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UNIFORMLY STABLE SOLUTION OF A NONLOCAL PROBLEM OF COUPLED SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper we are concerned with a nonlocal problem of a coupled system of differential equations. We study the local existence of the solution and its continuous dependence. The global existence and its uniform stability is being studied.

1. Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last decades. The reader is referred to ([1]-[16]) and ([18]-[19]) and references therein.

In [14] the authors studied the uniformly stable positive monotonic solution of a nonlocal Cauchy problem

$$x' = f(t,x(t)),\ t \in [0,T]$$

$$\sum_{j=1}^{m} b_j x(\eta_j) = x_1,\ \eta_j \in (0,a) \subset [0,T].$$

Here we are concerned with the nonlocal problem of the coupled system of differential equations of the type

$$\frac{dx}{dt} = f_1(t,y(t)),\ \ t \in (0,T] \tag{1.1}$$

$$\frac{dy}{dt} = f_2(t,x(t)),\ \ t \in (0,T], \tag{1.2}$$

with the nonlocal conditions

$$x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = x_0,\ \ a_k > 0,\ \tau_k \in (0,\ T) \tag{1.3}$$

$$y(0) + \sum_{j=1}^{m} b_j y(\eta_j) = y_0,\ \ b_j > 0,\ \eta_j \in (0,\ T). \tag{1.4}$$

The local and global existence of solutions of the nonlocal problem (1.1)-(1.4) is proved. The continuous dependence on x_0, y_0 and the uniform stability are studied.

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2. Integral Representation

Let X be the class of all columns vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, $x,y \in C(0,T]$ with the norm

$$\left|\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|\right|_X = ||x|| + ||y|| = \sup_{t\in[0,T]} | x(t) | + \sup_{t\in[0,T]} | y(t) | .$$

Let $N > 0$ be given, $t_0 = \max\{\tau_k, \eta_j\}$, and let Y be the class of all column vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, $x,y \in C[t_0,T]$ with the norm

$$\left|\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|\right|_Y = ||x||^* + ||y||^* = \sup_{t\in[t_0,T]} e^{-Nt} | x(t) | + \sup_{t\in[t_0,T]} e^{-Nt} | y(t) | .$$

Throughout the paper we assume that the following assumptions hold:

(H1) $f_i : [0,T] \times R \to R,\ i = 1,2$ are continuous.;

(H2) f_i satisfy the Lipschitz condition with Lipschitz constant l

$$| f_i(t,x) - f_i(t,y) | \leqslant l \, | x - y | .$$

Now we have the following lemma.

LEMMA 1. *The solution of the nonlocal problem* (1.1)-(1.4) *can be expressed by the system of the integral equations*

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s,y(s))ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))ds \\ \\ by_0 + \int_0^t f_2(s,x(s))ds - b \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x(s))ds \end{pmatrix}.$$

Proof. Integrating equation (1.1), we obtain

$$x(t) = x(0) + \int_0^t f_1(s,y(s))ds.$$

Then

$$x(\tau_k) = x(0) + \int_0^{\tau_k} f_1(s,y(s))ds,$$

and

$$\sum_{k=1}^{n} a_k x(\tau_k) = \sum_{k=1}^{n} a_k x(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y(s))ds,$$

$$x_0 - x(0) = \sum_{k=1}^{n} a_k x(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s))ds,$$

$$(1 + \sum_{k=1}^{n} a_k)x(0) = x_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s))\, ds.$$

Hence

$$x(t) = a\left(x_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s))ds\right) + \int_0^t f_1(s, y(s))ds.$$

Similarly, we can obtain

$$y(t) = b\left(y_0 - \sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s, x(s))ds\right) + \int_0^t f_2(s, x(s))ds,$$

where

$$\left(1 + \sum_{k=1}^{n} a_k\right)^{-1} = a, \left(1 + \sum_{j=1}^{m} b_j\right)^{-1} = b.$$

Thus, the proof is completed. □

Now, define the operator F by

$$F\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s, y(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s, y(s))ds \\ \\ by_0 + \int_0^t f_2(s, x(s))ds - b\sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s, x(s))ds \end{pmatrix} = \begin{pmatrix} F_1 y(t) \\ F_2 x(t) \end{pmatrix}.$$

LEMMA 2. $F : X \to X$.

