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RESONANT BOUNDARY VALUE PROBLEMS FOR SINGULAR MULTI–TERM FRACTIONAL DIFFERENTIAL EQUATIONS

YUJI LIU AND XIAOHUI YANG

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Abstract. In this article, we present and study three class of resonant boundary value problems for singular fractional differential equations with sup-multiplicative-like operators. The existence results for solutions of these boundary-value problems are established. Our analysis relies on the well known coincidence degree theory. Here the nonlinearity terms in fractional differential equations depend on $D^{\gamma}_{0+}u$ and may be singular at $t=0$ or $t=1$. The results obtained generalize and enrich known results to some extent from the literature.

1. Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes and in engineering and have been of great interest recently. In its turn, mathematical aspects of studies on fractional differential equations were discussed by many authors, see the text books [20, 29, 34] and papers [1, 2, 3, 6, 14, 20, 32, 39, 40, 9, 24, 30, 37, 38, 28, 26] and the references therein. The methods are based upon fixed point theorems in cones in Banach spaces.

Recently the coincidence degree theory [27] has been used to study the existence of solutions to boundary value problems for fractional differential equations [31, 35, 42, 7, 4, 27, 18, 10, 17, 23].

In paper [31], authors studied the existence of solutions in $C^{\alpha-1}[0,1]=\{u:u(t)=I^{\alpha-1}_{0+}x(t),t\in[0,1],x\in C^0[0,1]\}$ of the following boundary value problem of fractional order differential equations at resonance

$$\begin{cases} D^{\alpha}_{0+}u(t)+f(t,u(t))=0, & t\in(0,1),\\ u(0)=0,\\ u(1)=\frac{1}{\eta^{\alpha-1}}u(\eta), \end{cases} \tag{1}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order $1<\alpha\leqslant 2$, $0<\eta<1$, $f\in C([0,1]\times\mathbb{R})$. Using intermediate value theorem, sufficient conditions

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for the existence of the solutions for the above fractional order differential equations were established.

In [17], the authors studied the existence of solutions in $C^1[0,1]$ of the following periodic boundary value problem for fractional differential equation

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) = f(t,u(t),u'(t)), \quad t \in [0,1], \\ u(0) = u(1), \\ u'(0) = u'(1), \end{cases} \tag{2}$$

where ${}^cD_{0+}^{\alpha}$ is a Caputo fractional derivative of order α, $f:[0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous.

In [10], the author investigated the existence of solutions in $C^{\alpha-1}[0,1] = \{u : u(t) = I_{0+}^{\alpha-1}x(t), t \in [0,1], x \in C^0[0,1]\}$ of the following fractional order differential equation boundary value problem at resonance

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t),D_{0+}^{\alpha-1}u(t)) + e(t), \quad t \in (0,1), \\ I_{0+}^{2-\alpha}u(0) = 0, \\ D_{0+}^{\alpha-1}u(1) = \sum\limits_{k=1}^{m} \beta_m D_{0+}^{\alpha-1}u(\xi_k), \end{cases} \tag{3}$$

where D_{0+}^{*} and I_{0+}^{*} are the standard Riemann-Liouville differentiation and integration of order $*$, $1 < \alpha \leqslant 2$, $f:[0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $e \in L^1(0,1)$, $\xi_i \in (0,1)$ and $\beta_i \in \mathbb{R}$ with $\sum\limits_{k=1}^{m} \beta_i = 1$.

In recent paper [23], authors established existence results of solutions of the following fractional order differential equation boundary value problem at resonance

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,u(t),D_{0+}^{\mu}u(t)) + e(t), \quad t \in (0,1), \\ (I_{0+}^{2-\alpha}u)'(0) = 0, \\ D_{0+}^{\mu}u(1) = \sum\limits_{k=1}^{m} \beta_m D_{0+}^{\mu}u(\xi_k), \end{cases} \tag{4}$$

where D_{0+}^{*} is the standard Riemann-Liouville differentiation of order $*$, $1 < \alpha \leqslant 2$, $\mu \in (0,\alpha-1]$, $f:(0,1)\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function and may be singular at $t = 0,1$, $e \in L^1(0,1)$, $\xi_i \in (0,1)$ and $\beta_i \in \mathbb{R}$ with $\sum\limits_{k=1}^{m} \beta_i \xi_i^{\alpha-\mu-2} = 1$.

In [37], the authors studied the existence of solutions of the following periodic value problem of fractional order differential equation

$$\begin{cases} D_{0+}^{2\alpha}u(t) = f(t,u(t),D_{0+}^{\alpha}u(t)), \quad t \in (0,T], \\ t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}, \\ t^{1-\alpha}D_{0+}^{\alpha}u(t)|_{t=0} = t^{1-\alpha}D_{0+}^{\alpha}u(t)|_{t=T}, \end{cases}$$

where $T > 0$ is a constant, D_{0+}^{*} is the standard Riemann-Liouville differentiation of order $*$, $f:[0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous. The methods used in [37] are based upon the upper and lower solutions and Schauder fixed point theorem.

The turbulent flow in a porous medium is a fundamental mechanics phenomenon. For studying this type of phenomena, Leibenson (see [21]) introduced the p-Laplacian equation as follows

$$(\phi_p(u'(t)))' = f(t,u(t),u'(t)),$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. Obviously, ϕ_p is invertible and $\phi_p^{-1} = \phi_q$, where $q > 1$ such that $1/p + 1/q = 1$. Since the p-Laplacian operator and fractional calculus arises from many applied fields, such as turbulent filtration in porous media, blood flow problems, rheology, modeling of viscoplasticity, material science, it is worth studying the fractional p-Laplacian equations.

In [12], T. Chen and W. Liu studied an anti-periodic boundary value problem for the fractional p-Laplacian equation:

$$\begin{cases} {}^cD_{0+}^{\beta}[\phi_p(D_{0+}^{\alpha}u(t))] = f(t,u(t)), \quad t \in [0,1], \\ u(0) = -u(1), \\ {}^cD_{0+}^{\alpha}u(0) = -{}^cD_{0+}^{\alpha}u(1), \end{cases} \tag{5}$$

where $0 < \alpha, \beta \leqslant 1$ with $1 < \alpha + \beta \leqslant 2$, ${}^cD_{0+}^{*}$ is a Caputo fractional derivative of order $*$, and $p > 1$, $\phi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [13], the authors studied the existence of solutions of the following resonant boundary value problem for fractional differential equation

$$\begin{cases} {}^cD_{0+}^{\beta}[\phi_p(D_{0+}^{\alpha}u(t))] = f(t,u(t),D_{0+}^{\alpha}u(t)), \quad t \in [0,1], \\ {}^cD_{0+}^{\alpha}u(0) = 0, \\ {}^cD_{0+}^{\alpha}u(1) = 0, \end{cases} \tag{6}$$

where $0 < \alpha, \beta \leqslant 1$ with $1 < \alpha + \beta \leqslant 2$, ${}^cD_{0+}^{*}$ is a Caputo fractional derivative of order $*$, and $p > 1$, $\phi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

In [36], authors studied the solvability of the following two-point boundary value problem for fractional p-Laplace differential equation

$$\begin{cases} {}^cD_{0+}^{\beta}[\phi_p({}^cD_{0+}^{\alpha}u(t))] = f(t,u(t),{}^cD_{0+}^{\alpha}u(t)), \quad t \in [0,1], \\ u(0) = 0, \\ {}^cD_{0+}^{\alpha}u(1) = {}^cD_{0+}^{\alpha}u(0), \end{cases} \tag{7}$$

where ${}^cD_{0+}$ denotes the Caputo fractional derivatives, $0 < \alpha, \beta \leqslant 1$, $1 < \alpha + \beta \leqslant 2$, $\phi_p(s) = |s|^{p-2}s$ is a p-Laplacian operator, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. By using the coincidence degree theory, the existence of solutions for above fractional boundary value problem was obtained.

Integral boundary conditions have various applications in applied fields such as underground water flow, blood flow problems, chemical engineering, thermo-elasticity, population dynamics, and so on. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [5, 8, 11, 33] and the references therein.

