Differential
Equations
& Applications
Volume 5, Number 3 (2013), 377–393
doi:10.7153/dea-05-24

BOUNDARY BLOW–UP RATES OF LARGE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH CONVENTION TERMS

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(*Communicated by Hiroyuki Usami*)

Abstract. We use Karamata regular variation theory to study the exact asymptotic behavior of large solutions near the boundary to a class of quasilinear elliptic equations with convection terms

$$\begin{cases} \Delta_p u \pm |\nabla u|^{q(p-1)} = b(x)f(u), & x \in \Omega, \\ u(x) = +\infty, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^{\mathbb{N}}$. The weight function $b(x)$ is a non-negative continuous function in the domain, $f(u) \in C^2[0,+\infty)$ is increasing on $[0,\infty)$, and regularly varying at infinity with index $\rho > p-1$.

1. Introduction and main results

In this paper, we will investigate the exact asymptotic behavior of large solutions near the boundary for the following quasilinear elliptic problems:

$$\begin{cases} \Delta_p u \pm |\nabla u|^{q(p-1)} = b(x)f(u), & x \in \Omega, \\ u(x) = +\infty, & x \in \partial\Omega, \end{cases} \qquad (P_\pm)$$

where Ω is a smooth bounded domain of $\mathbb{R}^{\mathbb{N}}(\mathbb{N} > 2)$ with smooth boundary $\partial\Omega$, $1 < p < \infty$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. The nonlinear terms f and $b(x)$ satisfy:

(f_1) $f \in C^2[0,+\infty), f(0) = 0, f$ is increasing on $[0,+\infty)$;

(b_1) $b(x) \in C(\Omega)$ is a non-negative function .

A large(or explosive) of $(P_\pm)$ we mean that $u(x) \to +\infty$ as $d(x) = \mathrm{dist}(x,\partial\Omega) \to 0^+$. Such problem arise in the study of the subsonic motion of a gas [26], the electric potential in some bodies [19], and Riemannian geometry [5].

Mathematics subject classification (2010): 35J25, 35J40, 35J92.
Keywords and phrases: elliptic equation, regular variation, large solutions, boundary blow-up rate, convection terms.
The Project Supported by the National Natural Science Foundation of China(No.11171092). Project Supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB110005).

The study of large solution goes back many years. Various authors have investigated the existence, asymptotic boundary behavior and uniqueness of solution to the problem:

$$\begin{cases} \Delta u = g(x)f(u), & \text{in } \Omega, \\ u(x) \to \infty, & \text{as } x \to \partial\Omega. \end{cases} \tag{1.1}$$

See references [2, 3, 7, 8, 18, 20, 24, 29, 30]. Still, let us review the following model

$$\begin{cases} \Delta_p u = g(x)f(u), & \text{in } \Omega, \\ u(x) \to \infty, & \text{as } x \to \partial\Omega. \end{cases} \tag{1.2}$$

This problem also has been studied by several authors, see e.g. [12, 13, 23] and the references therein. Gladiali and Porru [13] study boundary asymptotic of solutions of this equation under some condition on f and when $g(x) \equiv 1$. Related problems on asymptotic behavior and uniqueness and also studies in [12]. Ahmed Mohammed in [23] establish boundary asymptotic estimate for solution of this equation under appropriate conditions on g and the nonlinearity f. They still allow g to be unbounded on Ω or to vanish on $\partial\Omega$.

More recently, many authors studied the exact asymptotic behavior of solution near the boundary to the following model:

$$\begin{cases} \Delta_p u + \lambda |\nabla u|^{q(p-1)} = k(x)f(u), & x \in \Omega \\ u(x) = +\infty, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

when $k(x) = e^{u(x)}$, by a perturbation method and by constructing comparison functions, C.L.Liu and Z.D.Yang [21] show the exact asymptotic behavior of solution near the boundary of problem (1.3). Furthermore, under suitable growth assumptions on k near the boundary and on f both at zero at infinity, the authors [22] show the existence of at least one solution in $C^1(\Omega)$ base on the method of explosive sub-supersolutions, which permits positive weights $k(x)$ which are unbounded and/or oscillatory near the boundary.

In a different direction, F.C. Cîrstea [6] opened a unified new approach, they using the Karamata regular variation theory to study the uniqueness and asymptotic behavior of large solutions to the semilinear elliptic equation $\Delta u + au = b(x)f(u)$ where f $\Gamma-$varying at ∞ they show that when f grows faster than any $u^p(p > 1)$ then the vanishing rate of b at $\partial\Omega$ enters into the completion with the grows of f at ∞. Still in [8, 9, 10, 25] many authors concerned with the existence, uniqueness and exact asymptotic behavior of solutions for the following quasilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u), & \text{in } \Omega \\ u = +\infty, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The uniqueness of such solution for a large class of functions $f(u)$ (including u^p or e^u as a special case and more general function) follows as a consequence of the exact blow-up rate.

More recently, the following model:

$$\begin{cases} -\Delta u+|\nabla u|^q=b(x)f(u), & x\in\Omega \\ u=+\infty, & x\in\partial\Omega \end{cases} \tag{1.5}$$

were studied widely. When $b(x)=1$, by the ordinary differential equation theory and the comparison principle,Bandle and Giarrusso [1] studied the existence and asymptotic boundary behavior of solution to (1.5) with two classical nonlinearities $f(u)=u^p, p>1$ and $f(u)=e^u$. In [15], S.B.Huang and Q.Y.Tian use Karamata regular variation theory, a perturbation method and constructing comparison function to show the exact asymptotic behavior of large solutions to the problem (1.5).

Motivated by the results of the above cited papers, we use Karamata regular variation theory to further deal with the boundary blow-up rate of large solutions of $(P_\pm)$ for more general nonlinear term $f(u)$, the results of the semilinear equation are extended to the quasilinear ones.

This paper is organized as follows. In the next section, we give some useful definitions and prove some properties from regular variation theory. The proof of the main Theorem carried out in the third section.

