = \\ \int_{A_1\cap A_2} \Psi_n(t) f(\alpha\Phi_1 + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt + \\ \int_{A_1\cap([0,1]/A_2)} \Psi_n(t) f(\alpha\Phi_1 + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt + \\ \int_{([0,1]/A_1)\cap A_2} \Psi_n(t) f(\alpha\Phi_1 + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt + \\ \int_{([0,1]/A_1)\cap([0,1]/A_2)} \Psi_n(t) f(\alpha\Phi_1 + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt.\end{aligned}$$

Since w_1 and w_2 are bounded, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}\lim_{\alpha\to\infty} \int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt = \\ f(\infty,\infty)\int_{A_1\cap A_2} \Psi_n(t)dt + f(\infty,-\infty)\int_{A_1\cap([0,1]/A_2)} \Psi_n(t)dt \\ + f(-\infty,\infty)\int_{([0,1]/A_1)\cap A_2} \Psi_n(t)dt + f(-\infty,-\infty)\int_{([0,1]/A_1)\cap([0,1]/A_2)} \Psi_n(t)dt = J_1.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{\alpha\to-\infty} \int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt = \\ f(-\infty,-\infty)\int_{A_1\cap A_2} \Psi_n(t)dt + f(-\infty,\infty)\int_{A_1\cap([0,1]/A_2)} \Psi_n(t)dt \\ + f(\infty,-\infty)\int_{([0,1]/A_1)\cap A_2} \Psi_n(t)dt + f(\infty,\infty)\int_{([0,1]/A_1)\cap([0,1]/A_2)} \Psi_n(t)dt = J_2.\end{aligned}$$

Without loss of generality, we assume $J_2 < 0 < J_1$.

Based on the above calculations, there is some $\alpha_0 \geqslant m$ where $m = \sup\{|f(s,t)| : (s,t) \in \mathbb{R}^2\}$ and some $J,$ such that for all $\alpha \geqslant \alpha_0,$

$$\int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt \geqslant J$$

and

$$\int_0^1 \Psi_n(t) f(-\alpha\Phi_1(t) + w_1(x(t)), -\alpha\Phi_2(t) + w_2(x(t)))dt \leqslant -J.$$

Then for all $t \in \mathbb{R},$ for $\alpha \geqslant \alpha_0$ and $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty),$

$$\begin{aligned} H_2(0,\alpha,x) &= \alpha - \left(\int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt \right) \\ &\leqslant \alpha - J \\ &\leqslant \alpha. \end{aligned}$$

Similarly, for $\alpha \geqslant \alpha_0$ and $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty),$ $H_2(0,-\alpha,x) \geqslant -\alpha.$

Clearly, since $\varepsilon = 0,$ there exists constants M_1, M_2 such that for all $(\alpha,x) \in \mathbb{R} \times (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty),$

$$\|H_1(\alpha,x)\|_\infty \leqslant M_1|\alpha| + M_2.$$

Letting $\delta = \alpha_0 + (m+J),$ define $\mathscr{B} = \{(\alpha,x) \in \mathbb{R} \times (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty) : |\alpha| \leqslant \delta \text{ and } \|x\|_\infty \leqslant M_1\delta + M_2\}.$

Now if $\alpha \in [\alpha_0, \delta],$ for all $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty),$ we have

$$\begin{aligned} H_2(0,\alpha,x) &= \alpha - \int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt \\ &\geqslant \alpha - \int_0^1 |\Psi_n(t)||f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))|dt \\ &\geqslant \alpha - m \\ &\geqslant \alpha - \alpha_0 \\ &\geqslant -J \\ &\geqslant -\delta \end{aligned}$$

and

$$\begin{aligned} H_2(0,-\alpha,x) &= -\alpha - \int_0^1 \Psi_n(t) f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))dt \\ &\leqslant -\alpha + \int_0^1 |\Psi_n(t)||f(\alpha\Phi_1(t) + w_1(x(t)), \alpha\Phi_2(t) + w_2(x(t)))|dt \\ &\leqslant -\alpha + m \\ &\leqslant -\alpha + \alpha_0 \\ &\leqslant J \\ &\leqslant \delta. \end{aligned}$$

Thus, for all $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ and $\alpha \in [\alpha_0, \delta]$, $H_2(0,\alpha,x)$, $H_2(0,-\alpha,x) \in [-\alpha,\alpha] \subseteq [-\delta,\delta]$.