In paper [25], Liu and Lu studied the following boundary value problem of integral type for fractional p-Laplace differential equation

$$\begin{cases} {}^cD_{0^+}^{\beta}[\phi_p(D_{0^+}^{\alpha}u(t))] = f(t,u(t),D_{0^+}^{\alpha}u(t)), \quad t\in[0,1], \\ u(0) = \mu\int_0^1 u(s)ds + \lambda u(\xi), \\ {}^cD_{0^+}^{\alpha}u(0) = k^cD_{0^+}^{\alpha}u(\eta), \end{cases} \tag{8}$$

where ${}^cD_{0^+}$ denotes the Caputo fractional derivatives, $0<\alpha,\beta\leqslant 1$, $1<\alpha+\beta\leqslant 2$, $\mu,\lambda,k\in\mathbb{R}$, $\xi,\eta\in[0,1]$, $\phi_p(s)=|s|^{p-2}s$ is a p-Laplacian operator, $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous. By using the coincidence degree theory, the existence of solutions for above fractional boundary value problem was obtained.

In [22], Liu investigated the existence of solutions of the following boundary value problem of integral type for the nonlinear fractional differential equation with the nonlinearity depending on $D_{0^+}^{\alpha}u$

$$\begin{cases} D_{0^+}^{\beta}[\rho(t)\Phi(D_{0^+}^{\alpha}u(t))] + q(t)f(t,u(t),D_{0^+}^{\alpha}u(t)) = 0, \quad t\in(0,1), \\ \lim_{t\to 0}t^{1-\alpha}u(t) + \sum_{i=1}^m a_iu(\xi_i) = \int_0^1 g(s,u(s),D_{0^+}^{\alpha}u(s))ds, \\ \lim_{t\to 0}\Phi^{-1}(t^{1-\beta}\rho(t))D_{0^+}^{\alpha}u(t) + \sum_{i=1}^m b_iD_{0^+}^{\alpha}u(\xi_i) = \int_0^1 h(s,u(s),D_{0^+}^{\alpha}u(s))ds \end{cases} \tag{9}$$

where $0<\alpha,\beta\leqslant 1$, $D_{0^+}^{*}$ is the Riemann-Liouville fractional derivative of order $*$, $\Phi(s)=|s|^{p-2}s$ with $s>1$ a p-Laplacian, $0<\xi_1<\xi_2<\cdots<\xi_m\leqslant 1$, $a_i,b_i(i=1,2,\cdots,m)$ are nonnegative numbers, $\rho\in C^0(0,1)$ is positive and satisfies that there exists $\sigma_1\in\mathbb{R}$ such that $\sigma_1(q-1)<\alpha$ and

$$t^{1-\beta}\rho(t)\geqslant t^{\sigma_1},\ t\in(0,1),$$

q defined on $(0,1)$ is nonnegative and satisfies that there exists $\sigma_2>-\beta$ such that

$$I_{0^+}^{\beta}q\in C^0[0,1],\ \ q(t)\leqslant t^{\sigma_2},\ t\in(0,1),$$

f,g,h defined on $[0,1]\times R\times R$ are nonnegative Caratheodory functions.

In almost all known papers, f is supposed to be continuous and be dependent on x, $D_{0^+}^{\alpha-1}x$ or x' or $D_{0^+}^{\alpha}$ or even $D_{0^+}^{p}$ with $p<\alpha-1$. We find the assumptions $p<\alpha-1$ and $\alpha+\beta>1$ are convenient for proving the completely continuous property of the defined nonlinear operator. So it is interesting to establish existence results of solutions of BVPs without the assumptions $p<\alpha-1$ or $\alpha+\beta>1$. Our paper also fill the researching gap mentioned.

The equations in (2)-(9) are called multi-term fractional differential equations [15]. Such class of equations have many applications. Bagley-Torvik equation and Basset equation are examples see [15]. Differential equations governed by nonlinear differential operators have been widely studied. In this setting the most investigated operator is the classical p Laplacian see citellllll, that is $\Phi_p(x)=|x|^{p-2}x$ with $p>1$, which, in recent years, has been generalized to other types of differential operators, that preserve the monotonicity of the p-Laplacian, but are not homogeneous. These more general operators, which are usually referred to as $\Phi-$Laplacian (or quasi-Laplacian), are involved in some models, e.g. in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces. The related nonlinear differential equation has the form

$$[\Phi(x')]' = f(t,x,x'),\ \ t \in (-\infty,+\infty),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\Phi(0) = 0$. More recently, equations involving other types of differential operators have been studied from a different point of view arising from other types of models, e.g. reaction diffusion equations with non-constant diffusivity and porous media equations, see [18].

In this paper, we are concerned with the following boundary value problems of integral type for multi-term fractional differential equations with Riemann-Liouville fractional derivatives and sup-multiplicative-like functions

$$\begin{cases} D_{0^+}^{\beta}[\Phi(\rho_1(t)D_{0^+}^{\alpha}u(t))] = q_1(t)f_1(t,u(t),D_{0^+}^{\gamma_1}u(t)), \\ \qquad\qquad\qquad t \in (0,1), \gamma_1 \in (0,\alpha), \alpha,\beta \in (0,1), \\ \lim_{t\to 0} t^{1-\beta}\Phi(\rho_1(t)D_{0^+}^{\alpha}u(t)) = \int_0^1 g_1(s,u(s),D_{0^+}^{\gamma_1}u(s))ds, \\ \lim_{t\to 1} t^{1-\beta}\Phi(\rho_1(t)D_{0^+}^{\alpha}u(t)) = \int_0^1 h_1(s,u(s),D_{0^+}^{\gamma_1}u(s))ds, \end{cases} \tag{10}$$

$$\begin{cases} D_{0^+}^{\beta}[D_{0^+}^{\alpha}u(t)] = q_2(t)f_2(t,u(t),D_{0^+}^{\gamma_2}u(t)), \\ \qquad\qquad\qquad t \in (0,1), \gamma_2 \in (\alpha,\alpha+\beta), \alpha,\beta \in (0,1), \\ \lim_{t\to 0} t^{1-\alpha}u(t) = \int_0^1 g_2(s,u(s),D_{0^+}^{\gamma_2}u(s))ds, \\ u(1) - \frac{1}{\eta^{\alpha+\beta-1}}u(\eta) = \int_0^1 h_2(s,u(s),D_{0^+}^{\gamma_2}u(s))ds, 0<\eta<1, \end{cases} \tag{11}$$

and

$$\begin{cases} D_{0^+}^{\delta}u(t))] = q_3(t)f_3(t,u(t),D_{0^+}^{\gamma_3}u(t)), \\ \qquad\qquad\qquad t \in (0,1), \gamma_3 \in (\delta-1,\delta), \delta \in (1,2], \\ \lim_{t\to 0} t^{2-\delta}u(t) - u(1) = \int_0^1 g_3(s,u(s),D_{0^+}^{\gamma_3}u(s))ds, \\ \lim_{t\to 0} t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}u(t) - D_{0^+}^{\gamma_3}u(1) = \int_0^1 h_3(s,u(s),D_{0^+}^{\gamma_3}u(s))ds, \end{cases} \tag{12}$$

where

(•) $D_{0^+}^{*}$ is the Riemann-Liouville fractional derivative of order $*$,

(•) $\Phi : \mathbb{R} \to \mathbb{R}$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1} : \mathbb{R} \to \mathbb{R}$ with supporting function ν (see definition in Section 2),

(•) $\rho_1 \in C^0(0,1)$ is positive, continuous, and satisfies assumptions given in Sections 3, respectively,

(•) $q_1, q_2, q_3 \in C^0(0,1)$ is nonnegative, continuous and satisfies assumptions given in Sections 3, 4 and 5, respectively,

(•) f_1, f_2, f_3 are defined on $(0,1)\times\mathbb{R}^2$ and satisfy assumptions given in Sections 3, 4 and 5, respectively,

(•) $g_i, h_i (i = 1,2,3)$ are defined on $(0,1)\times\mathbb{R}^2$ and satisfy assumptions given in Sections 3, 4 and 5, respectively.

We obtain the results on the existence of at least one solution (see definitions in Section 2) of BVP(10), BVP(11) and BVP(12), respectively. The methods used in the proofs of obtained theorems are based upon the coincidence degree theory. The results obtained generalize and enrich some known results to some extent from the literature.

We clarify the structure of sequential fractional differential equations. Due to the lack of commutativity of the fractional derivatives, this represents an interesting complication that does not arise in the integer-order setting. In problems (10) and (11), we necessarily have a composition of two fractional derivatives, which gives rise to a sequential problem. Problem (12) is a natural generalized form of ordinary differential equation $x''(t) = q(t)f(t,x(t),x'(t))$. Consequently, we feel these to be interesting, if at this point minor, contributions.