Before give out our main theorem, now we introduce a class of functions used to describe the behavior of weight function $b(x)$. Let κ_1 denote (as in [14, 23]) the set of all positive, monotonic functions $k\in L^1(0,\tau)\bigcap C^1(0,\tau)$, for some $\tau>0$, which satisfy:

$$\lim_{t\to 0+} K(t)/k(t)=0, \quad \lim_{t\to 0+}\frac{d}{dt}\big(K(t)/k(t)\big)=l, \text{ where } K(t)=\int_0^t k(s)ds.$$

We point out that $l\in[0,1]$ if k is non-decreasing and $l\in[1,\infty)$ if k is non-increasing. For more propositions of κ_1 refer to [7, 11]. Some examples of functions $k\in\kappa_1$ are:

(1) $k(t)=t^q$ for $q>-1$ with $l=\frac{1}{1+q}$;

(2) $k(t)=(-lnt)^q$ for $q<0$ with $l=1$;

(3)$k(t)=exp(-t^q)$ for $q<0$ with $l=0$.

DEFINITION. A positive measurable function f defined on $[A,\infty)$ for some $A>0$ is called regularly varying (at infinity) with index $\rho\in\mathbb{R}$ (written $f\in RV\rho$) if for all $\xi>0, \lim\limits_{u\to\infty} f(\xi u)/f(u)=\xi^\rho$.

We modify the methods developed in [15], which give the following theorems

THEOREM 1.1. *Let $f\in RV_\rho(\rho>p-1)$ with $q(p-1)<\rho$ satisfy (f_1), $b(x)$ satisfy (b_1).*

(1) *If $b(x)$ satisfies*

(b_2) $\lim\limits_{d(x)\to 0}\frac{b(x)}{k^p\big(d(x)\big)}=c_0>0$, for $k(x)\in\kappa_1$ with $0<l<\infty$, and

$$0<q<\frac{p^{p-1}[p-1+l(\rho+1-p)]}{p+l(\rho+1-p)} \tag{1.6}$$

then every solution $u_\pm \in C^1(\Omega)$ *to problem* $(P_\pm)$ *satisfies*

$$\lim_{d(x)\to 0_+} \frac{u_\pm(x)}{\phi\big(d(x)\big)} = \xi_1, \tag{1.7}$$

where ϕ *is uniquely determined by*

$$f\big(\phi(t)\big)K^p(t) = \phi^{p-1}(t), \tag{1.8}$$

and

$$\xi_1 = \Big[\frac{p^{p-1}(p-1)(p-lp+l\rho+l)}{c_0(\rho+1-p)^p}\Big]^{\frac{1}{\rho+1-p}}.$$

(2) *If* $b(x)$ *satisfies*

(b_3) $\lim_{d(x)\to 0} \frac{b(x)}{k^{q(p-1)}\big(d(x)\big)} = c_q > 0$, *for* $k(x) \in \kappa_1$ *with* $0 < l < \infty$, *and* q *satisfies*

$$q(p-1) - p + \frac{q(p-1)[1-q(p-1)]}{l\big[q(p-1)-\rho\big]} > 0 \text{ and } q(p-1) < \rho, \tag{1.9}$$

then every solution $u_+ \in C^1(\Omega)$ *to problem* (P_+) *satisfies*

$$\lim_{d(x)\to 0_+} u_+(x)/\varphi\big(d(x)\big) = \xi_2, \tag{1.10}$$

where φ *is uniquely determined by*

$$f\big(\varphi(t)\big)K^{q(p-1)} = \varphi^{q(p-1)}(t), \tag{1.11}$$

and

$$\xi_2 = \Big[\frac{1}{c_q}\Big(\frac{q(p-1)}{\rho - q(p-1)}\Big)^{q(p-1)}\Big]^{\frac{1}{\rho-q(p-1)}}$$

(3) *If* $b(x)$ *satisfies* (b_3) *and* q *satisfies*

$$1 < q < \frac{p}{p-1} \text{ and } \rho\alpha l + q(p-1)(p-1-l) > q(p-1)(\alpha - p + 1). \tag{1.12}$$

then every solution $u_- \in C^1(\Omega)$ *to problem* (P_-) *satisfies*

$$\lim_{d(x)\to 0_+} u_-(x)d^{-\alpha}(x) = \xi_3, \tag{1.13}$$

where $\alpha = \frac{q(p-1)-p}{q(p-1)-p+1}$ *and*

$$\xi_3 = \Big[\frac{(\alpha-1)(p-1)}{\alpha^{(q-1)(p-1)}}\Big]^{\frac{1}{(q-1)(p-1)}}$$

REMARK. When $b(x)$ satisfies conditions b_1 and b_2, for example, we can choose $b(x) = c_0 k^p\big(d(x)\big) + o\big(k^p(d(x))\big)$; When $b(x)$ satisfies conditions b_1 and b_3, we can choose $b(x) = c_q k^{q(p-1)}\big(d(x)\big) + o\big(k^{q(p-1)}(d(x))\big)$.

2. Preliminaries

In this section we give some preliminary considerations on various assumptions and properties needed for our main result. We start with some basic definitions and properties of regular variation theory which was initiated by Jovan and Karamata in a well-known paper of [16] in 1930. For detailed accounts of the theory of regular variation, its extensions and many of its applications, we refer the interested reader to [4, 17, 27, 28].

DEFINITION 2.1. A positive measurable function L defined on $[A,+\infty)$ for some $A>0$ is called slowly varying at infinity if for all $\xi>0$, $\lim\limits_{u\to\infty} L(\xi u)/L(u)=1$.

From the above definition we easily deduce that if L varies slowly, then $u^{\rho}L(u)\in RV_{\rho}$.

DEFINITION 2.2. A function $f(u)$ defined for $u>A$ is called a normalized regularly varying function of index ρ (in short $f\in NRV_{\rho}$) if it is C^1 and satisfies

$$\lim_{u\to\infty}\frac{uf'(u)}{f(u)}=\rho.$$

The concept of normalized regular variation can be applied at zero as follows:

DEFINITION 2.3. We say that f is normalized regularly varying at (the right of) zero with index $\rho\in R$ (written $f\in NRV_{\rho}(0+)$) if $u\to f(1/u)$ is normalized regularly varying at ∞ with index $-\rho$.