Proof. Let $x, y \in C(0,T]$, $t_1, t_2 \in (0,T]$. Then from assumption $(H1)$ and for every $\varepsilon > 0$, $\delta > 0$ s.t $\mid t_2 - t_1 \mid < \delta$, we have

$$\mid F_1 y(t_2) - F_1 y(t_1) \mid \leqslant \int_{t_1}^{t_2} f_1(s, y(s))ds < \varepsilon/2.$$

Similarly

$$\mid F_2 x(t_2) - F_2 x(t_1) \mid \leqslant \int_{t_1}^{t_2} f_2(s, x(s))ds < \varepsilon/2.$$

Then $F_1, F_2 : C[0,T] \to C[0,T]$. Hence $F : X \to X$.

Thus, the proof is completed. □

By the same way, we can prove the following lemma

LEMMA 3. $F : Y \to Y$.

3. Local existence

THEOREM 1. *Consider that assumptions* $(H1)$-$(H2)$ *are satisfied, if* $2lT < 1$, *then the nonlocal problem* (1.1)-(1.4) *has a unique solution* $z \in X$.

Proof. Let

$$FU(t) = F\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s,y_1(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_1(s))ds \\ \\ by_0 + \int_0^t f_2(s,x_1(s))ds - a\sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x_1(s))ds \end{pmatrix},$$

$$FV(t) = F\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s,y_2(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f_1(s,y_2(s))ds \\ \\ by_0 + \int_0^t f_2(s,x_2(s))ds - a\sum_{j=1}^{m} b_j \int_0^{\eta_j} f_2(s,x_2(s))ds \end{pmatrix},$$

then

$$FU(t) - FV(t) =$$

$$\begin{pmatrix} \int_0^t [f_1(s,y_1(s)) - f_1(s,y_2(s))]ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f_1(s,y_1(s)) - f_1(s,y_2(s))]ds \\ \\ \int_0^t [f_2(s,x_1(s)) - f_2(s,x_2(s))]ds - b\sum_{j=1}^{m} b_j \int_0^{\eta_j} [f_2(s,x_1(s)) - f_2(s,x_2(s))]ds \end{pmatrix}.$$

Now

$$\left| \int_0^t [f_1(s,y_1(s)) - f_1(s,y_2(s))]ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f_1(s,y_1(s)) - f_1(s,y_2(s))]ds \right|$$

$$\leqslant \int_0^t \mid [f_1(s,y_1(s)) - f_1(s,y_2(s))] \mid ds$$

$$+ \mid a \mid \sum_{k=1}^{n} \mid a_k \mid \int_0^{\tau_k} \mid [f_1(s,y_1(s)) - f_1(s,y_2(s))] \mid ds$$

$$\leqslant l \int_0^t \mid y_1(s) - y_2(s) \mid ds + a\sum_{k=1}^{n} a_k l \int_0^{\tau_k} \mid y_1(s) - y_2(s) \mid ds$$

$$\leqslant l \parallel y_1 - y_2 \parallel T + a \sum_{k=1}^{n} a_k l \parallel y_1 - y_2 \parallel T$$
$$= lT \parallel y_1 - y_2 \parallel [1 + a \sum_{k=1}^{n} a_k] \leqslant 2lT \parallel y_1 - y_2 \parallel .$$

Similarly

$$\left| \int_0^t [f_2(s, x_1(s)) - f_2(s, x_2(s))] ds \right.$$
$$\left. + b \sum_{j=1}^{m} b_j \int_0^{\eta_j} [f_2(s, x_1(s)) - f_2(s, x_2(s))] ds \right| \leqslant 2lT \parallel x_1 - x_2 \parallel .$$

Hence

$$\parallel F\,U - F\,V \parallel_X \leqslant\ 2lT \parallel y_1 - y_2 \parallel + 2lT \parallel x_1 - x_2 \parallel \leqslant 2lT \parallel U - V \parallel_X .$$

Then F is a contraction [17] and there exists a unique solution $z \in X$ of the nonlocal problem (1.1)-(1.4).

Thus, the proof is completed. □

4. Continuous dependence

Consider the nonlocal problem of the system of equations (1.1) and (1.2) with the nonlocal conditions

$$x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = \tilde{x}_0,\ \tau_k \in (0, T), \tag{4.1}$$

$$y(0) + \sum_{j=1}^{m} b_j y(\eta_j) = \tilde{y}_0,\ \eta_j \in (0, T). \tag{4.2}$$

Here, we study the continuous dependence (on the data x_0, y_0) of the solution of the coupled system of differential equations (1.1) and (1.2).

Let $\tilde{z}(t) = \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix}$ be the solution of the the nonlocal problem (1.1), (1.2), (4.1) and (4.2).