The remainder of this paper is as follows: in section 2, we present preliminary results. In section 3, existence result for BVP(10) is established. The existence result for BVP(11) is established in Section 4. The main existence theorem for BVP(12) is established in Section 5.

2. Preliminary results

To obtain the main results, we need some notations and an abstract existence theorem by Kilbas, M.Srivastava, Trujillo [20], Gaines and Mawhin [27].

Denote the Gamma function and Beta function respectively by

$$\Gamma(\alpha_1) = \int_0^{+\infty} s^{\alpha_1 - 1} e^{-s} ds, \ \mathbf{B}(\alpha_1, \beta_1) = \int_0^1 (1-x)^{\alpha_1 - 1} x^{\beta_1 - 1} dx, \alpha_1, \beta_1 > 0.$$

DEFINITION 2.1. (see [20]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side exists.

DEFINITION 2.2. (see [20]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 < \alpha \leqslant n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

LEMMA 2.1. (see [20]). *For $\alpha > 0$, the general solution of fractional differential equation $D_{0+}^{\alpha} x(t) = 0$ is given by $x(t) = c_0 t^{\alpha-n} + c_1 t^{\alpha-n+1} + c_2 t^{\alpha-2} + \ldots + c_{n-1} t^{\alpha-1}$, where $c_i \in R$, $i = 0,1,2,\ldots,n-1$ $n-1 < \alpha \leqslant n$.*

DEFINITION 2.3. (see [27]). Let E and Z be Banach spaces. $L :\ D(L) \subset E \to Z$ is called a Fredholm operator of index zero if $\mathrm{Im}L$ is closed in E and $\dim \mathrm{Ker}L = \mathrm{co}\dim \mathrm{Im}L < +\infty$.

It is easy to see that if L is a Fredholm operator of index zero, then there exist the projectors $P: E \to E$, and $Q: Z \to Z$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \ \operatorname{Ker} Q = \operatorname{Im} L, \ X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \ Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

If $L: D(L) \subset E \to Z$ is called a Fredholm operator of index zero, the inverse of

$$L|_{D(L)\cap \operatorname{Ker} P}: D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is denoted by K_p.

DEFINITION 2.4. (see [27]). Suppose that $L: D(L) \subset E \to Z$ is a Fredholm operator of index zero. For nonempty open bounded subset Ω of E satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$, the continuous map $N: \overline{\Omega} \to Z$ is called $L-$compact if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ is bounded and relatively compact.

LEMMA 2.2. (see [27]). *Let L be a Fredholm operator of index zero and N be $L-$compact on each open nonempty set Ω centered at zero. Assume that the following conditions are satisfied:*

(i). *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [D(L) \setminus KerL) \cap \partial\Omega] \times (0,1)$;*

(ii). *$Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;*

(iii). *$deg(\wedge^{-1}QN\big|_{KerL}, \ \Omega \cap KerL, 0) \neq 0$, where $\wedge^{-1}: Y/ImL \to KerL$ is the inverse of the isomorphism $\wedge: KerL \to Y/ImL$.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

DEFINITION 2.5. (see [13]). An odd homeomorphism Φ of the real line R onto itself is called a sup-multiplicative-like function (or operator) if there exists a homeomorphism ω of $[0,+\infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \geqslant 0$ it holds

$$\Phi(v_1 v_2) \geqslant \omega(v_1)\Phi(v_2). \tag{13}$$

ω is called the supporting function of Φ.

REMARK 2.1. Note that any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in \mathbb{R}$$

is a sup-multiplicative-like operator, provided that $c_j \geqslant 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, \quad u\}$, $u \geqslant 0$.

REMARK 2.2. It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and moreover their inverses Φ^{-1} and ν respectively are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leqslant \nu(w_1)\Phi^{-1}(w_2), \tag{14}$$

for all $w_1, w_2 \geqslant 0$ and ν is called the supporting function of Φ^{-1}.

REMARK 2.3. If $\phi_p^{-1}(x)=|x|^{p-2}x$ for $p>1$, we call ϕ_p a one-dimensional p-Laplacian. By Remark 2.1, ϕ_p is a sup-multiplicative-like function with its supporting function $\omega(x)=|x|^{p-2}x$ and its inverse function $\phi_p^{-1}(x)=|x|^{q-2}x$. The supporting function of ϕ_p^{-1} is $\nu(x)=|x|^{q-2}x$. Here q satisfies $1/p+1/q=1$. It is easy to see that

$$\lim_{t\to 0}\frac{1}{\nu(t^{1-\beta})\nu(t^{\beta-1})}=1.$$

In this paper we always suppose that Φ is a sup-multiplicative-like function with its supporting function ω, the inverse function Φ^{-1} has its supporting function ν.

3. Solvability of BVP(10)

Suppose that

(H1) $f_1:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items:
- $t\to f_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)$ is continuous on $(0,1)$ for each $(x,y)\in\mathbb{R}^2$,
- $(x,y)\to f_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t\in(0,1)$,
- for each $r>0$, there exists a nonnegative $M_r\geqslant 0$ such that

$$\left|f_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)\right|\leqslant M_r, t\in(0,1),|x|,|y|\leqslant r.$$

(H2) $g_1:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items:
- $t\to g_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)$ is continuous on $(0,1)$ for each $(x,y)\in\mathbb{R}^2$,
- $(x,y)\to g_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t\in(0,1)$,
- for each $r>0$, there exists a nonnegative function $\phi_r\in L^1(0,1)$ such that

$$\left|g_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)\right|\leqslant \phi_r(t), t\in(0,1),|x|,|y|\leqslant r.$$

(H3) $h_1:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items:
- $t\to h_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)$ is continuous on $(0,1)$ for each $(x,y)\in\mathbb{R}^2$,
- $(x,y)\to h_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t\in(0,1)$,
- for each $r>0$, there exists a nonnegative function $\psi_r\in L^1(0,1)$ such that

$$\left|h_1\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_1-1}y\right)\right|\leqslant \psi_r(t), t\in(0,1),|x|,|y|\leqslant r.$$

(H4) q_1 satisfies that there exist numbers $k_1>-\alpha$ and $l_1\in(-\beta,0)$ with $1+k_1+l_1>0$ such that $q_1(t)\leqslant t^{k_1}(1-t)^{l_1}$ for all $t\in(0,1)$ and $q(t)\not\equiv 0$ on $(0,1)$.

(H5) ρ_1 satisfies that there exists a number $k>-\alpha$ such that $\rho_1(t)\geqslant t^{-k}\nu(t^{\beta-1})$ for all $t\in(0,1)$.

(H6) $\Phi:\mathbb{R}\to\mathbb{R}$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1}:\mathbb{R}\to\mathbb{R}$ with supporting function ν with

$$\lim_{t\to 0}\frac{1}{\nu(t^{1-\beta})\nu(t^{\beta-1})}=1.$$

REMARK 3.1. From (H4), we know that

$$0<\int_0^1(1-s)^{\beta-1}q_1(s)s^{\alpha-1}ds\leqslant\int_0^1(1-s)^{\beta-1}s^{k_1}(1-s)^{l_1}s^{\alpha-1}ds$$
$$=\mathbf{B}(\beta+l_1,\alpha+k_1)<\infty,$$

and

$$0<\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds\leqslant\int_0^1(1-s)^{\beta-1}s^{k_1}(1-s)^{l_1}s^{1-\beta}ds$$
$$=\mathbf{B}(\beta+l_1,2-\beta+k_1)<\infty.$$

Define

$$x(t)=u(t),\;\; y(t)=\Phi(\rho_1(t)D_{0^+}^{\alpha}x(t)).$$

Then BVP(10) is transformed to

$$\begin{cases}\frac{\rho_1(t)}{v(t^{\beta-1})}D_{0^+}^{\alpha}x(t)=\frac{\Phi^{-1}(y(t))}{v(t^{\beta-1})},t\in(0,1),\\ \frac{D_{0^+}^{\beta}y(t)}{q_1(t)}=f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right),t\in(0,1),\\ \lim\limits_{t\to 0}t^{1-\beta}y(t)=\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt,\\ \lim\limits_{t\to 1}t^{1-\beta}y(t)=\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt.\end{cases} \tag{15}$$

It is easy to see that if (x,y) is a solution of BVP(15), then x is a solution of BVP(10).