PROPOSITION 2.1. *If L is slowly varying, then:*

(1) *For any $\alpha>0$, $u^{\alpha}L(u)\to\infty$, $u^{-\alpha}L(u)\to 0$ as $u\to\infty$.*
(2) *$\big(L(u)\big)^{\alpha}$ varies slowly for every $\alpha\in R$.*
(3) *If L_1 varies slowly, so do $L(u)L_1(u)$ and $L(u)+L_1(u)$.*

PROPOSITION 2.2. (Representation theorem) *The function $L(u)$ is slowly varying if and only if it can be written in the form*

$$L(u)=M(u)exp\Big\{\int_B^u\frac{\omega(t)}{t}dt\Big\}\ \ (u\geqslant B),$$

for some $B>0$, where $\omega\in C[B,\infty)$ satisfies $\lim\limits_{u\to\infty}\omega(u)=0$ and $M(u)$ is measurable on $[B,\infty)$ such that $\lim\limits_{u\to\infty}M(u)=M_0\in(0,\infty)$.

PROPOSITION 2.3. (see [21] Weak Comparison Principle) *Let Ω be a bounded domain in $\mathbb{R}^{\mathbb{N}}(\mathbb{N}>2)$ with a smooth boundary $\partial\Omega$ where $\varphi:(0,a)\to(0,a)$ is continuous and non-decreasing. Let $u_1,u_2\in W^{1,p}(\Omega)$ satisfy*

$$\int_{\Omega}|\nabla u_1|^{p-1}\nabla u_1\nabla\psi dx+\int_{\Omega}\varphi u_1\psi dx\leqslant\int_{\Omega}|\nabla u_2|^{p-1}\nabla u_2\nabla\psi dx+\int_{\Omega}\varphi u_2\psi dx$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$. *Then the inequality*

$$u_1 \leqslant u_2, \quad on \quad \partial\Omega$$

implies that

$$u_1 \leqslant u_2, \quad in \quad \partial\Omega.$$

LEMMA 2.1. *The function* ϕ *given by (1.8) is well defined and*

(1) $\lim_{t\to 0} \frac{t\phi'(t)}{\phi(t)} = \frac{p}{l(p-1-\rho)}$;

(2) $\lim_{t\to 0} \frac{\left[\phi'(t)\right]^{q(p-1)-p+2}}{\phi''(t)} = 0$;

(3) $\lim_{t\to 0} \frac{\phi'(t)}{\phi''(t)} = \lim_{t\to 0} \frac{\phi(t)}{\phi'(t)} = \lim_{t\to 0} \frac{\phi(t)}{\phi''(t)} = 0$;

(4) $\lim_{t\to 0} \frac{|\phi'(t)|^{p-2}\phi''(t)}{k^p(t)f\left(\phi(t)\right)} = \frac{p^{p-1}[p-lp-l\rho+l]}{(\rho+1-p)^p}$.

Proof. (1) As $f \in RV_\rho(\rho > p-1)$ with $q(p-1) < \rho$ satisfy (f_1),it is easy to know that the function ϕ given by (1.8) is well defined. By (1.8), we find

$$(p-1)\phi^{p-2}(t)\phi'(t) = f'\left(\phi(t)\right)K^p(t)\phi'(t) + pf\left(\phi(t)\right)K^{p-1}(t)k(t)$$

Thus

$$\phi'(t) = \frac{f'\left(\phi(t)\right)K^p(t)\phi'(t)}{(p-1)\phi^{p-2}(t)} + \frac{pf\left(\phi(t)\right)K^{p-1}(t)k(t)}{(p-1)\phi^{p-2}(t)}. \tag{2.1}$$

Then

$$\lim_{t\to 0} \frac{t\phi'(t)}{\phi(t)} = \lim_{t\to 0} \frac{1}{p-1} \frac{f'\left(\phi(t)\right)}{f\left(\phi(t)\right)} \frac{t\phi'(t)}{\phi(t)} + \frac{p}{p-1} \lim_{t\to 0} \frac{tk(t)}{K(t)}$$

From this, we obtain that

$$\lim_{t\to 0} \frac{t\phi'(t)}{\phi(t)} = \frac{p}{l(p-1-\rho)}. \tag{2.2}$$

(2) By a direct calculation of (2.1), we have

$$\begin{aligned}(p-1)\phi^{p-2}(t)\phi''(t) &+ (p-1)(p-2)\phi^{p-3}(t)\left[\phi'(t)\right]^2 \\ &= f''\left(\phi(t)\right)K^p(t)\left[\phi'(t)\right]^2 + 2pf'\left(\phi(t)\right)K^{p-1}(t)k(t)\phi'(t) \\ &\quad + f'\left(\phi(t)\right)K^p(t)\phi''(t) + p(p-1)f\left(\phi(t)\right)K^{p-2}(t)k^2(t) \\ &\quad + pf\left(\phi(t)\right)K^{p-1}(t)k'(t),\end{aligned}$$

which implies that

$$\phi''(t) = \frac{1}{(p-1)\phi^{p-2}(t) - f'\big(\phi(t)\big)K^p(t)}\Big[f''(\phi(t))K^p(t)[\phi'(t)]^2 + 2pf'(\phi(t))K^{p-1}(t)k(t)\phi'(t) + p(p-1)f(\phi(t))K^{p-2}(t)k^2(t) + pf\big(\phi(t)\big)K^{p-1}(t)k'(t) - (p-1)(p-2)\phi^{p-3}(t)[\phi'(t)]^2\Big]. \quad (2.3)$$

Taking into account (2.1), we still have

$$\phi'(t) = \frac{pf\big(\phi(t)\big)K^{p-1}(t)k(t)}{(p-1)\phi^{p-2}(t) - f'\big(\phi(t)\big)K^p(t)},$$

it follows that

$$\phi''(t) = \frac{f''\big(\phi(t)\big)K(t)\big[\phi'(t)\big]^3}{pf\big(\phi(t)\big)k(t)} + \frac{2f'\big(\phi(t)\big)[\phi'(t)]^2}{f\big(\phi(t)\big)} + \frac{(p-1)k(t)\phi'(t)}{K(t)} + \frac{k'(t)\phi'(t)}{k(t)} - \frac{(p-1)(p-2)\phi^{p-3}(t)\big[\phi'(t)\big]^3}{pf\big(\phi(t)\big)K^{p-1}(t)k(t)}.$$