Furthermore, if $0 \leqslant \alpha < \alpha_0$, for all $x \in (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$,

$$\begin{aligned}|H_2(0,\pm\alpha,x)| &\leqslant |\pm\alpha| + \int_0^1 |\Psi_n(t)||f(\alpha\Phi_1(t)+w_1(x(t)),\alpha\Phi_2(t)+w_2(x(t)))|dt \\ &\leqslant \alpha_0 + m \\ &\leqslant \delta + J.\end{aligned}$$

We have shown that H_2 maps $[-\delta,\delta] \times (\mathscr{C}([0,1],\mathbb{R}^n), \|\cdot\|_\infty)$ into $[-\delta,\delta]$. From this it follows that $H(\mathscr{B}) \subseteq \mathscr{B}$. For if $(\alpha,x) \in \mathscr{B}$, then $H_2(0,\alpha,x) \in [-\delta,\delta]$, while

$$\begin{aligned}|H_1(0,\alpha,x)| &\leqslant M_1|\alpha| + M_2 \\ &\leqslant M_1\delta + M_2.\end{aligned}$$

Since H maps $\mathscr{B}$ into itself, $H(0,\alpha,x)$ has a fixed point by Schauder's Fixed Point Theorem. □

4. Solvability of (1), (2)

Recall that we define $A_1 = \{t \in [0,1] : \Phi_1(t) > 0\}$ and $A_2 = \{t \in [0,1] : \Phi_2(t) > 0\}$, and we use the notation J_1 and J_2 as

$$\begin{aligned}J_1 = f(\infty,\infty)\int_{A_1\cap A_2}\Psi_n(t)dt + f(\infty,-\infty)\int_{A_1\cap([0,1]/A_2)}\Psi_n(t)dt \\ + f(-\infty,\infty)\int_{([0,1]/A_1)\cap A_2}\Psi_n(t)dt + f(-\infty,-\infty)\int_{([0,1]/A_1)\cap([0,1]/A_2)}\Psi_n(t)dt\end{aligned}$$

and

$$\begin{aligned}J_2 = f(-\infty,-\infty)\int_{A_1\cap A_2}\Psi_n(t)dt + f(-\infty,\infty)\int_{A_1\cap([0,1]/A_2)}\Psi_n(t)dt \\ + f(\infty,-\infty)\int_{([0,1]/A_1)\cap A_2}\Psi_n(t)dt + f(\infty,\infty)\int_{([0,1]/A_1)\cap([0,1]/A_2)}\Psi_n(t)dt.\end{aligned}$$

THEOREM 1. *Suppose that*

(i) $\dim(\sum_{k=0}^{N} B_k\Gamma(t_k)) = 1$ *where* $\Gamma(t)$ *is the principal matrix solution of* $\dot{x}(t) = A(t)x(t)$;

(ii) $f : \mathbb{R}^2 \to \mathbb{R}$ *is continuous;*

(iii) $f(\infty,\infty)$, $f(\infty,-\infty)$, $f(-\infty,\infty)$ *and* $f(-\infty,-\infty)$ *exist;*

(iv) $J_1 J_2 < 0$;

(v) $G : \mathscr{C}^{(n-1)}([0,1],\mathbb{R}^n) \to \mathscr{C}^{(n-1)}([0,1],\mathbb{R})$ *is continuous.*

Then, there exists an ε_0 *such that for* $\varepsilon \in [0,\varepsilon_0]$, *there is at least one solution of*

$$y^{(n)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t) = f(y(t),y'(t))+\varepsilon G(y,y',\cdots,y^{(n-1)})(t)$$

that satisfies

$$\sum_{j=1}^{n} b_{ij}(0)y^{(j-1)}(0)+\sum_{j=1}^{n} b_{ij}(t_1)y^{(j-1)}(t_1)+\cdots+\sum_{j=1}^{n} b_{ij}(N)y^{(j-1)}(1) = 0$$

for $i = 1,2,\cdots,n$.