Let $C(0,1]$ or $C[0,1]$ be the set of all continuous functions on $(0,1]$ or $[0,1]$. We use the Banach spaces:

$$X=\Big\{x\in C(0,1]:D_{0^+}^{\gamma_1}x\in C(0,1],\;\text{ there exist the limits }\lim_{t\to 0}t^{1-\alpha}x(t),\lim_{t\to 0}t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x(t)\Big\}$$

with the norm

$$||x||_X=\max\left\{\sup_{t\in(0,1]}t^{1-\alpha}|x(t)|,\lim_{t\in(0,1]}t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x(t)\right\},\; x\in X,$$

and

$$Y=\left\{y\in C(0,1]:\;\text{ there exists the limit }\;\lim_{t\to 0}t^{1-\beta}y(t)\right\}$$

with the norm

$$||y||_Y=\sup\nolimits_{t\in(0,1]}t^{1-\beta}|y(t)|,y\in Y,$$

and $C[0,1]$ with the norm $||u||_\infty=\max\limits_{t\in[0,1]}|u(t)|$.

Choose $E = X \times Y$ with the norm $||(x,y)|| = \max\{||x||_X, ||y||_Y\}, (x,y) \in E$. Choose $Z = C[0,1] \times C[0,1] \times \mathbb{R}^2$ with the norm

$$||(u,v,a,b)|| = \max\{||u||_\infty,\ ||v||_\infty,\ |a|,\ |b|\}, (x,y,a,b) \in Z.$$

Choose

$$D(L) = \left\{(x,y) \in E : \tfrac{\rho_1(t)}{\nu(t^{\beta-1})} D_{0+}^{\alpha} x \in C[0,1],\ \tfrac{D_{0+}^{\beta}}{q_1(t)} y \in C[0,1]\right\}$$

and define the linear operator L on $E \cap D(L)$ by

$$L(x,y)(t) = \left(\frac{\rho_1(t)}{\nu(t^{\beta-1})} D_{0+}^{\alpha} x(t),\ \frac{D_{0+}^{\beta} y(t)}{q_1(t)},\ \lim_{t\to 0} t^{1-\beta} y(t),\ \lim_{t\to 1} t^{1-\beta} y(t)\right)$$

for $(x,y) \in E \cap D(L)$.

Define the nonlinear operator N on E by

$$N(x,y)(t) = \left(\frac{\Phi^{-1}(y(t))}{\nu(t^{\beta-1})},\ f_1\left(t, x(t), D_{0+}^{\gamma_1} x(t)\right), \int_0^1 g_1\left(t, x(t), D_{0+}^{\gamma_1} x(t)\right) dt,\right.$$
$$\left.\int_0^1 h_1\left(t, x(t), D_{0+}^{\gamma_1} x(t)\right) dt\right) \text{ for } (x,y) \in E.$$

Then BVP(15) can be written as

$$L(x,y) = N(x,y),\ \ (x,y) \in E \cap D(L).$$

REMARK 3.2. It is easy to see that $L : E \bigcap D(L) \to Z$ is well defined. Furthermore, $N : E \to Z$ is well defined.

In fact, for $(x,y) \in E$, then $||(x,y)|| = r < \infty$. Then (H1)-(H3) imply that there exists $M_r \geqslant 0$ such that

$$\begin{aligned}|f_1(t, x(t), D_{0+}^{\gamma_1} x(t))| &= \left|f_1\left(t, t^{\alpha-1}[t^{1-\alpha} x(t)], t^{\alpha-\gamma_1-1}[t^{1+\gamma_1-\alpha} D_{0+}^{\gamma} x(t)]\right)\right| \\ &\leqslant M_r, t \in [0,1],\end{aligned}$$

and there exist $\phi_r, \psi_r \in L^1(0,1)$ such that

$$|g_1(t, x(t), D_{0+}^{\gamma_1} x(t))| \leqslant \phi_r(t), t \in (0,1),$$
$$|h_1(t, x(t), D_{0+}^{\gamma_1} x(t))| \leqslant \psi_r(t), t \in (0,1).$$

From (H1), we know that

$$t \to f_1(t, x(t), D_{0+}^{\gamma_1} x(t)) = f_1\left(t, t^{\alpha-1}[t^{1-\alpha} x(t)], t^{\alpha-\gamma_1-1}[t^{1+\gamma_1-\alpha} D_{0+}^{\gamma} x(t)]\right)$$

is continuous on $[0,1]$.

Since Φ is a sup-multiplicative-like function, we get that

$$\frac{\Phi^{-1}(t^{1-\beta}x)}{\nu(t^{1-\beta})\nu(t^{\beta-1})} \leqslant \frac{\Phi^{-1}(x)}{\nu(t^{\beta-1})} \leqslant \Phi^{-1}(t^{1-\beta}x),\ x \geqslant 0,$$

$$\frac{\Phi^{-1}(t^{1-\beta}x)}{\nu(t^{1-\beta})\nu(t^{\beta-1})} \geqslant \frac{\Phi^{-1}(x)}{\nu(t^{\beta-1})} \geqslant \Phi^{-1}(t^{1-\beta}x),\ x \leqslant 0.$$

Then for $y \in Y$, we have

$$\frac{\Phi^{-1}(t^{1-\beta}y(t))}{\nu(t^{\beta-1})\nu(t^{\beta-1})} \leqslant \frac{\Phi^{-1}(y(t))}{\nu(t^{\beta-1})} \leqslant \Phi^{-1}(t^{1-\beta}y(t)),\ \ y(t) \geqslant 0,$$

$$\frac{\Phi^{-1}(t^{1-\beta}y(t))}{\nu(t^{\beta-1})\nu(t^{\beta-1})} \leqslant \frac{\Phi^{-1}(y(t))}{\nu(t^{\beta-1})} \leqslant \Phi^{-1}(t^{1-\beta}y(t)),\ \ y(t) \leqslant 0.$$

Together with (H6) we see that $\lim_{t\to 0}\frac{\Phi^{-1}(y(t))}{\nu(t^{\beta-1})}$ exists. Hence $t \to \frac{\Phi^{-1}(y(t))}{\nu(t^{\beta-1})}$ is continuous on $[0,1]$. Since

$$\left|\int_0^1 g_1(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\right| \leqslant \int_0^1 \phi_r(s)ds < \infty,$$

$$\left|\int_0^1 h_1(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\right| \leqslant \int_0^1 \psi_r(s)ds < \infty.$$

Then $N(x,y) \in Z$. So $N : E \to Z$ is well defined.

LEMMA 3.1. *Suppose that (H1)-(H6) hold. Then L is a Fredholm operator of index zero and $N: X \to Y$ is $L-$compact.*

Proof. First, for $(x,y) \in E\bigcap D(L)$, we see that $L(x,y) \in Z$. To prove that L is a Fredholm operator of index zero, we should do the following four steps.

Step (i) Prove that

$$\mathrm{Ker}L = \{(ct^{\alpha-1},0):\ c \in \mathbb{R}\}. \tag{16}$$

We know that $(x,y) \in \mathrm{Ker}L$ if and only if

$$\begin{cases} \frac{\rho_1(t)}{\nu(t^{\beta-1})}D_{0^+}^{\alpha}x(t) = 0, \\ \frac{D_{0^+}^{\beta}y(t)}{q_1(t)} = 0, \\ \lim_{t\to 0} t^{1-\beta}y(t) = 0, \\ \lim_{t\to 1} t^{1-\beta}y(t) = 0. \end{cases}$$

Hence $x(t) = ct^{\alpha-1}$ and $y(t) = 0$. Note that $|\Gamma(0)| = \infty$ and $\frac{1}{\Gamma(0)} = 0$. So

$$D_{0^+}^{\gamma_1}x(t) = c\frac{\gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}.$$

It is easy to see that $x \in X$ and $y \in Y$. Furthermore, we have

$$\frac{\rho_1(t)}{\nu(t^{\beta-1})} D_{0+}^{\alpha} x(t) = 0 \in C[0,1], \ \frac{D_{0+}^{\beta}}{q_1(t)} y(t) = 0 \in C[0,1].$$

Then $(x,y) \in \mathrm{Ker}L$ if and only if $x(t) = ct^{\alpha-1}$ for some $c \in \mathbb{R}$ and $y(t) = 0, t \in [0,1]$. Thus $\mathrm{Ker}L = \{(ct^{\alpha-1}, 0) \in E, \ c \in \mathbb{R}\}$.