Thus

$$\frac{\phi''(t)}{\big[\phi'(t)\big]^{q(p-1)-p+2}} = \frac{f''\big(\phi(t)\big)K(t)\big[\phi'(t)\big]^{q-qp+p+1}}{pf\big(\phi(t)\big)k(t)} + \frac{2f'\big(\phi(t)\big)\big[\phi'(t)\big]^{q-qp+p}}{f\big(\phi(t)\big)} + \frac{(p-1)k(t)\big[\phi'(t)\big]^{q-qp+p-1}}{K(t)} + \frac{k'(t)\big[\phi'(t)\big]^{q-qp+p-1}}{k(t)} - \frac{(p-1)(p-2)\phi^{p-3}(t)\big[\phi'(t)\big]^{q-qp+p+1}}{pf\big(\phi(t)\big)K^{p-1}(t)k(t)}$$
$$= E_1 + E_2 + E_3 + E_4 - E_5.$$

By (1.8), we get

$$\lim_{t\to 0} E_1 = \frac{1}{p}\lim_{t\to 0}\frac{\phi^2(t)f''\big(\phi(t)\big)}{f\big(\phi(t)\big)}\Big(\frac{t\phi'(t)}{\phi(t)}\Big)^{q-qp+p+1}\frac{K(t)}{tk(t)}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)}$$

Thanks to Proposition 2.2, $f \in RV_\rho$, which is equivalent to $f(u)$ being of the form

$$f(u) = M(u)u^\rho exp\Big\{\int_B^u \frac{\omega(t)}{t}dt\Big\} \quad (u \geqslant B),$$

where $\omega \in C[B,\infty)$ satisfies $\lim_{u\to\infty}\omega(u) = 0$ and $M(u)$ is measurable on $[B,\infty)$ such that $\lim_{u\to\infty} M(u) = M_0 \in (0,\infty)$. This fact implies that

$$\lim_{t\to 0}\frac{\phi^2(t)f''\big(\phi(t)\big)}{f\big(\phi(t)\big)} = \rho(\rho-1). \quad (2.4)$$

So,

$$\lim_{t\to 0} E_1 = \rho(\rho-1)l\Big[\frac{p}{l(p-1-\rho)}\Big]^{q-qp+p+1}\lim_{t\to 0}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)}, \tag{2.5}$$

Similarly, we have:

$$\begin{aligned}\lim_{t\to 0} E_2 &= 2\lim_{t\to 0}\frac{\phi(t)f'\big(\phi(t)\big)}{f\big(\phi(t)\big)}\Big(\frac{t\phi'(t)}{\phi(t)}\Big)^{q-qp+p}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)}\\ &= 2\rho\left[\frac{p}{l(p-1-\rho)}\right]^{q-qp+p}\lim_{t\to 0}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)},\end{aligned} \tag{2.6}$$

$$\lim_{t\to 0} E_3 = \frac{p-1}{l}\left[\frac{p}{l(p-1-\rho)}\right]^{q-qp+p-1}\lim_{t\to 0}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)}, \tag{2.7}$$

$$\lim_{t\to 0} E_4 = (1-l)\left[\frac{p}{l(p-1-\rho)}\right]^{q-qp+p-1}\lim_{t\to 0}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)}, \tag{2.8}$$

$$\lim_{t\to 0} E_5 = \frac{l(p-1)(p-2)}{p}\left[\frac{p}{l(p-1-\rho)}\right]^{q-qp+p+1}\lim_{t\to 0}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)}. \tag{2.9}$$

On the other hand, (1.6) and Proposition 2.1 show that

$$\lim_{t\to 0}\phi^{q-qp+p-1}(t)t^{-(q-qp+p)} = \infty.$$

Taking into account (2.6)-(2.9), we have

$$\lim_{t\to 0}\frac{\phi''(t)}{\big[\phi'(t)\big]^{q(p-1)-p+2}} = \lim_{t\to 0}E_1 + \lim_{t\to 0}E_2 + \lim_{t\to 0}E_3 + \lim_{t\to 0}E_4 - \lim_{t\to 0}E_5 = \infty.$$

That is,

$$\lim_{t\to 0}\frac{\big[\phi'(t)\big]^{q(p-1)-p+2}}{\phi''(t)} = 0.$$

(3) As $\lim_{t\to 0}\frac{t\phi'(t)}{\phi(t)} = \frac{p}{l(p-1-\rho)}$, this implies that

$$\lim_{t\to 0}\frac{\phi(t)}{\phi'(t)} = 0. \tag{2.10}$$

By $\lim_{t\to 0}\frac{\big[\phi'(t)\big]^{q(p-1)-p+2}}{\phi''(t)} = 0$ and (1.6), we get

$$\lim_{t\to 0}\frac{\phi'(t)}{\phi''(t)} = 0, \tag{2.11}$$

using (2.10)-(2.11), we obtain

$$\lim_{t\to 0}\frac{\phi(t)}{\phi''(t)}=\lim_{t\to 0}\frac{\phi(t)}{\phi'(t)}\frac{\phi'(t)}{\phi''(t)}=0.$$

(4) By (1.8) and (2.3), we get

$$\begin{aligned}\lim_{t\to 0}\frac{|\phi'(t)|^{p-2}\phi''(t)}{k^p(t)f\big(\phi(t)\big)}&=\lim_{t\to 0}\frac{|\phi'(t)|^{p-2}[\phi'(t)]^3f''\big(\phi(t)\big)K(t)}{pf^2\big(\phi(t)\big)k^{p+1}(t)}+\lim_{t\to 0}\frac{2|\phi'(t)|^pf'\big(\phi(t)\big)}{f^2\big(\phi(t)\big)k^p(t)}\\&+\lim_{t\to 0}(p-1)\frac{|\phi'(t)|^{p-2}\phi'(t)}{k^{p-1}(t)K(t)f\big(\phi(t)\big)}+\lim_{t\to 0}\frac{|\phi'(t)|^{p-2}\phi'(t)k'(t)}{k^{p+1}(t)f\big(\phi(t)\big)}\\&-\lim_{t\to 0}\frac{(p-1)(p-2)|\phi'(t)|^{p-2}\big[\phi'(t)\big]^3\big[\phi(t)\big]^{p-3}}{pk^{p+1}(t)f^2\big(\phi(t)\big)K^{p-1}(t)}\\&=F_1+F_2+F_3+F_4-F_5.\end{aligned}$$

$$\begin{aligned}F_1&=\lim_{t\to 0}\frac{\phi^2(t)f''\big(\phi(t)\big)}{pf\big(\phi(t)\big)}\left|\frac{t\phi'(t)}{\phi(t)}\right|^{p-2}\left(\frac{t\phi'(t)}{\phi(t)}\right)^3\left(\frac{K(t)}{tk(t)}\right)^{p+1}\\&=-\frac{\rho(\rho-1)}{p}\left[\frac{p}{\rho+1-p}\right]^{p+1}.\end{aligned}\tag{2.12}$$

