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ON SENSITIVITY OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS USING FRIEDRICHS CONSTANT

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Abstract. We treat nonresonance of quasilinear elliptic equations with nonlinear boundary conditions via Friedrichs constant $c_\star$. Here we give a sensitivity result for nonlinear perturbations of the right hand side and provide an explicit estimate for $c_\star$ in convex domains. Finally, to illustrate our results we discuss the p-Laplacian case.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ. Consider the Dirichlet eigenvalue problem $-\Delta u = \lambda u$ in Ω ; $u = 0$ on Γ and for given $\rho \in \mathbb{R}$ the related Dirichlet problem $P_{\rho,dir}$: For given f, find u such that there holds

$$-\Delta u = \rho u + f \text{ in } \Omega \,; \; u = 0 \text{ on } \Gamma. \tag{1.1}$$

The problem $P_{\rho,dir}$ is called nonresonant, if for every $f \in H^{-1}(\Omega)$ there exists at least one solution $u \in H^1(\Omega)$; otherwise it is resonant. Since the seminal paper of Landesman and Lazer [12] there was much work dealing with existence conditions for resonant and nonresonant elliptic boundary value problems. Let us state some existence results for quasilinear elliptic Dirichlet problems

$$Qu = f \text{ in } \Omega; \; u = \varphi \text{ on } \Gamma \tag{1.2}$$

from literature. Sufficient conditions for nonresonance are presented e.g. in [14, 15, 11] in the semilinear case and in [2, 8] for the p-Laplacian. [3] investigates nonresonance to the right of the first eigenvalue of the one dimensional p-Laplacian. Existence conditions at resonance are given by [12, 17] for semilinear problems and by [5, 9] for the p-Laplacian.

In [10] De Figueiredo and Gossez deal with nonresonance below the first eigenvalue in the semilinear case. We will focus on this type of nonresonance in the following note.

Consider the linear problem (1.1) again. The Lax-Milgram Theorem and Friedrichs' inequality tell us that there is a nonresonant state of (1.1) if $\rho < \lambda_1$; where λ_1 is the

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principal eigenvalue of $\big(-\Delta, H_0^1(\Omega)\big)$. Alternatively, nonresonance below λ_1 can be described by the associated Friedrichs constant

$$c_F(\Omega) = \sup_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega)}} = \frac{1}{\sqrt{\lambda_1}},$$

i.e. the optimal constant in Friedrichs' inequality $\|u\|_{L^2(\Omega)} \leqslant c_F \|\nabla u\|_{L^2(\Omega)}$ for $u \in H_0^1(\Omega)$. Observe that the relation $\rho \leqslant \lambda_1$ yields uniform ellipticity of the operator in the variational form of (1.1). In the nonlinear case (1.2) uniform ellipticity is more restrictive than nonresonance below the first eigenvalue presented e.g. in [10]. One the other hand we are able to state sufficient conditions not only for the existence solutions, but also for uniqueness. Moreover, we obtain explicit a priori bounds and sensitivity results with respect to nonlinear perturbations of the right hand side. We will apply this concept to the following quasilinear boundary value problem P_ρ in divergence form with a nonlinear boundary condition:

$$\begin{aligned} \operatorname{div}\big(A(\cdot,\nabla u_\rho)\big) &= \rho\,\alpha(\cdot,u_\rho) + f \quad \text{in } \Omega \\ -A(\cdot,\nabla u_\rho)n &= \beta(\cdot,u_\rho) \quad \text{on } \Gamma. \end{aligned} \tag{1.3}$$

Here, A, α, β are Carathéodory functions satisfying certain growth conditions. The existence of solutions to such problems under general conditions is provided by [1] in the semilinear case. [13, 19] treat the behaviour of (1.3) for the p-Laplacian at resonance. Our aim is to complement investigations on (1.3) by an explicit sensitivity estimate for nonlinear perturbations of the right hand side of (1.3).