DEFINITION 1. The solution of the nonlocal problem (1.1)-(1.4), $z \in X$ is *continuously dependent* (on the data x_0, y_0) if $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $| x_0 - \tilde{x}_0 | \leqslant \delta/2$ and $| y_0 - \tilde{y}_0 | \leqslant \delta/2$ implies that $\parallel z - \tilde{z} \parallel_X \leqslant \varepsilon$

THEOREM 2. *Consider that assumptions $(H1)$-$(H2)$ are satisfied. Then the solution of the nonlocal problem (1.1)-(1.4) is continuously dependent.*

Proof. Let

$$U(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}ax_0+\int_0^t f_1(s,y(s))ds-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}f_1(s,y(s))ds\\ by_0+\int_0^t f_2(s,x(s))ds-b\sum_{j=1}^{m}b_j\int_0^{\eta_j}f_2(s,x(s))ds\end{pmatrix}$$

be the solution of the nonlocal problem (1.1)-(1.4) and

$$\tilde{U}(t)=\begin{pmatrix}\tilde{x}(t)\\ \tilde{y}(t)\end{pmatrix}=\begin{pmatrix}a\tilde{x}_0+\int_0^t f_1(s,\tilde{y}(s))ds-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}f_1(s,\tilde{y}(s))ds\\ b\tilde{y}_0+\int_0^t f_2(s,\tilde{x}(s))ds-b\sum_{j=1}^{m}b_j\int_0^{\eta_j}f_2(s,\tilde{x}(s))ds\end{pmatrix}$$

be the solution of the nonlocal problem (1.1), (1.2), (4.1) and (4.2). Then

$$U(t)-\widetilde{U}(t)=\begin{pmatrix}V_1(t)-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}[f_1(s,y(s))-f_1(s,\tilde{y}(s))]ds\\ \\ V_2(t)-b\sum_{j=1}^{m}b_j\int_0^{\eta_j}[f_2(s,x(s))-f_2(s,\tilde{x}(s))]ds\end{pmatrix},$$

where

$$V_1(t)=a(x_0-\tilde{x}_0)+\int_0^t[f_1(s,y(s))-f_1(s,\tilde{y}(s))]ds, \tag{4.3}$$

$$V_2(t)=b(y_0-\tilde{y}_0)+\int_0^t[f_2(s,x(s))-f_2(s,\tilde{x}(s))]ds. \tag{4.4}$$

Now

$$\begin{aligned}&\left|V_1(t)-a\sum_{k=1}^{n}a_k\int_0^{\tau_k}[f_1(s,y(s))-f_1(s,\tilde{y}(s))]ds\right|\\ &\quad\leqslant|a(x_0-\tilde{x}_0)|\\ &\qquad+\left|\int_0^t[f_1(s,y(s))-f_1(s,\tilde{y}(s))]ds+a\sum_{k=1}^{n}a_k\int_0^{\tau_k}[f_1(s,y(s))-f_1(s,\tilde{y}(s))]ds\right|\\ &\quad\leqslant a\delta/2+2l\parallel y-\tilde{y}\parallel T.\end{aligned}$$

Similarly

$$\left|V_2(t)-b\sum_{j=1}^{m}b_j\int_0^{\eta_j}[f_2(s,x(s))-f_2(s,\tilde{x}(s))]ds\right|\leqslant b\delta/2+2lT\parallel x-\tilde{x}\parallel.$$

Therefore

$$\| U-\tilde{U} \|_X = \left\| \begin{pmatrix} V_1(t) - a\sum\limits_{k=1}^{n} a_k \int\limits_0^{\tau_k} [f_1(s,y(s)) - f_1(s,\tilde{y}(s))]ds \\ V_2(t) - b\sum\limits_{j=1}^{m} b_j \int\limits_0^{\eta_j} [f_2(s,x(s)) - f_2(s,\tilde{x}(s))]ds \end{pmatrix} \right\|_X$$

$$\leqslant \left\| V_1(t) - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f_1(s,y(s)) - f_1(s,\tilde{y}(s))]ds \right\|$$

$$+ \left\| V_2(t) - b\sum_{j=1}^{m} b_j \int_0^{\eta_j} [f_2(s,x(s)) - f_2(s,\tilde{x}(s))]ds \right\|$$

$$\leqslant (a+b)(\delta/2) + 2lT \| U-\tilde{U} \|_X,$$

where $V_1(t)$ and $V_2(t)$ are defined in (4.3) and (4.4). Hence

$$\| U-\tilde{U} \|_X \leqslant (a+b)(\delta/2)/(1-2lT) \leqslant \delta/(1-2lT) = \varepsilon.$$

Thus, the proof is completed. □

5. Global existence

THEOREM 3. *If assumptions* $(H1)$-$(H2)$ *are satisfied, then the nonlocal problem* (1.1)-(1.4) *has a unique solution* $z \in Y$.