Step (ii) Prove that

$$\mathrm{Im}L = \left\{ (u,v,a,b) \in Z, \ \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} q_1(s) v(s) ds = b - a \right\}. \tag{17}$$

For $(u,v,a,b) \in Z$, we know that $(u,v,a,b) \in \mathrm{Im}L$ if and only if there exist $(x,y) \in E \bigcap D(L)$ such that

$$\begin{cases} \frac{\rho_1(t)}{\nu(t^{\beta-1})} D_{0+}^{\alpha} x(t) = u(t), \\ \frac{D_{0+}^{\beta} y(t)}{q_1(t)} = v(t), \\ \lim\limits_{t\to 0} t^{1-\beta} y(t) = a, \\ \lim\limits_{t\to 1} t^{1-\beta} y(t) = b. \end{cases}$$

It follows that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\nu(s^{\beta-1})}{\rho_1(s)} u(s) ds + \Psi t^{\alpha-1} \text{ with } \Psi \in \mathbb{R},$$

$$D_{0+}^{\gamma_1} x(t) = \int_0^t \frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} \frac{\nu(s^{\beta-1})}{\rho_1(s)} u(s) ds + \Psi \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} t^{\alpha-\gamma_1-1},$$

$$y(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} q_1(s) v(s) ds + \Upsilon t^{\beta-1} \text{ with } \Upsilon \in \mathbb{R}.$$

We note that (H5) implies that

$$\begin{aligned} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\nu(s^{\beta-1})}{\rho_1(s)} u(s) ds \right| &\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{\nu(s^{\beta-1})}{s^{-k}\nu(s^{\beta-1})} ||u||_\infty ds \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k ds ||u||_\infty \\ &= t^{\alpha+k} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw ||u||_\infty < \infty, \ t \in (0,1], \end{aligned}$$

and

$$\left| \int_0^t \frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} \frac{\nu(s^{\beta-1})}{\rho_1(s)} u(s) ds \right| \leqslant t^{\alpha+k-\gamma_1} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^k dw ||u||_\infty < \infty,$$

where $t \in (0,1]$. Then $x, D_{0+}^{\gamma_1} x \in C(0,1]$ and $y \in C(0,1]$. Since

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})}{\rho_1(s)}u(s)ds\right| \leqslant t^{1+k}\int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw||u||_\infty \to 0 \text{ as } t \to 0,$$

$$t^{1+\gamma_1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\frac{\nu(s^{\beta-1})}{\rho_1(s)}u(s)ds\right| \leqslant t^{1+k}\int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw||u||_\infty \to \text{ as } t \to 0,$$

then $x \in X$.

Since (H4) implies that

$$\begin{aligned} t^{1-\beta}\left|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)v(s)ds\right| &\leqslant t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{k_1}(1-s)^{l_1}||v||_\infty ds \\ &\leqslant t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{k_1}(t-s)^{l_1}ds||v||_\infty \\ &= ||v||_\infty t^{1+k_1+l_1}\int_0^1 \frac{(1-w)^{\beta+l_1-1}}{\Gamma(\beta)}w^{k_1}dw \\ &\to 0 \text{ as } t \to 0, \end{aligned}$$

then $\lim_{t\to 0} t^{1-\beta}y(t) = a$ implies $y \in Y$ and $\Upsilon = a$. From $\lim_{t\to 1} t^{1-\beta}y(t) = b$, we get

$$\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)v(s)ds + a = b. \tag{18}$$

On the other hand, if (18) holds, we get $(u,v,a,b) \in \mathrm{Im}L$. Hence $(u,v,a,b) \in \mathrm{Im}L$ if and only if (18) holds. Then (17) is valid.

Step (iii) Prove that $\mathrm{Im}L$ is closed in X and $\dim \mathrm{Ker}L =$ co dim $\mathrm{Im}L < +\infty$.

From (17) $\mathrm{Im}L$ is closed in Z.

It follows from $\mathrm{Ker}L = \{(ct^{\alpha-1}, 0) \in E,\ c \in \mathbb{R}\}$ that $\dim \mathrm{Ker}L = 1$. Define the projector $P: E \to E$ by

$$P(x,y)(t) = \left(\frac{\int_0^1 (1-s)^{\alpha-1}q_1(s)x(s)ds}{\int_0^1 (1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}t^{\alpha-1},\ 0\right) \text{ for } (x,y) \in E. \tag{19}$$

It is easy to prove that $\mathrm{Im}\,P \subseteq \mathrm{Ker}L$.

For $(ct^{\alpha-1}, 0) \in \mathrm{Ker}L$, choose $x_c(t) = ct^{\alpha-1}$ and $y_c(t) = 0$. It follows that

$$\frac{\rho_1(t)}{\nu(t^{\beta-1})}D_{0+}^{\alpha}x_c(t) = 0 \quad \text{and} \quad \frac{D_{0+}^{\beta}y_c(t)}{q_1(t)} = 0.$$

One can show that $(x_c, 0) \in E$ and

$$P(x_c,0)(t) = \left(\frac{\int_0^1 (1-s)^{\alpha-1}q_1(s)cs^{\alpha-1}ds}{\int_0^1 (1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}t^{\alpha-1},\ 0\right) = (ct^{\alpha-1}, 0).$$

So $\mathrm{Im}\,P \supseteq \mathrm{Ker}\,L$. Hence $\mathrm{Im}\,P = \mathrm{Ker}\,L$.

For $(x,y)\in E$, we have

$$
\begin{aligned}
&P((x,y)-P(x,y))\\
&\qquad = P\left(x-\frac{\int_0^1(1-s)^{\alpha-1}q_1(s)x(s)ds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}t^{\alpha-1},\ 0\right)\\
&\qquad = \left(\frac{\int_0^1(1-s)^{\alpha-1}q_1(s)\left(x(s)-\frac{\int_0^1(1-s)^{\alpha-1}q_1(s)x(s)ds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}s^{\alpha-1}\right)ds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}t^{\alpha-1},\ 0\right)\\
&\qquad = (0,0).
\end{aligned}
$$

So $X=\mathrm{Ker}L+\mathrm{Ker}P$. For $(x,y)\in\mathrm{Ker}L\cap\mathrm{Ker}P$, we get $x(t)=0$ and $y(t)=0$. Hence $X=\mathrm{Ker}L\oplus\mathrm{Ker}P$.

Define the projector $Q:\ Z\to Z$ by

$$Q(u,v,a,b)(t)=\left(0,\ \frac{\int_0^1(1-s)^{\beta-1}q_1(s)v(s)ds-\Gamma(\beta)(b-a)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},\ 0,\ 0\right) \tag{20}$$

for $(u,v,a,b)\in Z$.

It is easy to show that $\mathrm{Ker}Q\subseteq\mathrm{Im}L$.

For $(u,v,a,b)\in\mathrm{Im}L$, we have $\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}q_1(s)v(s)=b-a$. Then

$$Q(u,v,a,b)=\left(0,\frac{\int_0^1(1-s)^{\beta-1}q_1(s)v(s)ds-\Gamma(\beta)(b-a)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},0,0\right)=(0,0,0,0).$$

It follows that $\mathrm{Im}L\subseteq\mathrm{Ker}Q$. Hence $\mathrm{Im}L=\mathrm{Ker}Q$.

For $(u,v,a,b)\in Z$, we have

$$
\begin{aligned}
&(u,v,a,b)-\left(0,\ \frac{\int_0^1(1-s)^{\beta-1}q_1(s)v(s)ds-\Gamma(\beta)(b-a)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},\ 0,\ 0\right)\\
&\qquad = \left(u,\ v-\frac{\int_0^1(1-s)^{\beta-1}q_1(s)v(s)ds-\Gamma(\beta)(b-a)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},\ a,\ b\right).
\end{aligned}
$$

It is easy to see that

$$\frac{1}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}q_1(s)\left(v(s)-\frac{\int_0^1(1-s)^{\beta-1}q_1(s)v(s)ds-\Gamma(\beta)(b-a)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}s^{1-\beta}\right)ds=b-a.$$

Hence

$$\left(u,\ v-\frac{\int_0^1(1-s)^{\beta-1}q_1(s)v(s)ds-\Gamma(\beta)(b-a)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},\ a,\ b\right)\in\mathrm{Im}L.$$

Thus $Y=\mathrm{Im}Q+\mathrm{Im}L$. It is easy to show that $\mathrm{Im}Q\cap\mathrm{Im}L=(0,0,0,0)$. Hence $Y=\mathrm{Im}Q\oplus\mathrm{Im}L$.

From above discussion, we see that $\dim \mathrm{Ker}L = \mathrm{co}\dim \mathrm{Im}L = \dim \mathrm{Im}Q = 1 < +\infty$. So L is a Fredholm operator of index zero.

Step (iv). We prove that N is $L-$compact. This is divided into three sub-steps.