Similarly,

$$F_2=2\rho\left[\frac{p}{\rho+1-p}\right]^p,\tag{2.13}$$

$$F_3=-(p-1)\left[\frac{p}{\rho+1-p}\right]^{p-1},\tag{2.14}$$

$$F_4=-(1-l)\left[\frac{p}{\rho+1-p}\right]^{p-1},\tag{2.15}$$

$$F_5=-\frac{(p-1)(p-2)}{p}\left[\frac{p}{\rho+1-p}\right]^{p+1}.\tag{2.16}$$

Thus, taking into account (2.12)-(2.16), we get

$$\lim_{t\to 0}\frac{|\phi'(t)|^{p-2}\phi''(t)}{k^p(t)f\big(\phi(t)\big)}=F_1+F_2+F_3+F_4-F_5=\frac{p^{p-1}[p-lp-l\rho+l]}{(\rho+1-p)^p}.$$

The proof of Lemma 2.1 is now complete. □

In a similar way we can prove that:

LEMMA 2.2. *The function ϕ given by (1.11) is well defined and*

(1) $\lim_{t\to 0}\frac{t\varphi'(t)}{\varphi(t)}=\frac{q(p-1)}{l\left[q(p-1)-\rho\right]}$;

(2) $\lim_{t\to 0}\frac{\varphi'(t)}{\varphi''(t)}=\lim_{t\to 0}\frac{\varphi(t)}{\varphi'(t)}=\lim_{t\to 0}\frac{\varphi(t)}{\varphi''(t)}=0$;

(3) $\lim_{t\to 0}\frac{|\varphi'(t)|^{p-2}\varphi''(t)}{k^{q(p-1)}(t)f\big(\varphi(t)\big)}=0$.

(4) $\lim_{t\to 0}\frac{|\varphi'(t)|^{q(p-1)}}{k^{q(p-1)}(t)f\big(\varphi(t)\big)}=\big[\frac{q(p-1)}{\rho-q(p-1)}\big]^{q(p-1)}$.

3. Proof of the Main Theorem

In this section, we are now ready to present the proof of Theorem 1.1.

Proof. Give $\delta>0,\forall\beta\in(0,\delta)$, denote

$$\Omega_\delta=\{x\in\Omega,0<d(x)<\delta\},\ \ \partial\Omega_\delta=\{x\in\Omega,d(x)=\delta\},$$

$$\Omega_\beta^-=\Omega_{2\delta}\setminus\overline{\Omega}_\beta,\ \ \Omega_\beta^+=\Omega_{2\delta-\beta}.$$

Proof of (1): *Case 1*: k is non-decreasing, then $l\in(0,1]$.

Refer to the reference [21] and [30], we can choose a $\delta>0$ sufficiently small such that

(1) $d(x)\in C^2(\Omega_{2\delta})$, and $|\nabla d(x)|=1$ in $\Omega_{2\delta}$;

(2) $k(x)$ is non-decreasing on $(0,2\delta)$;

(3) $c_0k^p\big(d(x)-\beta\big)<b(x)<c_0k^p\big(d(x)+\beta\big)$ for all $x\in\Omega_{2\delta}$ with $\beta\in(0,\delta)$ being arbitrary.

Set $u_\beta^\pm(x)=\xi^\pm\phi\big(d(x)\pm\beta\big),x\in\Omega_\beta^\pm$, where

$$\xi^\pm=\Big[\frac{p^{p-1}(p-1)(p-lp+l\rho+l)}{(c_0\pm\varepsilon)(\rho+1-p)^p}\Big]^{\frac{1}{\rho+1-p}},$$

$$\nabla u_\beta^\pm=\xi^\pm\phi'\big(d(x)\pm\beta\big)\nabla d(x),$$

and

$$\operatorname{div}(|\nabla u_\beta^\pm|^{p-2}\nabla u_\beta^\pm)=(\xi^\pm)^{p-1}\big[(p-1)|\phi'\big(d(x)\pm\beta\big)|^{p-2}\phi''\big(d(x)\pm\beta\big)|\nabla d(x)|^p + |\phi'\big(d(x)\pm\beta\big)|^{p-2}\phi'\big(d(x)\pm\beta\big)\Delta_p d(x)\big].$$

This implies that

$$\begin{aligned}\Delta_p u_\beta^+ &\pm|\nabla u_\beta^+|^{q(p-1)}-b(x)f(u_\beta^+)\\ &\geqslant k^p\big(d(x)+\beta\big)f(u_\beta^+)\Big[\mathrm{A}_1^+(d(x)+\beta)+\mathrm{A}_2^+(d(x)+\beta)\Delta_p d(x)\\ &\qquad \pm\mathrm{A}_3^+(d(x)+\beta)-c_0\Big],\end{aligned}$$

$$\Delta_p u_\beta^- \pm |\nabla u_\beta^-|^{q(p-1)} - b(x)f(u_\beta^-)$$
$$\leqslant k^p\big(d(x)-\beta\big)f(u_\beta^-)\big[\mathrm{A}_1^-(d(x)-\beta)+\mathrm{A}_2^-(d(x)-\beta)\Delta_p d(x)$$
$$\pm \mathrm{A}_3^-(d(x)-\beta)-c_0\big],$$

where we set

$$\mathrm{A}_1^\pm(t)=\frac{(p-1)(\xi^\pm)^{p-1}|\phi'(t)|^{p-2}\phi''(t)}{k^p(t)f\big(\xi^\pm\phi(t)\big)}$$

$$\mathrm{A}_2^\pm(t)=\frac{(\xi^\pm)^{p-1}|\phi'(t)|^{p-2}\phi'(t)}{k^p(t)f\big(\xi^\pm\phi(t)\big)},\ \ \mathrm{A}_3^\pm(t)=\frac{(\xi^\pm)^{q(p-1)}|\phi'(t)|^{q(p-1)}}{k^p(t)f\big(\xi^\pm\phi(t)\big)}$$

$$\lim_{t\to0}\mathrm{A}_1^\pm(t)=\lim_{t\to0}(p-1)(\xi^\pm)^{p-1}\frac{f\big(\phi(t)\big)}{f\big(\xi^\pm\phi(t)\big)}\frac{|\phi'(t)|^{p-2}\phi''(t)}{k^p(t)f\big(\xi^\pm\phi(t)\big)}$$
$$=(p-1)(\xi^\pm)^{p-\rho-1}p^{p-1}\frac{p-lp+l\rho+l}{(p-\rho-1)^p};$$

and

$$\lim_{t\to0}\mathrm{A}_2^\pm(t)=\lim_{t\to0}\frac{\phi'(t)}{\phi''(t)}\mathrm{A}_1^\pm(t)=0;$$

$$\lim_{t\to0}\mathrm{A}_3^\pm(t)=\lim_{t\to0}(\xi^\pm)^{(q-1)(p-1)}\frac{\big(\phi'(t)\big)^{q(p-1)-p+2}}{\phi''(t)}\mathrm{A}_1^\pm(t)=0.$$