Proof. Let

$$FU(t) = F\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int\limits_0^t f_1(s,y_1(s))ds - a\sum\limits_{k=1}^{n} a_k \int\limits_0^{\tau_k} f_1(s,y_1(s))ds \\ \\ by_0 + \int\limits_0^t f_2(s,x_1(s))ds - a\sum\limits_{j=1}^{m} b_j \int\limits_0^{\eta_j} f_2(s,x_1(s))ds \end{pmatrix},$$

$$FV(t) = F\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int\limits_0^t f_1(s,y_2(s))ds - a\sum\limits_{k=1}^{n} a_k \int\limits_0^{\tau_k} f_1(s,y_2(s))ds \\ \\ by_0 + \int\limits_0^t f_2(s,x_2(s))ds - a\sum\limits_{j=1}^{m} b_j \int\limits_0^{\eta_j} f_2(s,x_2(s))ds \end{pmatrix},$$

then

$$FU(t) - FV(t) =$$

$$\begin{pmatrix} \int\limits_0^t [f_1(s,y_1(s)) - f_1(s,y_2(s))]ds - a\sum\limits_{k=1}^{n} a_k \int\limits_0^{\tau_k} [f_1(s,y_1(s)) - f_1(s,y_2(s))]ds \\ \\ \int\limits_0^t [f_2(s,x_1(s)) - f_2(s,x_2(s))]ds - b\sum\limits_{j=1}^{m} b_j \int\limits_0^{\eta_j} [f_2(s,x_1(s)) - f_2(s,x_2(s))]ds \end{pmatrix}.$$

Now

$$e^{-Nt}\left|\int_0^t [f_1(s,y_1(s))-f_1(s,y_2(s))]ds - a\sum_{k=1}^n a_k\int_0^{\tau_k}[f_1(s,y_1(s))-f_1(s,y_2(s))]ds\right|$$

$$\leqslant \int_0^t e^{-Nt} \mid [f_1(s,y_1(s))-f_1(s,y_2(s))] \mid ds$$

$$+ \mid a \mid \sum_{k=1}^n \mid a_k \mid \int_0^{\tau_k} e^{-Nt} \mid [f_1(s,y_1(s))-f_1(s,y_2(s))] \mid ds$$

$$\leqslant l\int_0^t e^{-Nt+Ns}\sup e^{-Ns} \mid y_1(s)-y_2(s) \mid ds$$

$$+ a\sum_{k=1}^n a_k l\int_0^{\tau_k} e^{-Nt+Ns}\sup e^{-Ns} \mid y_1(s)-y_2(s) \mid ds$$

$$\leqslant l \parallel y_1-y_2 \parallel^* \int_0^t e^{-Nt+Ns}ds + a\sum_{k=1}^n a_k l \parallel y_1-y_2 \parallel^* \int_0^{\tau_k} e^{-Nt+Ns}ds$$

$$\leqslant l \parallel y_1-y_2 \parallel^* [e^{-Nt+Ns}/N]_0^t + a\sum_{k=1}^n a_k\, l \parallel y_1-y_2 \parallel^* [e^{-Nt+Ns}/N]_0^{\tau_k}$$

$$\leqslant (l/N) \parallel y_1-y_2 \parallel^* [1-e^{-Nt}] + (al/N) \parallel y_1-y_2 \parallel^* \sum_{k=1}^n a_k[e^{-N(t-\tau_k)}-e^{-Nt}]$$

$$\leqslant (l/N) \parallel y_1-y_2 \parallel^* [1-e^{-Nt}] + (al/N) \parallel y_1-y_2 \parallel^* \sum_{k=1}^n a_k\, [e^{-N(t-t_0)}-e^{-Nt}]$$

$$\leqslant (l/N) \parallel y_1-y_2 \parallel^* [1-e^{-Nt}] + (l/N) \parallel y_1-y_2 \parallel^* \sum_{k=1}^n a_k$$

$$\leqslant (l/N) \parallel y_1-y_2 \parallel^* [1-e^{-Nt}](1+a\sum_{k=1}^n a_k\,) \leqslant (2l/N) \parallel y_1-y_2 \parallel^* .$$