Proof.
As above, we define mappings

$$\begin{aligned}
&H_1 : \mathbb{R}\times\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R} \to \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\\
&H_2 : \mathbb{R}\times\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R} \to \mathbb{R}\\
&H : \mathbb{R}\times\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R} \to \mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty)\times\mathbb{R}
\end{aligned}$$

by

$$H_1(\varepsilon,\alpha,x) = \Gamma(t)[\alpha p+v_x]+$$
$$\Gamma(t)\int_0^t \Gamma^{-1}(s)[[\mathscr{F}(x(s))+\varepsilon\mathscr{G}(x(s))]-(\int_0^1 \Psi^T(u)[\mathscr{F}(x(u))+\varepsilon\mathscr{G}(x(u))]du)\Psi(s)]ds,$$
$$H_2(\varepsilon,\alpha,x) = \alpha-(\int_0^1 \Psi_n(t)f(\alpha\Phi_1(t)+w_1(\varepsilon,x(t),\alpha\Phi_2(t)+w_2(\varepsilon x(t))))dt+$$
$$\varepsilon\int_0^1 \Psi_n(t)g(\alpha\Phi_1(t)+w_1(\varepsilon,x(t)))dt),$$

and

$$H(\varepsilon,\alpha,x) = (H_1(\varepsilon,\alpha,x),H_2(\varepsilon,\alpha,x)).$$

It is evident that for $\varepsilon = 0$, there exists constants M_1 and M_2 so that

$$\|H_1(0,\alpha,x)\|_\infty \leqslant M_1|\alpha|+M_2.$$

By the proof of Lemma 2, defining $\alpha_0 \geqslant m+J$ and $\delta = \alpha_0+m+J$ for some fixed real number J, we can create a nonempty, convex set $\mathscr{B} = \{(\alpha,x)\in\mathbb{R}^2\times\mathscr{C}([0,1],\mathbb{R}^n,\|\cdot\|_\infty) : |\alpha|\leqslant\delta \text{ and } \|x\|_\infty \leqslant M_1|\delta|+M_2+J\}$ such that, when $\varepsilon = 0$, the following hold true:

(i) for all $\alpha \geqslant \alpha_0$, $H_2(0,\alpha,x) \leqslant \alpha-J$ and $H_2(0,\alpha,x) \geqslant -\alpha+J$;

(ii) for $\alpha \in [\alpha_0,\delta]$, $H_2(0,\alpha,x) \geqslant -J$ and $H_2(0,\alpha,x) \leqslant J$;

(iii) for $0 \leqslant \alpha < \alpha_0$, $|H_2(0,\alpha,x)| \leqslant \delta+J$; and

(iv) $\|H_1(0,\alpha,x)\|_\infty \leqslant M_1\delta+M_2$.

It follows that

$$\inf_{(\alpha,x)\in\mathscr{B}}\mathrm{dist}(H(0,\alpha,x),\partial\mathscr{B}) > 0;$$

that is, when $\varepsilon = 0$, there is a positive distance between the boundary of the set $\mathscr{B}$ and the set of $H(0,\alpha,x)$ for $(\alpha,x) \in \mathscr{B}$. Since $\{\Phi(\cdot)\alpha + w(x(\cdot))|(\alpha,x) \in \mathscr{B}\}$ is equicontinuous and uniformly bounded, it is compact by Arzela-Ascoli's Theorem. This implies that if we choose a positive value, $\tilde{\varepsilon}$, so that we restrict ε to the interval $[0,\tilde{\varepsilon}]$, the map $(\varepsilon,\alpha,x) \mapsto H(\varepsilon,\alpha,x)$ is uniformly continuous on $\mathscr{B}$. From this it follows that there exists ε_0 such that if $|\varepsilon| \leqslant \varepsilon_0$,

$$H(\varepsilon,\alpha,x) \in \mathscr{B}$$

for all $(\alpha,x) \in \mathscr{B}$. The solvability of (1), (2) is now a consequence of Schauder's Fixed Point Theorem. □