This yields

$$\lim_{t\to0}\big[\mathrm{A}_1^+(t)+\mathrm{A}_2^+(t)\Delta_p d(x)\pm\mathrm{A}_3^+(t)-c_0\big]=\varepsilon$$

$$\lim_{t\to0}\big[\mathrm{A}_1^-(t)+\mathrm{A}_2^-(t)\Delta_p d(x)\pm\mathrm{A}_3^-(t)-c_0\big]=-\varepsilon$$

Thus we can choose $\delta>0$ small enough so that

$$\begin{cases}\Delta_p u_\beta^+ \pm |\nabla u_\beta^+|^{q(p-1)}-b(x)f(u_\beta^+)\geqslant 0,\ \ x\in\Omega_\beta^+,\\ \Delta_p u_\beta^- \pm |\nabla u_\beta^-|^{q(p-1)}-b(x)f(u_\beta^-)\leqslant 0,\ \ x\in\Omega_\beta^-.\end{cases}$$

Set $M(\delta)=\max_{d(x)=2\delta}u_\pm(x)$, $N(\delta)=\xi_1\phi(2\delta)$, where $u_\pm(x)$ is a non-negative solution of problem $(P_\pm)$. It can be easily seen that

$$u_\pm(x)\leqslant M(\delta)+u_\beta^-,x\in\partial\Omega_\beta^-,$$

$$u_\beta^+\leqslant N(\delta)+u_\pm(x),\ x\in\partial\Omega_\beta^+$$

Consequently, taking into account (f_1) and proposition 2.3, we obtain

$$u_\pm(x)\leqslant M(\delta)+u_\beta^-,x\in\Omega_\beta^-,$$

$$u_\beta^+ \leqslant N(\delta) + u_\pm(x),\ x \in \Omega_\beta^+.$$

This implies that

$$u_\beta^+ - N(\delta) \leqslant u_\pm(x) \leqslant M(\delta) + u_\beta^-,\ \ x \in \Omega_\beta^- \bigcap \Omega_\beta^+$$

We conclude that, for all $x \in \Omega_\beta^- \bigcap \Omega_\beta^+$,

$$\frac{u_\beta^+}{\phi\big(d(x)\pm\beta\big)} - \frac{N(\delta)}{\phi\big(d(x)\pm\beta\big)} \leqslant \frac{u_\pm(x)}{\phi\big(d(x)\pm\beta\big)} \leqslant \frac{M(\delta)}{\phi\big(d(x)\pm\beta\big)} + \frac{u_\beta^-}{\phi\big(d(x)\pm\beta\big)}. \quad (3.1)$$

Recall that,

$$x \in \Omega_\beta^- \bigcap \Omega_\beta^+ = (\Omega_{2\delta} \setminus \bar{\Omega}_\beta) \bigcap \Omega_{2\delta-\beta} = \Omega_{2\delta-\beta} \setminus \bar{\Omega}_\beta.$$

So $d(x) \to 0$ implies $\beta \to 0$. Letting $d(x) \to 0$ and $\varepsilon \to 0$ in (3.1), we have thus proven that (1.7) holds.

Case 2: k is non-increasing, then $l \in [1,+\infty)$.

In order to prove Theorem 1.1, in this case, we define $\phi_1(t)$ as

$$f\big(\phi_1(t)\big)t^p = \phi_1^{p-1}(t). \quad (3.2)$$

It can be easily seen that $\phi_1(K(t)) = \phi(t)$. Diminish $\delta > 0$, such that

(1) $d(x) \in C^2(\Omega_{2\delta})$, and $|\nabla d(x)| = 1$ in $\Omega_{2\delta}$;

(2) $k(x)$ is non-decreasing on $(0,2\delta)$;

(3) $(c_0-\varepsilon)k^p\big(d(x)\big) < b(x) < (c_0+\varepsilon)k^p\big(d(x)\big)$ for all $x \in \Omega_{2\delta}$ with $\beta \in (0,\delta)$ being arbitrary.

Define $u_\beta^\pm(x) = \xi^\pm \phi_1\big(K(d(x)\big) + K(\beta)),\ \ x \in \Omega_\beta^\pm$, here

$$\xi^\pm = \Big[\frac{p^{p-1}(p-1)(p+l-pl+p\rho)}{(c_0 \pm 2\varepsilon)(p-\rho-1)^p}\Big]^{\frac{1}{\rho-p+1}}.$$

A simple calculation yields

$$\nabla u_\beta^\pm(x) = \xi^\pm \phi_1'\big(K(d(x)) \pm K(\beta)\big)k\big(d(x)\big)\nabla d(x),$$

and

$$\begin{aligned}
&\operatorname{div}(|\nabla u_\beta^\pm|^{p-2}\nabla u_\beta^\pm)(x)\\
&\quad = (\xi^\pm)^{p-1}[(p-1)|\phi_1'\big(K(d(x)) \pm K(\beta)\big)|^{p-2}\\
&\qquad\qquad \times \phi_1''\big(K(d(x)) \pm K(\beta)\big)|k(d(x))|^p|\nabla d(x)|^p\\
&\quad + (p-1)|\phi_1'\big(K(d(x)) \pm K(\beta)\big)|^{p-2}\\
&\qquad\qquad \times \phi_1'\big(K(d(x)) \pm K(\beta)\big)|k(d(x))|^{p-2})k'(d(x))|\nabla d(x)|^p
\end{aligned}$$

$$+|\phi_1'(K(d(x))\pm K(\beta))|^{p-2}\phi_1'(K(d(x))\pm K(\beta))|k(d(x))|^{p-2}k(d(x))\triangle_p d(x)].$$