Then

$$\left\|\int_0^t [f_1(s,y_1(s))-f_1(s,y_2(s))]ds + a\sum_{k=1}^n a_k\int_0^{\tau_k}[f_1(s,y_1(s))-f_1(s,y_2(s))]ds\right\|^* \leqslant (2l/N) \parallel y_1-y_2 \parallel^* .$$

Similarly

$$\left\|\int_0^t [f_2(s,x_1(s))-f_2(s,x_2(s))]ds + b\sum_{j=1}^m b_j\int_0^{\eta_j}[f_2(s,x_1(s))-f_2(s,x_2(s))]ds\right\|^*$$

$$\leqslant (2l/N) \parallel x_1 - x_2 \parallel^* .$$

And so $\parallel FU - FV \parallel_Y \leqslant (2l/N) \parallel U - V \parallel_Y$.

Choose N large enough such that $2l/N < 1$. Then F is a contraction [17] and there exists a global solution $z \in Y$.

Thus, the proof is completed. □

6. Stability of the solution

Here, we study the uniform stability of the solution $z \in Y$ of the nonlocal problem (1.1)-(1.4).

DEFINITION 2. The solution $z \in Y$ of the nonlocal problem (1.1)-(1.4) is *uniformly stable* if $\forall \varepsilon > 0, \exists \delta > 0$ such that $| x_0 - \widetilde{x}_0 | \leqslant \delta/2$ and $| y_0 - \widetilde{y}_0 | \leqslant \delta/2$ implies that $\parallel z - \tilde{z} \parallel_Y \leqslant \varepsilon$.

THEOREM 4. *Consider assumptions* $(H1)$-$(H2)$ *are satisfied, then the solution* $z \in Y$ *of the nonlocal problem* (1.1)-(1.4) *is uniformly stable.*

Proof. Let

$$U(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ax_0 + \int_0^t f_1(s, y(s))ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, y(s))ds \\ by_0 + \int_0^t f_2(s, x(s))ds - b \sum_{j=1}^m b_j \int_0^{\eta_j} f_2(s, x(s))ds \end{pmatrix}$$

be the solution of the nonlocal problem (1.1)-(1.4). Now

$$\tilde{U}(t) = \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} a\tilde{x}_0 + \int_0^t f_1(s, \tilde{y}(s))ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} f_1(s, \tilde{y}(s))ds \\ b\tilde{y}_0 + \int_0^t f_2(s, \tilde{x}(s))ds - b \sum_{j=1}^m b_j \int_0^{\eta_j} f_2(s, \tilde{x}(s))ds \end{pmatrix}$$

be the solution of the nonlocal problem (1.1), (1.2), (4.1) and (4.2), then

$$U(t) - \widetilde{U}(t) = \begin{pmatrix} V_1(t) - a \sum_{k=1}^n a_k \int_0^{\tau_k} [f_1(s, y(s)) - f_1(s, \tilde{y}(s))]ds \\ \\ V_2(t) - b \sum_{j=1}^m b_j \int_0^{\eta_j} [f_2(s, x(s)) - f_2(s, \tilde{x}(s))]ds \end{pmatrix},$$

where $V_1(t)$ and $V_2(t)$ are defined in (4.3) and (4.4), and

$$\parallel U - \tilde{U} \parallel_Y \leqslant |a(x_0 - \tilde{x}_0)|$$

$$+ \| \int_0^t [f_1(s,y(s)) - f_1(s,\tilde{y}(s))]ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f_1(s,y(s)) - f_1(s,\tilde{y}(s))]ds \|^*$$
$$+ |b(y_0 - \tilde{y}_0)|$$
$$+ \| \int_0^t [f_2(s,x(s)) - f_2(s,\tilde{x}(s))]ds - b\sum_{j=1}^{m} b_j \int_0^{\eta_j} [f_2(s,x(s)) - f_2(s,\tilde{x}(s))]ds \|^*$$
$$\leqslant a\delta/2 + (2l/N) \| y_1 - y_2 \|^* + b\delta/2 + (2l/N) \| x_1 - x_2 \|^* .$$

Then

$$\| U - \tilde{U} \|_Y \leqslant (a+b)(\delta/2)/(1-2l/N) \leqslant \delta/(1-2l/N) = \varepsilon.$$

Thus, the proof is completed. □

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