The paper is organized as follows. In section 2 we define the Friedrichs constant $c_\star$ induced by a problem specific norm $\|\cdot\|_\star$ on $W^{1,p}(\Omega)$. Then we give sufficient conditions for uniform ellipticity of (1.3) in terms of $\|\cdot\|_\star$ and $c_\star$ and provide an explicit bound on u_ρ in the norm $\|\cdot\|_\star$. We call this state of (1.3) subresonant. In section 3 we study the sensitivity of subresonant solutions for $\rho \to 0$. An explicit estimate for $c_\star$ in convex domains shows the dependence of the sensitivity on parameters of the boundary value problem. In particular we reveal the antagonistic character of the flow β over the boundary Γ and the source term α in Ω. This is -to our knowledge- a new result. Section 4 concludes with the p-Laplace operator to illustrate the obtained results. For physical applications we refer to [6, 7].

2. Formulation of the subresonance conditions

2.1. Setup of the boundary value problem

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$ be a bounded domain with a Lipschitz boundary $\partial\Omega =: \Gamma$. For given $\rho \in \mathbb{R}$, $p \in (1,\infty)$ and $f \in \big(W^{1,p}(\Omega)\big)^*$ we consider the following nonlinear boundary value problem P_ρ in divergence form:

$$\begin{aligned} -\operatorname{div}\big(A(\cdot,\nabla u_\rho)\big) &= \rho\,\alpha(\cdot,u_\rho) + f \quad \text{in } \Omega \\ -A(\cdot,\nabla u_\rho)n &= \beta(\cdot,u_\rho) \quad \text{on } \Gamma \end{aligned} \tag{2.1}$$

where n denotes the outer normal on Γ. We assume that $A : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory map that satisfies the growth condition

$$\exists C_A < \infty : \ |A(x,\zeta)| \leqslant C_A\big(g_A(x) + |\zeta|^{p-1}\big) \quad \text{for almost every } x \in \Omega\,;\ \forall \zeta \in \mathbb{R}^d \tag{2.2}$$

for some $g_A \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, and the monotonicity condition

$$\exists c_A > 0 \ : \ \big(A(x,\zeta_1) - A(x,\zeta_2)\big)\big(\zeta_1 - \zeta_2\big) \geqslant c_A\,|\zeta_1 - \zeta_2|^p \quad \text{for a.e. } x \in \Omega \text{ and } \zeta_1, \zeta_2 \in \mathbb{R}^d. \tag{2.3}$$

Further we assume that $\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory map that satisfies $\alpha(\cdot,0) = 0$ and a Lipschitz-condition

$$\exists L_\alpha < \infty : \ |\alpha(x,s_1) - \alpha(x,s_2)| \leqslant L_\alpha\,|s_1 - s_2|^{p-1} \ \text{ for a.e. } x \in \Omega\,;\ \ s_1, s_2 \in \mathbb{R}. \tag{2.4}$$

Finally we assume that $\beta : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory map that satisfies the growth condition

$$\exists C_\beta < \infty : \ |\beta(x,s)| \leqslant C_\beta\big(g_\beta(x) + |s|^{p-1}\big) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}. \tag{2.5}$$

for given $g_\beta \in L^{p'}(\Gamma)$; and the monotonicity condition

$$\exists c_\beta > 0 \ : \ \big(\beta(x,s_1) - \beta(x,s_2)\big)(s_1 - s_2) \geqslant c_\beta\,|s_1 - s_2|^p \quad \text{for a.e. } x \in \Gamma \text{ and } s_1, s_2 \in \mathbb{R}. \tag{2.6}$$

Setting of an adaquate norm on $W^{1,p}(\Omega)$ In order to get concise terms for the following estimates we define the norm

$$\|v\|_\star^p := \|\nabla v\|^p_{L^p(\Omega)} + \frac{c_\beta}{c_A}\,\|v\|^p_{L^p(\Gamma)}\,.$$

Note that $\|\cdot\|_\star$ is equivalent to the canonical $W^{1,p}$-norm. We refer to a paper of Mikhlin, [16].