5. Example

We now present an example to illustrate the main theorem of this paper. We consider the differential equation

$$y'' + 3y' + 2y = f(y(t),y'(t)) + \int_0^t w(t,s)g(s,y(s),y'(s))ds \tag{7}$$

subject to boundary conditions

$$\begin{aligned} &y(0) + y'(0) + y\left(\frac{J-1}{2}\right) + y'\left(\frac{J-1}{2}\right) = 0, \\ &y(J) + y'(J) = 0, \end{aligned} \tag{8}$$

where $J \geqslant 1$.

In system form, (7),(8) becomes

$$\begin{aligned} &x'(t) = A(t)x(t) + F(x(t)) + G(x(t)) \\ &B_0 x(0) + B_{\frac{J-1}{2}} x\left(\frac{J-1}{2}\right) + B_J x(J) = 0 \end{aligned} \tag{9}$$

where $x_1(t) = y(t),\ x_2(t) = y'(t)$,

$$A(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ for all t,}$$

$$B_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ B_{\frac{J-1}{2}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ B_J = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix},$$

$$F(x(t)) = \begin{bmatrix} 0 \\ f(x(t)) \end{bmatrix}, \text{ and } G(x(t)) = \begin{bmatrix} 0 \\ \int_0^t w(t,s)g(s,x(s))ds \end{bmatrix}.$$

It is easy to verify that

$$\Gamma(t) = \begin{bmatrix} e^{-2t}(-1+2e^t) & e^{-2t}(-1+e^t) \\ -2e^{-2t}(-1+e^t) & -e^{-2t}(-2+e^t) \end{bmatrix}$$

and

$$B_0 + B_{\frac{J-1}{2}}\Gamma\left(\frac{J-1}{2}\right) + B_J\Gamma(J) = \begin{bmatrix} 1+e^{1-J} & 1+e^{1-J} \\ -e^{-2J} & -e^{-2J} \end{bmatrix},$$

which gives

$$\ker\left(B_0 + B_{\frac{J-1}{2}}\Gamma\left(\frac{J-1}{2}\right) + B_J\Gamma(J)\right) = \text{ span } \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$$

From this, we conclude that

$$\phi(t) = \Gamma(t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}.$$

It follows that $A_1 = \{t \in [0,J] : \phi_1(t) > 0\} = [0,J]$ and $A_2 = \{t \in [0,J] : \phi_2(t) > 0\} = \emptyset$.

The computations above give

$$\psi(t) = \begin{cases} \begin{pmatrix} -e^{2t} \\ -e^{2t} \end{pmatrix} & \text{for } 0 \leqslant t \leqslant \frac{J-1}{2}, \\ \begin{pmatrix} -e^{2t} - e^{2t+1-J} \\ -e^{2t} - e^{2t+1-J} \end{pmatrix} & \text{for } \frac{J-1}{2} < t \leqslant J. \end{cases},$$

$$J_1 = f(\infty, -\infty)\left(1 - \frac{1}{2}e^{2J} - \frac{1}{2}e^{J+1}\right),$$

and

$$J_2 = f(-\infty, \infty)\left(1 - \frac{1}{2}e^{2J} - \frac{1}{2}e^{J+1}\right).$$

It can easily be shown that if $J \geqslant 1$ and $f(\infty,-\infty)f(-\infty,\infty) < 0$, we are guaranteed that $J_1J_2 < 0$. Then, according to the proof of Theorem 1, if $\int_0^t |w(t,s)g(s)|ds < \min\{|J_1|, |J_2|\}$ for all $t \in [0,J]$, (7), (8) will have a solution.

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