Substep (iv1) We prove that N is continuous. Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We will show that $N(x_n,y_n)\to N(x_0,y_0)$ as $n\to\infty$.

In fact, we have $r>0$ such that $||(x_n,y_n)||\leqslant r<+\infty$ and

$$\begin{aligned}&\sup_{t\in(0,1]} t^{1-\alpha}|x_n(t)-x_0(t)|\to 0, n\to\infty,\\ &\sup_{t\in(0,1]} t^{1+\gamma_1-\alpha}|D_{0^+}^{\gamma_1}x_n(t)-D_{0^+}^{\gamma_1}x_0(t)|\to 0, n\to\infty,\\ &\sup_{t\in(0,1]} t^{1-\beta}|y_n(t)-y_0(t)|\to 0,\ n\to\infty.\end{aligned}\tag{21}$$

By

$$\begin{aligned}N(x_n,y_n)(t)=\Bigg(&\frac{\Phi^{-1}\left(y_n(t)\right)}{\nu(t^{\beta-1})},\ f_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right),\\ &\int_0^1 g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt,\int_0^1 h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt\Bigg)\end{aligned}$$

for $(x,y)\in E$. Since Φ is a sup-multiplicative-like function, we get that

$$\begin{aligned}&\frac{\Phi^{-1}(y)}{\nu(t^{1-\beta})\nu(t^{\beta-1})}\leqslant\frac{\Phi^{-1}(t^{\beta-1}y)}{\nu(t^{\beta-1})}\leqslant\Phi^{-1}(y),\ \ y\geqslant 0,\\ &\frac{\Phi^{-1}(y)}{\nu(t^{1-\beta})\nu(t^{\beta-1})}\geqslant\frac{\Phi^{-1}(t^{\beta-1}y)}{\nu(t^{\beta-1})}\geqslant\Phi^{-1}(y),\ \ y\leqslant 0.\end{aligned}\tag{22}$$

Then

$$(t,y)\to\frac{\Phi^{-1}(t^{\beta-1}y)}{\nu(t^{\beta-1})}\ \text{ is continuous on } [0,1]\times R.$$

It follows that $\frac{\Phi^{-1}(t^{\beta-1}x)}{\nu(t^{\beta-1})}$ is uniformly continuous on $[0,1]\times[-r,r]$.

For any $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\frac{\Phi^{-1}(t^{\beta-1}u_1)}{\nu(t^{\beta-1})}-\frac{\Phi^{-1}(t^{\beta-1}u_2)}{\nu(t^{\beta-1})}\right|<\varepsilon,\ \ t\in[0,1],|u_1-u_2|<\delta.$$

From (21), there exists N_1 such that

$$\begin{aligned}&t^{1-\alpha}|x_n(t)-x_0(t)|<\delta,\ \ t\in(0,1),n>N_1,\\ &t^{1+\gamma_1-\alpha}|D_{0^+}^{\gamma_1}x_n(t)-D_{0^+}^{\gamma_1}x_0(t)|<\delta,\ \ t\in(0,1),n>N_1,\\ &t^{1-\beta}|y_n(t)-y_0(t)|<\delta,\ \ t\in(0,1),n>N_1.\end{aligned}\tag{23}$$

Hence,

$$\left|\frac{\Phi^{-1}(y_n(t))}{\nu(t^{\beta-1})}-\frac{\Phi^{-1}(y_0(t))}{\nu(t^{\beta-1})}\right|=\left|\frac{\Phi^{-1}\left(t^{\beta-1}[t^{1-\beta}y_n(t)]\right)}{\nu(t^{\beta-1})}-\frac{\Phi^{-1}\left(t^{\beta-1}[t^{1-\beta}y_0(t)]\right)}{\nu(t^{\beta-1})}\right|$$
$$<\varepsilon,\ \text{ for } t\in(0,1),\, n>N_1.$$

It follows that

$$\lim_{n\to\infty}\sup_{t\to(0,1)}\left|\frac{\Phi^{-1}(y_n(t))}{\nu(t^{\beta-1})}-\frac{\Phi^{-1}(y_0(t))}{\nu(t^{\beta-1})}\right|=0.$$

Similarly we can show that

$$\lim_{n\to\infty}\sup_{t\to(0,1)}\left|f_1(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t))-f_1(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t))\right|=0,$$

$$\lim_{n\to\infty}\int_0^1 g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt=\int_0^1 g_1\left(t,x_0(t),D_{0^+}^{\gamma_1}x_0(t)\right)dt,$$

$$\lim_{n\to\infty}\int_0^1 h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt=\int_0^1 h_1\left(t,x_0(t),D_{0^+}^{\gamma_1}x_0(t)\right)dt.$$

Then

$$||N(x_n,y_n)-N(x_0,y_0)||\to 0,\, n\to\infty.$$

It follows that N is continuous.

Let Ω be a bounded open subset of E. We have that there exists $r>0$ such that $||(x,y)||\leqslant r$ for all $(x,y)\in\overline{\Omega}$. Since (H1)-(H3) hold, then there exists $M_r\geqslant 0$, $\phi_r,\psi_r\in L^1(0,1)$ such that

$$\begin{aligned}&\left|f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\right|\leqslant M_r, t\in(0,1),(x,y)\in\overline{\Omega},\\&\left|g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\right|\leqslant \phi_r(t), t\in(0,1),(x,y)\in\overline{\Omega},\\&\left|h\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\right|\leqslant \psi_r(t), t\in(0,1),(x,y)\in\overline{\Omega}.\end{aligned}\tag{24}$$

Substep (iv2) Prove that $QN(\overline{\Omega})$ is bounded.

One has

$$\begin{aligned}&QN(x,y)(t)\\&=Q\Bigg(\frac{\Phi^{-1}(y(t))}{\nu(t^{\beta-1})},\ f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right),\\&\qquad\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt,\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\Bigg)\\&=\Bigg(0,\ \frac{\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x(s),D_{0^+}^{\gamma_1}x(s)\right)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta}\end{aligned}$$

$$-\frac{\Gamma(\beta)\left(\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt-\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},\ 0,\ 0\Bigg). \tag{25}$$

It is easy to see from (25), (26) and Remark 3.1 that

$$\begin{aligned}&\Bigg|\frac{\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x(s),D_{0^+}^{\gamma_1}x(s)\right)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta}\\&\qquad-\frac{\Gamma(\beta)\left(\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt-\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta}\Bigg|\\&\leqslant M_r\frac{\int_0^1(1-s)^{\beta-1}q_1(s)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}+\frac{\Gamma(\beta)\left(\int_0^1\phi_r(t)dt+\int_0^1\psi_r(t)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}<+\infty.\end{aligned}$$

It follows $QN(\overline{\Omega})$ is bounded.

Substep (iv3) Prove that $K_P(I-Q)N:\overline{\Omega}\to E$ is compact, i.e., prove that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact.

Let $P:X\to X$ and $Q:Y\to Y$ be defined by (19) and (20). For $(u,v,a,b)\in\mathrm{Im}L$, let

$$\begin{aligned}K_P(u,v,a,b)(t)=\Bigg(&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{v(s^{\beta-1})}{\rho_1(s)}u(s)ds-c(u,v)t^{\alpha-1},\\&\frac{\int_0^t(t-s)^{\beta-1}q_1(s)v(s)ds}{\Gamma(\beta)}+at^{\beta-1}\Bigg),\end{aligned} \tag{26}$$

where $c(u,v)$ is a constant defined by:

$$c(u,v)=\frac{\int_0^1(1-s)^{\alpha-1}\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{v(w^{\beta-1})}{\rho_1(w)}u(w)dwds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}$$

One sees $K_P(u,v,a,b)\in E\bigcap D(L)$ and

$$\begin{aligned}&PK_P(u,v,a,b)(t)\\&=P\Bigg(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{v(s^{\beta-1})}{\rho_1(s)}u(s)ds-c(u,v)t^{\alpha-1},\\&\qquad\frac{\int_0^t(t-s)^{\beta-1}q_1(s)v(s)ds}{\Gamma(\beta)}+at^{\beta-1}\Bigg)\\&=\left(\frac{\int_0^1(1-s)^{\alpha-1}q_1(s)\left[\int_0^s\frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\frac{v(r^{\beta-1})}{\rho_1(r)}u(r)dr-c(u,v)s^{\alpha-1}\right]ds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}t^{\alpha-1},0\right)\\&=(0,0).\end{aligned}$$

It follows that $K_p(u,v,a,b)\in \mathrm{Ker}P$. Then $K_P:\ \mathrm{Im}\,L\to D(L)\cap \mathrm{Ker}P$ is well defined.