2.2. Existence and uniqueness in $W^{1,p}(\Omega)$

Variational form of P_ρ. For $u, v \in W^{1,p}(\Omega)$, we define the nonlinear operator $\mathscr{A}_\rho$ by

$$\big\langle \mathscr{A}_\rho u\,,\, v\big\rangle = \int_\Omega A(x,\nabla u)\,\nabla v\,\mathrm{d}x + \int_\Gamma \beta(x,u)\,v\,\mathrm{d}\sigma_x - \rho\int_\Omega \alpha(x,u)\,v\,\mathrm{d}x \tag{2.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\big(W^{1,p}(\Omega)\big)^*$ and $W^{1,p}(\Omega)$. The growth conditions on A and β in (2.2), (2.5) and the Lipschitz-condition on α in (2.4) imply the mapping property $\mathscr{A}_\rho : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^*$. Hence the variational form of (2.1) reads: Find $u_\rho \in W^{1,p}(\Omega)$ such that

$$\big\langle \mathscr{A}_\rho u_\rho\,,\, v\big\rangle = \langle f\,,\, v\rangle\,, \quad \forall v \in W^{1,p}(\Omega). \tag{2.8}$$

Well-posedness of P_ρ. Next we formulate a sufficient condition for subresonance of P_ρ . To this end we define the Friedrichs constant

$$c_\star := \sup_{v \in W^{1,p}(\Omega)\setminus\{0\}} \big(\|v\|_{L^p(\Omega)} / \|v\|_\star\big)$$

and the dual norm $\|f\|_* = \sup_{\|v\|_\star \leqslant 1} |\langle f, v\rangle|$.

PROPOSITION 1. *Let $|\rho| < \frac{c_A}{L_\alpha c_\star^p}$. Then, for all $f \in \big(W^{1,p}(\Omega)\big)^*$ there exists a unique solution $u_\rho \in W^{1,p}(\Omega)$ of (2.1) which is bounded by*

$$\big(c_A - |\rho|\, L_\alpha\, c_\star^p\big)\, \|u_\rho\|_\star^{p-1} \leqslant \|f\|_* + \|A(\,\cdot\,,0)\|_{L^{p'}(\Omega)} + \sqrt[p]{\frac{c_A}{c_\beta}}\, \|\beta(\cdot,0)\|_{L^{p'}(\Gamma)}\,.$$

Proof. (i) *Existence and uniqueness.* It follows from the Theorem of Browder and Minty for monotone operators, [20], in virtue of the uniform ellipticity of $\mathscr{A}_\rho$ for $|\rho| < \frac{c_A}{L_\alpha c_\star^p}$:

$$\begin{aligned}\big\langle \mathscr{A}_\rho u - \mathscr{A}_\rho v\,,\, u - v \big\rangle &\geqslant c_A\, \|\nabla(u-v)\|^p_{L^p(\Omega)} + c_\beta\, \|u - v\|^p_{L^p(\Gamma)} - |\rho|\, L_\alpha\, \|u - v\|^p_{L^p(\Omega)} \\ &\geqslant \big(c_A - |\rho|\, L_\alpha\, c_\star^p\big)\, \|u - v\|_\star^p \quad \forall u, v \in W^{1,p}(\Omega)\,.\end{aligned}$$

(ii) *boundedness.* We have $\big\langle\, f\,,\, u_\rho\, \big\rangle \leqslant \|f\|_*\, \|u_\rho\|_\star$; and on the other hand

$$\begin{aligned}\big\langle \mathscr{A}_\rho\, u_\rho\,,\, u_\rho \big\rangle \geqslant c_A\, \|\nabla u_\rho\|^p_{L^p(\Omega)} &- \|A(\,\cdot\,,0)\|_{L^{p'}(\Omega)}\, \|\nabla u_\rho\|_{L^p(\Omega)} \\ &+ c_\beta\, \|u_\rho\|^p_{L^p(\Gamma)} - \|\beta(\cdot,0)\|_{L^{p'}(\Gamma)}\, \|u_\rho\|_{L^p(\Gamma)} - |\rho|\, L_\alpha\, \|u_\rho\|^p_{L^p(\Omega)}.\end{aligned}$$