This implies that

$$\begin{aligned}\Delta_p u_\beta^+ \pm |\nabla u_\beta^+|^{q(p-1)} - b(x)f(u_\beta^+) \\ \geqslant k^p(d(x))f(u_\beta^+)\big[\mathrm{B}_1^+(K(d(x))+K(\beta)) \\ +\mathrm{B}_2^+(K(d(x))+K(\beta)) + \mathrm{B}_3^+(K(d(x))+K(\beta))\Delta_p d(x) \\ \pm \mathrm{B}_4^+(K(d(x))+K(\beta)) - (c_0+\varepsilon)\big],\end{aligned}$$

$$\begin{aligned}\Delta_p u_\beta^- \pm |\nabla u_\beta^-|^{q(p-1)} - b(x)f(u_\beta^-) \\ \leqslant k^p(d(x))f(u_\beta^-)\big[\mathrm{B}_1^-(K(d(x))-K(\beta)) \\ +\mathrm{B}_2^-(K(d(x))-K(\beta)) + \mathrm{B}_3^-(K(d(x))-K(\beta))\Delta_p d(x) \\ \pm \mathrm{B}_4^-(K(d(x))-K(\beta)) - (c_0-\varepsilon)\big]\end{aligned}$$

where we set

$$\begin{aligned}\mathrm{B}_1^\pm(t) &= \frac{(p-1)(\xi^\pm)^{p-1}|\phi_1'(t)|^{p-2}\phi_1''(t)}{f(\xi^\pm\phi_1(t))};\\ \mathrm{B}_2^\pm(t) &= \frac{(p-1)(\xi^\pm)^{p-1}|\phi_1'(t)|^{p-2}\phi'(t)k'(d(x))}{k^2(d(x))f(\xi^\pm\phi_1(t))};\\ \mathrm{B}_3^\pm(t) &= \frac{(\xi^\pm)^{p-1}|\phi_1'(t)|^{p-2}\phi_1'(t)}{k(d(x))f(\xi^\pm\phi_1(t))};\\ \mathrm{B}_4^\pm(t) &= \frac{(\xi^\pm)^{q(p-1)}|\phi_1'(t)|^{q(p-1)}}{k^{p-q(p-1)}(d(x))f(\xi^\pm\phi_1(t))}.\end{aligned}$$

It is worth pointing out that here $t=K(d(x))\pm K(\beta)$, and

$$\lim_{d(x)\to 0}(K(d(x))\pm K(\beta))=0,\quad x\in\Omega_\beta^\pm.$$

Next, we are going to find

$$\lim_{d(x)\to 0}\mathrm{B}_i^\pm(t)\quad (i=1,2,3,4).$$

By virtue of (3.2), we have

$$f'(\phi_1(t))\phi_1'(t)t^p + pf(\phi_1(t))t^{p-1} = (p-1)\phi_1^{p-2}(t)\phi_1'(t).$$

Thus, taking into account (3.2) and $f\in RV_\rho$, we obtain

$$\rho\lim_{t\to 0}\frac{t\phi_1'(t)}{\phi_1(t)} + p = (p-1)\lim_{t\to 0}\frac{t\phi_1'(t)}{\phi_1(t)}.$$

Then

$$\lim_{t\to 0}\frac{t\phi_1'(t)}{\phi_1(t)}=\frac{p}{p-\rho-1}.$$

Similar arguments of the proof of ϕ show that

$$\lim_{t\to 0}\frac{|\phi_1'(t)|^{p-2}\phi_1''(t)t^p}{\phi_1^{p-1}(t)}=\frac{p^{p-1}(\rho+1)}{(\rho-p+1)^p}.$$

So,

$$\begin{aligned}\lim_{d(x)\to 0}\mathrm{B}_1^{\pm}(t)&=\lim_{d(x)\to 0}(p-1)(\xi^{\pm})^{p-1}\frac{f\big(\phi_1(t)\big)}{f\big(\xi^{\pm}\phi_1(t)\big)}\frac{|\phi_1'(t)|^{p-2}\phi_1''(t)t^p}{\phi_1^{p-1}(t)}\\&=\frac{(p-1)p^{p-1}(\rho+l)}{(\xi^{\pm})^{\rho+1-p}(\rho-p+1)^p};\end{aligned}$$

$$\begin{aligned}\lim_{d(x)\to 0}\mathrm{B}_2^{\pm}(t)&=\lim_{t\to 0}(p-1)(\xi^{\pm})^{p-1}\frac{f\big(\phi_1(t)\big)}{f\big(\xi^{\pm}\phi_1(t)\big)}\Big[\frac{t\phi_1'(t)}{(\phi_1(t))^{p-1}}\Big]^{p-1}\\&\quad\times\frac{d(x)k'\big(d(x)\big)}{k\big(d(x)\big)}\frac{t}{d(x)k\big(d(x)\big)}\\&=\frac{(1-l)(p-1)}{(\xi^{\pm})^{\rho+1-p}}\Big(\frac{p}{p-1-\rho}\Big)^{p-1}.\end{aligned}\tag{3.3}$$

In obtaining (3.3), we have used $t=K\big(d(x)\big)\pm K(\beta)$ and

$$\lim_{d(x)\to 0}\frac{t}{d(x)k\big(d(x)\big)}=\lim_{d(x)\to 0}\frac{K\big(d(x)\big)\pm K(\beta)}{d(x)k\big(d(x)\big)}=l$$

A simple calculation yields

$$\lim_{d(x)\to 0}\mathrm{B}_3^{\pm}(t)=\lim_{d(x)\to 0}\mathrm{B}_4^{\pm}(t)=0.$$

We conclude that

$$\lim_{t\to 0}\big[\mathrm{B}_1^{+}(t)+\mathrm{B}_2^{+}(t)+\mathrm{B}_3^{+}(t)\Delta_p d(x)\pm\mathrm{B}_4^{+}(t)-(c_0+\varepsilon)\big]=\varepsilon$$

$$\lim_{t\to 0}\big[\mathrm{B}_1^{-}(t)+\mathrm{B}_2^{-}(t)+\mathrm{B}_3^{-}(t)\Delta_p d(x)\pm\mathrm{B}_4^{-}(t)-(c_0-\varepsilon)\big]=-\varepsilon$$

Similarly to the proof of Case 1, we can obtain (1.7) holds.