Furthermore, for $(u,v,a,b)\in \mathrm{Im}L$, by direct computation, we have

$$(LK_P)(u,v,a,b)(t)=(u,v,a,b).$$

On the other hand, for $(x,y)\in \mathrm{Ker}\ P\cap E\bigcap D(L)$, by direct computation, we have

$$K_PL(x,y)(t)=(x(t),y(t)).$$

Then K_P is the inverse of $L:D(L)\bigcap \mathrm{Ker}P\to \mathrm{Im}L$. The isomorphism $\wedge:\ \mathrm{Ker}L\to Y/\mathrm{Im}L$ is given by

$$\wedge\,(ct^{\alpha-1},0)=(0,ct^{1-\beta},0,0),\ \ c\in\mathbb{R}. \tag{27}$$

Then

$$\begin{aligned}
&K_p(I-Q)N(x,y)(t)\\
&=K_p(I-Q)\left(\frac{\Phi^{-1}\left(y(t)\right)}{\nu(t^{\beta-1})},\ f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right),\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt,\right.\\
&\qquad\left.\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)\\
&=K_P\left(\frac{\Phi^{-1}\left(y(t)\right)}{\nu(t^{\beta-1})},\ f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right),\right.\\
&\qquad\left.\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt,\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)\\
&\quad-K_P\left(0,\ \frac{\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x(s),D_{0^+}^{\gamma_1}x(s)\right)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta}\right.\\
&\qquad\left.-\frac{\Gamma(\beta)\left(\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt-\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}t^{1-\beta},\ 0,\ 0\right)\\
&=(x_1(t),y_1(t))
\end{aligned}$$

with

$$x_1(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds-\frac{\int_0^1(1-s)^{\alpha-1}\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(w))}{\rho_1(w)}dwds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}t^{\alpha-1},$$

and

$$\begin{aligned}
y_1(t)=&\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q_1(s)f(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\\
&-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q_1(s)s^{1-\beta}ds\frac{\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x(s),D_{0^+}^{\gamma_1}x(s)\right)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\\
&+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q_1(s)s^{1-\beta}ds\frac{\Gamma(\beta)\left(\int_0^1 h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt-\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\\
&+t^{\beta-1}\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt.
\end{aligned}$$

To complete this step, we must prove that $K_P(I-Q)N(\overline{\Omega})$ is bounded and equi-continuous on each subinterval $[e,f]\subseteq(0,1]$ and equi-convergent at $t=0$.

Firstly, we have

$$t^{1-\alpha}x_1(t)=t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds-\frac{\int_0^1(1-s)^{\alpha-1}\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(w))}{\rho_1(w)}dwds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds},$$

and

$$t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t)=t^{1+\gamma_1-\alpha}\int_0^t\frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds-\frac{\int_0^1(1-s)^{\alpha-1}\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(w))}{\rho_1(w)}dwds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)},$$

and

$$\begin{aligned}
&t^{1-\beta}y_1(t)\\
&=\frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q_1(s)f(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\\
&\quad-\frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q_1(s)s^{1-\beta}ds\frac{\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x(s),D_{0^+}^{\gamma_1}x(s)\right)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\\
&\quad+\frac{t^{1-\beta}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}q_1(s)s^{1-\beta}ds\\
&\qquad\times\frac{\Gamma(\beta)\left(\int_0^1h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt-\int_0^1g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\\
&\quad+\int_0^1g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt.
\end{aligned}$$

It is easy to show from (25) that $K_P(I-Q)N(\overline{\Omega})$ is bounded.

Second, for each $[e,f]\subseteq(0,1]$, and $t_1,t_2\in[e,f]$ with $t_2>t_1$, we have

$$\begin{aligned}
&|t_1^{1-\alpha}x_1(t_1)-t_2^{1-\alpha}x_1(t_2)|\\
&\qquad\leqslant\left|t_1^{1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds-t_2^{1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds\right|,
\end{aligned}$$

and

$$
\begin{aligned}
&\left|t_1^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_1)-t_2^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_2)\right|\\
&\qquad\leqslant\left|t_1^{1+\gamma_1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds\right.\\
&\qquad\qquad\left.-t_2^{1+\gamma_1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|t_1^{1-\beta}y_1(t_1)-t_2^{1-\beta}y_1(t_2)|\\
&\leqslant\frac{1}{\Gamma(\beta)}\left|t_1^{1-\beta}\int_0^{t_1}(t_1-s)^{\beta-1}q_1(s)f(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\right.\\
&\quad\left.-t_2^{1-\beta}\int_0^{t_2}(t_2-s)^{\beta-1}q_1(s)f(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\right|\\
&\quad+\left|\frac{t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}q_1(s)s^{1-\beta}ds-\frac{t_2^{1-\beta}}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}q_1(s)s^{1-\beta}ds\right|\\
&\qquad\times\left|\frac{\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x(s),D_{0^+}^{\gamma_1}x(s)\right)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\right|\\
&\quad+\left|\frac{t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}q_1(s)s^{1-\beta}ds-\frac{t_2^{1-\beta}}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}q_1(s)s^{1-\beta}ds\right|\\
&\qquad\times\left|\frac{\Gamma(\beta)\left(\int_0^1h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt-\int_0^1g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\right|\\
&\leqslant\frac{1}{\Gamma(\beta)}\left|t_1^{1-\beta}\int_0^{t_1}(t_1-s)^{\beta-1}q_1(s)f(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\right.\\
&\quad\left.-t_2^{1-\beta}\int_0^{t_2}(t_2-s)^{\beta-1}q_1(s)f(s,x(s),D_{0^+}^{\gamma_1}x(s))ds\right|\\
&\quad+\left|\frac{t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}q_1(s)s^{1-\beta}ds-\frac{t_2^{1-\beta}}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}q_1(s)s^{1-\beta}ds\right|\\
&\qquad\times\frac{M_r\int_0^1(1-s)^{\beta-1}q_1(s)ds}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}\\
&\quad+\left|\frac{t_1^{1-\beta}}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}q_1(s)s^{1-\beta}ds-\frac{t_2^{1-\beta}}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}q_1(s)s^{1-\beta}ds\right|\\
&\qquad\times\frac{\Gamma(\beta)\left(\int_0^1[\psi_r(t)+\phi_r(t)]dt\right)}{\int_0^1(1-s)^{\beta-1}q_1(s)s^{1-\beta}ds}.
\end{aligned}
$$

We find that

$$
\left|t_1^{1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds-t_2^{1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(s))}{\rho_1(s)}ds\right|
$$

$$
\begin{aligned}
&\leqslant |t_1^{1-\alpha}-t_2^{1-\alpha}|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}\left(s^{\beta-1}|s^{1-\beta}y(s)|\right)}{\rho_1(s)}ds\\
&\quad+t_1^{1-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}\left(s^{\beta-1}|s^{1-\beta}y(s)|\right)}{\rho_1(s)}ds\\
&\quad+t_1^{1-\alpha}\int_0^{t_1}\frac{|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|}{\Gamma(\alpha)}\frac{\Phi^{-1}\left(s^{\beta-1}|s^{1-\beta}y(s)|\right)}{\rho_1(s)}ds\\
&\leqslant |t_1^{1-\alpha}-t_2^{1-\alpha}|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})}{\rho_1(s)}ds\Phi^{-1}\left(||y||_\infty\right)\\
&\quad+t_1^{1-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})}{\rho_1(s)}\Phi^{-1}\left(||y||_\infty\right)ds\\
&\quad+t_1^{1-\alpha}\int_0^{t_1}\frac{|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})}{\rho_1(s)}\Phi^{-1}\left(||y||_\infty\right)ds\\
&\leqslant |t_1^{1-\alpha}-t_2^{1-\alpha}|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds\Phi^{-1}(r)\\
&\quad+t_1^{1-\alpha}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds\Phi^{-1}(r)\\
&\quad+t_1^{1-\alpha}\int_0^{t_1}\frac{|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|}{\Gamma(\alpha)}s^k ds\Phi^{-1}(r)\\
&= |t_1^{1-\alpha}-t_2^{1-\alpha}|t_2^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\Phi^{-1}(r)\\
&\quad+t_1^{1-\alpha}t_2^{\alpha+k}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\Phi^{-1}(r)\\
&\quad+t_1^{1-\alpha}\left[t_1^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw-t_2^{\alpha+k}\int_0^{\frac{t_1}{t_2}}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\right]\Phi^{-1}(r)\\
&\leqslant |t_1^{1-\alpha}-t_2^{1-\alpha}|\max\{e^{\alpha+k},f^{\alpha+k}\}\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}\Phi^{-1}(r)\\
&\quad+f^{1-\alpha}\max\{e^{\alpha+k},f^{\alpha+k}\}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\Phi^{-1}(r)\\
&\quad+f^{1-\alpha}\left[|t_1^{\alpha+k}-t_2^{\alpha+k}|\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}\right.\\
&\qquad\qquad\left.+\max\{e^{\alpha+k},f^{\alpha+k}\}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\right]\Phi^{-1}(r)\\
&\quad\to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t_1\to t_2.
\end{aligned}
$$