By the definition of $\|\cdot\|_\star$ and $\|\cdot\|_{L^p(\Omega)} \leqslant c_\star \|\cdot\|_\star$ we arrive at

$$\big\langle \mathscr{A}_\rho\, u_\rho\,,\, u_\rho \big\rangle \geqslant \big(c_A - |\rho|\, L_\alpha\, c_\star^p\big)\, \|u_\rho\|_\star^p - \Big(\|A(\,\cdot\,,0)\|_{L^{p'}(\Omega)} + \sqrt[p]{\frac{c_A}{c_\beta}}\, \|\beta(\cdot,0)\|_{L^{p'}(\Gamma)}\Big)\, \|u_\rho\|_\star$$

which implies the assertion. □

REMARK 1. If - in addition - the Carathéodory map α satisfies the monotonicity condition

$$\big(\alpha(x,s_1) - \alpha(x,s_2)\big)\big(s_1 - s_2\big) \geqslant 0 \text{ for a.e. } x \in \Omega\,;\;\; s_1, s_2 \in \mathbb{R}$$

then problem (2.1) is well posed for every $\rho < \frac{c_A}{L_\alpha c_\star^p}$. The bound on $\|u_\rho\|_\star$ in Proposition 1 holds for every $\rho < \frac{c_A}{L_\alpha c_\star^p}$ and reads as

$$c_A\, \|u_\rho\|_\star^{p-1} \leqslant \|f\|_* + \|A(\,\cdot\,,0)\|_{L^{p'}(\Omega)} + \sqrt[p]{\frac{c_A}{c_\beta}}\, \|\beta(\cdot,0)\|_{L^{p'}(\Gamma)} \quad \text{for } \rho \leqslant 0\,.$$

3. Approximation of subresonant solutions

In this section we study the approximation of P_0 by P_ρ, i.e. of

$$-\mathrm{div}\big(A(\cdot,\nabla u_0)\big) = f \quad \text{in } \Omega\,; \; -A(\cdot,\nabla u_0)\,n = \beta(\cdot,u_0) \quad \text{on } \Gamma$$

and give an error estimate in the norm $\|\cdot\|_\star$.

3.1. Sensitivity results

PROPOSITION 2. *Let u_ρ, u_0 denote the solutions of the boundary value problems P_ρ, P_0, respectively. Then there holds* $\limsup\limits_{\rho\to 0}\big(|\rho|^{\frac{1}{1-p}}\,\|u_\rho - u_0\|_\star\big) < \infty$.

Proof. Consider the difference in the variational equations of P_ρ and P_0, i.e.

$$\big\langle \mathscr{A}_\rho u_\rho - \mathscr{A}_0 u_0\,,\, v\big\rangle = 0 \quad \forall v \in W^{1,p}(\Omega)\,.$$

This reads as

$$\int\limits_\Omega \big(A(x,\nabla u_\rho) - A(x,\nabla u_0)\big)\,\nabla v\,\mathrm{d}x + \int\limits_\Gamma \big(\beta(x,u_\rho) - \beta(x,u_0)\big)\,v\,\mathrm{d}\sigma_x$$
$$= \rho\int\limits_\Omega \alpha(x,u_\rho)\,v\,\mathrm{d}x\,. \qquad (3.1)$$

Set $v = u_\rho - u_0$ and we obtain $c_A\,\|u_\rho - u_0\|_\star^p \leqslant \rho\int\limits_\Omega \alpha(x,u_\rho)\,(u_\rho - u_0)\,\mathrm{d}x$. The Lipschitz-continuity of α and Hölder's inequality imply

$$c_A\,\|u_\rho - u_0\|_\star^p \leqslant |\rho|\,L_\alpha\,\|u_\rho\|_{L^p(\Omega)}^{p-1}\,\|u_\rho - u_0\|_{L^p(\Omega)}\,.$$

This gives $c_A\,\|u_\rho - u_0\|_\star^{p-1} \leqslant |\rho|\,L_\alpha\,c_\star^p\,\|u_\rho\|_\star^{p-1}$. Using the upper bound on $\|u_\rho\|_\star^{p-1}$ for $\rho \to 0$ from Proposition 1 concludes the proof. $\square$

If the solution u_0 is explicitly known, the following estimate becomes useful.