Proof of (2): *Case 1*: k is non-decreasing, then $l\in(0,1]$.

Diminish $\delta>0$ such that

(1) $d(x)\in C^2(\Omega_{2\delta})$, and $|\nabla d(x)|=1$ in $\Omega_{2\delta}$;

(2) $k(x)$ is non-decreasing on $(0,2\delta)$;

(3) $c_q k^{q(p-1)}\big(d(x)-\beta\big) < b(x) < c_q k^{q(p-1)}\big(d(x)+\beta\big)$ for all $x \in \Omega_{2\delta}$ with $\beta \in (0,\delta)$ being arbitrary.

Set $u_\beta^\pm(x) = \xi^\pm \varphi\big(d(x)\pm\beta\big), x \in \Omega_\beta^\pm$,

$$\xi^\pm = \Big[\frac{1}{c_q \pm \varepsilon}\frac{q(p-1)}{\rho - q(p-1)}\Big]^{\frac{1}{\rho - q(p-1)}},$$

where φ appears in (1.11), therefore,

$$\nabla u_\beta^\pm = \xi^\pm \varphi'\big(d(x)\pm\beta\big)\nabla d(x),$$

and

$$\begin{aligned}\operatorname{div}(|\nabla u_\beta^\pm|^{p-2}\nabla u_\beta^\pm) = (\xi^\pm)^{p-1}[(p-1)|\varphi'\big(d(x)\pm\beta\big)|^{p-2}\varphi''\big(d(x)\pm\beta\big)|\nabla d(x)|^p \\ + |\varphi'\big(d(x)\pm\beta\big)|^{p-2}\varphi'\big(d(x)\pm\beta\big)\Delta_p d(x)].\end{aligned}$$

Thus

$$\begin{aligned}&\Delta_p u_\beta^+ + |\nabla u_\beta^+|^{q(p-1)} - b(x)f(u_\beta^+) \\ &\quad \geqslant k^{q(p-1)}\big(d(x)+\beta\big)f(u_\beta^+)\Big[C_1^+\big(d(x)+\beta\big) \\ &\qquad + C_2^+\big(d(x)+\beta\big)\Delta_p d(x) + C_3^+\big(d(x)+\beta\big) - c_q\Big],\end{aligned}$$

$$\begin{aligned}&\Delta_p u_\beta^- + |\nabla u_\beta^-|^{q(p-1)} - b(x)f(u_\beta^-) \\ &\quad \leqslant k^{q(p-1)}\big(d(x)-\beta\big)f(u_\beta^-)\Big[C_1^-\big(d(x)-\beta\big) \\ &\qquad + C_2^-\big(d(x)-\beta\big)\Delta_p d(x) + C_3^-\big(d(x)-\beta\big) - c_q\Big],\end{aligned}$$

where we set:

$$C_1^\pm(t) = \frac{(p-1)(\xi^\pm)^{p-1}|\varphi'(t)|^{p-2}\varphi''(t)}{k^{q(p-1)}(t)f\big(\xi^\pm\varphi(t)\big)},$$

$$C_2^\pm(t) = \frac{(\xi^\pm)^{p-1}|\varphi'(t)|^{p-2}\varphi'(t)}{k^{q(p-1)}(t)f\big(\xi^\pm\varphi(t)\big)},\ \ C_3^\pm(t) = \frac{(\xi^\pm)^{q(p-1)}|\varphi'(t)|^{q(p-1)}}{k^{q(p-1)}(t)f\big(\xi^\pm\varphi(t)\big)}.$$

From Lemma 2, and direct computation shows that:

$$\lim_{t\to 0} C_1^\pm(t) = 0; \quad \lim_{t\to 0} C_2^\pm(t) = \lim_{t\to 0}\frac{\varphi'(t)}{\varphi''(t)}C_1^\pm(t) = 0;$$

and

$$\lim_{t\to 0} C_3^\pm(t) = \lim_{t\to 0}(\xi^\pm)^{q(p-1)}\Big[\frac{t\varphi'(t)}{\varphi(t)}\Big]^{q(p-1)}\Big[\frac{K(t)}{tk(t)}\Big]^{q(p-1)}\frac{f\big(\varphi(t)\big)}{f\big(\xi^\pm\varphi(t)\big)}$$

$$= \frac{1}{(\xi^{\pm})\rho - q(p-1)}\Big[\frac{q(p-1)}{\rho - q(p-1)}\Big]^{q(p-1)}.$$

This yields

$$\lim_{t\to 0}\big[C_1^+(t) + C_2^+(t)\Delta_p d(x) + C_3^+(t) - c_q\big] = \varepsilon$$

$$\lim_{t\to 0}\big[C_1^-(t) + C_2^-(t)\Delta_p d(x) + C_3^-(t) - c_q\big] = -\varepsilon$$

Similar arguments of the proof of (1) show that (1.10) holds.

Case 2: k is non-increasing. Then $l \in [1, +\infty)$.
Diminish $\delta > 0$, such that

(1) $d(x) \in C^2(\Omega_{2\delta})$, and $|\nabla d(x)| = 1$ in $\Omega_{2\delta}$;

(2) $k(x)$ is non-decreasing on $(0, 2\delta)$;

(3) $(c_q - \varepsilon)k^{q(p-1)}\big(d(x)\big) < b(x) < (c_q + \varepsilon)k^{q(p-1)}\big(d(x)\big)$ for all $x \in \Omega_{2\delta}$ with $\beta \in (0, \delta)$ being arbitrary.

Define $u_\beta^{\pm}(x) = \xi^{\pm}\varphi_1\big(K(d(x)) \pm K(\beta)\big)$, $x \in \Omega_\beta^{\pm}$, here

$$\xi^{\pm} = \Big[\frac{1}{c_q \pm 2\varepsilon}\frac{q(p-1)}{\rho - q(p-1)}\Big]^{\frac{1}{\rho - q(p-1)}}$$

where $\varphi_1(t)$ is given by as $f\big(\varphi_1(t)\big)t^{q(p-1)} = \varphi_1^{p-1}(t)$.

Since the argument follows the same ideas as for case 2 of (1), we omit the details.

Proof of (3). The proof of (3) is a slight variation of that of (1), therefore it will not given. □

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(Received November 18, 2012)
(Revised June 3, 2013)

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