PROPOSITION 3. *Let u_ρ, u_0 denote the solution of P_ρ, P_0, respectively. Then, for $|\rho| < \frac{c_A}{L_\alpha c_\star^p}$, there holds*

$$\big(c_A - |\rho|\,L_\alpha\,c_\star^p\big)\,\|u_\rho - u_0\|_\star^{p-1} \leqslant |\rho|\,L_\alpha\,c_\star^p\,\|u_0\|_\star^{p-1}\,.$$

Proof. As in the proof of Proposition 2 we have

$$c_A\,\|u_\rho - u_0\|_\star^p \leqslant |\rho|\int\limits_\Omega \big|\alpha(x,u_\rho)\big|\,\big|u_\rho - u_0\big|\,\mathrm{d}x.$$

Using the triangle inequality for the right hand side yields

$$c_A \|u_\rho - u_0\|_\star^p \leqslant |\rho| \int_\Omega \left(\left|\alpha(x,u_\rho) - \alpha(x,u_0)\right| + |\alpha(x,u_0)| \right) \left|u_\rho - u_0\right| \mathrm{d}x .$$

Lipschitz-continuity of α gives $|\alpha(x,u_0)| \leqslant L_\alpha |u_0|^{p-1}$ and Hölder's inequality implies

$$c_A \|u_\rho - u_0\|_\star^p \leqslant |\rho| L_\alpha \|u_\rho - u_0\|_{L^p(\Omega)}^p + |\rho| L_\alpha \|u_0\|_{L^p(\Omega)}^{p-1} \|u_\rho - u_0\|_{L^p(\Omega)} .$$

Using $\|\cdot\|_{L^p(\Omega)} \leqslant c_\star \|\cdot\|_\star$ yields the estimate. $\square$

3.2. An explicit estimate for $c_\star$ on convex domains

To arrive at a more precise approximation in Proposition 2, we establish

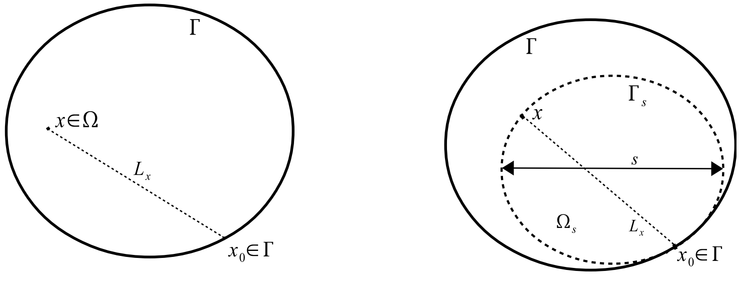
THEOREM 1. (Inhomogeneous Friedrichs inequality in $W^{1,p}(\Omega)$) *Assume that the set $\Omega \subset \mathbb{R}^d$ is bounded and convex and $p \in [1,\infty)$. Then, for $u \in W^{1,p}(\Omega)$ we have*

$$\|u\|_{L^p(\Omega)}^p \leqslant 2^{p-1} \left(\frac{\operatorname{diam}(\Omega)}{d} \|u\|_{L^p(\Gamma)}^p + \frac{\operatorname{diam}(\Omega)^p}{p} \|\nabla u\|_{L^p(\Omega)}^p \right).$$

Proof. Assume $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and $0 = x_0 \in \Gamma$. Otherwise translate $\tilde{x} := x - x_0$. The Mean value theorem for integration allows to choose $x_0 \in \Gamma$ such that

$$|u(x_0)|^p = \left| \frac{1}{|\Gamma|} \int_\Gamma u \,\mathrm{d}\sigma \right|^p \leqslant \frac{1}{|\Gamma|} \int_\Gamma |u|^p \,\mathrm{d}\sigma = \frac{1}{|\Gamma|} \|u\|_{L^p(\Gamma)}^p . \tag{3.2}$$

The estimate follows by Jensen's inequality.



For every $x \in \Omega$ define the line segment $L_x = \{t\,x;\ t \in (0,1)\} \subset \Omega$. Then there holds

$$\begin{aligned} |u(x) - u(0)| &= \left| \int_0^1 \frac{d(u \circ \gamma)(t)}{dt} \,\mathrm{d}t \right| \leqslant \int_0^1 |\langle \nabla u(\gamma(t)), \dot{\gamma}(t) \rangle| \,\mathrm{d}t \\ &\leqslant \int_0^1 |\nabla u(\gamma(t))| \, |\dot{\gamma}(t)| \,\mathrm{d}t = \int_{L_x} |\nabla u| \,\mathrm{d}\gamma \end{aligned}$$

where $\gamma : [0,1] \to L_x$ denotes a parametrization of L_x, thus $|u(x)| \leqslant |u(0)| + \int_{L_x} |\nabla u| \, \mathrm{d}\gamma$. The finite form of Jensen's inequality gives

$$|u(x)|^p \leqslant 2^{p-1} \big(|u(0)|^p + \big(\int_{L_x} |\nabla u| \, \mathrm{d}\gamma \big)^p \big).$$

The general form implies

$$|u(x)|^p \leqslant 2^{p-1} \left(|u(0)|^p + |L_x|^{p-1} \int_{L_x} |\nabla u|^p \, \mathrm{d}\gamma \right). \tag{3.3}$$

By $\Omega_s := \frac{s}{\operatorname{diam}(\Omega)} \Omega$, $s \in (0, \operatorname{diam}(\Omega))$, with its boundary Γ_s, we denote a homotopic contraction of Ω to $x_0 = 0$. This contraction exists since Ω is convex. Since $s = \operatorname{diam}(\Omega_s)$, we have

$$|\Gamma_s| = \frac{|\Gamma|}{\operatorname{diam}(\Omega)^{d-1}} s^{d-1}.$$

As $x \in \Gamma_s$ implies $|L_x| = |x| \leqslant s$ and since Ω_s is convex, an integration of (3.3) over Γ_s yields

$$\int_{\Gamma_s} |u|^p \, \mathrm{d}\sigma \leqslant 2^{p-1} \left(|\Gamma_s| \, u(0)^p + s^{p-1} \int_{\Omega_s} |\nabla u|^p \, \mathrm{d}x \right).$$

Using $\int_{\Omega_s} |\nabla u|^p \, \mathrm{d}x \leqslant \|\nabla u\|^p_{L^p(\Omega)}$, an integration over s provides via Cavalieri's principle

$$\|u\|^p_{L^p(\Omega)} \leqslant 2^{p-1} \left(\frac{|\Gamma| \, |u(0)|^p}{\operatorname{diam}(\Omega)^{d-1}} \int_0^{\operatorname{diam}(\Omega)} s^{d-1} \, \mathrm{d}s + \frac{\operatorname{diam}(\Omega)^p}{p} \|\nabla u\|^p_{L^p(\Omega)} \right).$$

Now (3.2) and an extension via density to arbitrary $u \in W^{1,p}(\Omega)$ finally imply the assertion. □

The definition of $\|\cdot\|_\star$ and Theorem 1 give the following

COROLLARY 1. (Estimate of $c_\star$ via scaling) *Let the assumptions of Theorem 1 hold. Moreover, assume that Ω is scaled with* $\operatorname{diam}(\Omega) = 1$. *Then, for every $u \in W^{1,p}(\Omega)$, we have*

$$\|u\|^p_{L^p(\Omega)} \leqslant 2^{p-1} \max \left(\frac{c_A}{d \, c_\beta}, \frac{1}{p} \right) \|u\|^p_\star ; \quad \text{hence } c_\star \leqslant \sqrt[p]{2^{p-1} \max \big(\frac{c_A}{d \, c_\beta}, \frac{1}{p} \big)}.$$

REMARK 2. Sharpness investigations on Theorem 1 are outstanding. Consider the case $u \in H^1_0(\Omega)$. Then Theorem 1 reads as $\|u\|_{L^2(\Omega)} \leqslant \operatorname{diam}(\Omega) \|\nabla u\|_{L^2(\Omega)}$. Due to a classical result of Payne and Weinberger [18] there even holds

$$\|u\|_{L^2(\Omega)} \leqslant \frac{\operatorname{diam}(\Omega)}{\pi} \|\nabla u\|_{L^2(\Omega)}.$$

This indicates that the estimate in Theorem 1 is not sharp. On the other hand it is possible to extend the result of Theorem 1 to non-convex domains, [7]. Hence it is fairly related to the result in [18] where we assume more restrictively $\int_\Omega u \, \mathrm{d}x = 0$ and the convexity of the domain Ω.

4. The p-Laplace equation with monotone boundary conditions - an example

In this section we consider the p-Laplace equation with monotone boundary conditions as an example for the problem P_ρ,

$$\begin{aligned} -\operatorname{div}\big(|\nabla u_\rho|^{p-2}\nabla u_\rho)\big) &= \rho\,\alpha(\cdot,u_\rho)+f \quad \text{in } \Omega \\ -|\nabla u_\rho|^{p-2}\nabla u_\rho\, n &= \beta(\cdot,u_\rho) \quad \text{on } \Gamma. \end{aligned} \tag{4.1}$$

Thus we have $A(\zeta)=|\zeta|^{p-2}\zeta$. The growth condition (2.2) is fulfilled. Following [4], condition (2.3) is satified for $p\geqslant 2$. In addition we assume the continuity, growth- and monotonicity conditions on α,β from section 2.1 to hold. For simplicity, let us assume $d=2$, $\operatorname{diam}(\Omega)=1$ and $c_\beta<c_A$. Thus by Corollary 1 we obtain

$$c_\star=\sqrt[p]{\frac{2^{p-2}c_A}{c_\beta}}.$$

COROLLARY 2. *Assume* $|\rho|<\frac{2^{2-p}c_\beta}{L_\alpha}$, $p\geqslant 2$. *Then, for all* $f\in\big(W^{1,p}(\Omega)\big)^*$ *there exists a unique solution* $u_\rho\in W^{1,p}(\Omega)$ *of* (4.1) *which is bounded by*

$$c_A\left(1-|\rho|\frac{L_\alpha}{2^{2-p}c_\beta}\right)\|u_\rho\|_\star^{p-1}\leqslant\|f\|_*+\|A(\cdot,0)\|_{L^{p'}(\Omega)}+\sqrt[p]{\frac{c_A}{c_\beta}}\,\|\beta(\cdot,0)\|_{L^{p'}(\Gamma)}.$$

The proof follows directly by the application of Proposition 1. The approximation result of Proposition 3 applied to (4.1) leads to

COROLLARY 3. *Let* u_ρ,u_0 *denote the solution of* P_ρ,P_0 *in* (4.1) *respectively. Then, for* $|\rho|<\frac{2^{2-p}c_\beta}{L_\alpha}$, $p\geqslant 2$, *there holds*

$$\left(\frac{2^{2-p}c_\beta}{L_\alpha}-|\rho|\right)\|u_\rho-u_0\|_\star^{p-1}\leqslant|\rho|\,\|u_0\|_\star^{p-1}.$$

REMARK 3. Observe that an increase of c_β extends the range of the subresonant state of (4.1) and reduces the difference $\|u_\rho-u_0\|_\star$ in Corollary 3. This implies that an increase of the flow β over the boundary Γ compensates an increase of the source term α in Ω.

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