Then we get

$$|t_1^{1-\alpha}x_1(t_1)-t_2^{1-\alpha}x_1(t_2)|\to 0 \text{ uniformly as } t_1\to t_2.$$

Similarly we can prove that

$$|t_1^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_1)-t_2^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_2)|\to 0 \text{ uniformly as } t_1\to t_2,$$
$$|t_1^{1-\beta}y_1(t_1)-t_2^{1-\beta}y_1(t_2)|\to 0 \text{ uniformly as } t_1\to t_2.$$

So $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f]\subseteq(0,1]$.

Third, we have

$$\begin{aligned}&\left|t^{1-\alpha}x_1(t)+\frac{\int_0^1(1-s)^{\alpha-1}\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(w))}{\rho_1(w)}dwds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}\right|\\
&\leqslant t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(|y(s)|)}{\rho_1(s)}ds\\
&\leqslant t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu(s^{\beta-1})}{\rho_1(s)}ds\Phi^{-1}(r)\\
&\leqslant t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^kds\Phi^{-1}(r)\\
&=t^{1+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^kdw\Phi^{-1}(r)\\
&\to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t\to 0.\end{aligned}$$

Similarly we can show that

$$\left|t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t)+\frac{\int_0^1(1-s)^{\alpha-1}\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y(w))}{\rho_1(w)}dwds}{\int_0^1(1-s)^{\alpha-1}q_1(s)s^{\alpha-1}ds}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\right|$$
$$\to 0 \quad \text{uniformly on } \overline{\Omega} \text{ as } t\to 0,$$

and

$$\left|t^{1-\beta}y_1(t)-\int_0^1 g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)dt\right|\to 0 \quad \text{uniformly on } \overline{\Omega} \text{ as } t\to 0.$$

Hence $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent at $t=0$.

So $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. Then N is $L-$compact. The proofs are completed.

A function $\prod:[0,\infty)\times[0,\infty)\to[0,\infty)$ is called a bi-non-increasing function if both $x\to\prod(x,y)$ and $y\to\prod(x,y)$ are non-increasing. Now, we prove that main theorem in this section. Suppose that

(H7) there exist nonnegative functions $\phi_g,\phi_h\in L^1(0,1)$ and bi-nondecreasing functions $\prod_f,\prod_g,\prod_h:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that

$$\begin{aligned}&\left|f_1\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_1-1}v\right)\right|\leqslant\prod_f(|u|,|v|),\\
&\left|g_1\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_1-1}v\right)\right|\leqslant\phi_g(t)\prod_g(|u|,|v|)\end{aligned}\tag{28}$$

or

$$\left|f_1\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_1-1}v\right)\right| \leqslant \prod_f(|u|,|v|),$$
$$\left|h_1\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_1-1}v\right)\right| \leqslant \phi_h(t)\prod_h(|u|,|v|) \tag{29}$$

hold for all $(u,v)\in\mathbb{R}^2$, $t\in(0,1)$.

(H8) there exists a constant $M>0$ such that for $x\in X$ with $t^{1-\alpha}|x(t)|>M$ for all $t\in(0,1)$ implies that

$$\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)-h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)$$
$$+g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\Big]dt\neq 0.$$

(H9) there exists a constant $M_0>0$ such that

$$c\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)$$
$$-h_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)$$
$$+g_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)\Big]dt>0 \tag{30}$$

holds for all $|c|>M_0$ or

$$c\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)$$
$$-h_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)$$
$$+g_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)\Big]dt<0 \tag{31}$$

holds for all $|c|>M_0$.

THEOREM 3.1. *Suppose that* $(H1)$-$(H9)$ *hold. Then BVP(10) has at least one solution if*

$$\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}r_f+\int_0^1\phi_g(t)dt\,r_g<1(\textit{ if } (28) \textit{ holds }),$$
$$2\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}r_f+\int_0^1\phi_h(t)dt\,r_h<1(\textit{ if } (29) \textit{ holds }), \tag{32}$$

where

$$A_0=\max\left\{M,\ M\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\right\},\ B_0=\max\left\{\frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)},\ \frac{\mathbf{B}(\alpha,k+1)+\mathbf{B}(\alpha-\gamma_1,k+1)}{\Gamma(\alpha-\gamma_1)}\right\},$$

$$r_f=\lim_{v\to+\infty}\frac{\prod_f(A_0+B_0\Phi^{-1}(v),A_0+B_0\Phi^{-1}(v))}{v},\ r_g=\lim_{v\to+\infty}\frac{\prod_g(A_0+B_0\Phi^{-1}(v),A_0+B_0\Phi^{-1}(v))}{v},$$

$$r_h=\lim_{v\to+\infty}\frac{\prod_h(A_0+B_0\Phi^{-1}(v)A_0+B_0\Phi^{-1}(v))}{v}.$$

Proof. Let E, Z L and N be defined above. By (H1)-(H6), from Lemma 3.1, L be a Fredholm operator of index zero and N be $L-$compact on each open nonempty set Ω centered at zero. We seek fixed point of the operator equation $L(x,y)=N(x,y)$. To apply Lemma 2.2, we should define an open bounded subset Ω of E centered at zero such that (i), (ii) and (iii) in Lemma 2.2 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1=\{(x,y)\in E\cap D(L)\setminus \text{Ker}L,\ L(x,y)=\lambda N(x,y)$ for some $\lambda\in(0,1)\}$. We prove that Ω_1 is bounded.

In fact, if Ω_1 is unbounded, then there exists two sequences $\{(x_n,y_n)\in E\bigcap D(L)\setminus \text{Ker}L\}$ and $\{\lambda_n\in[0,1]\}$ such that $L(x_n,y_n)=\lambda_n N(x_n,y_n)$ and $N(x_n,y_n)\in \text{Im}L$ and $||(x_n,y_n)||\to\infty$ as $n\to\infty$. Then

$$\begin{cases} \frac{\rho_1(t)}{\nu(t^{\beta-1})}D_{0^+}^{\alpha}x_n(t)=\lambda_n\frac{\Phi^{-1}(y_n(t))}{\nu(t^{\beta-1})}, t\in(0,1),\\ \frac{D_{0^+}^{\beta}y_n(t)}{q_1(t)}=\lambda_n f_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right), t\in(0,1),\\ \lim\limits_{t\to 0}t^{1-\beta}y_n(t)=\lambda_n\int_0^1 g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt,\\ \lim\limits_{t\to 1}t^{1-\beta}y_n(t)=\lambda_n\int_0^1 h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt, \end{cases}$$

and

$$\int_0^1(1-s)^{\beta-1}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds=\int_0^1 h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt \\ -\int_0^1 g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt. \quad (33)$$

It follows that

$$y_n(t)=\lambda_n\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds \\ +\lambda_n t^{\beta-1}\int_0^1 g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt, \quad (34)$$

or

$$y_n(t)=\lambda_n\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds \\ -\lambda_n\left[\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds \right. \\ \left. -\int_0^1 h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt\right]t^{\beta-1}, \quad (35)$$

and there exists a constant $\Upsilon\in\mathbb{R}$ such that

$$x_n(t)=\lambda_n\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y_n(s))}{\rho_1(s)}ds+\Upsilon t^{\alpha-1}. \quad (36)$$

From (H7) and (33), there exists $t_0 \in (0,1)$ such that $|t_0^{1-\alpha}x_n(t_0)| \leqslant M$. By (36), we have

$$t_0^{1-\alpha}x_n(t_0) = \lambda_n t_0^{1-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y_n(s))}{\rho_1(s)}ds + \Upsilon.$$

Then

$$\begin{aligned}
|\Upsilon| &\leqslant |t_0^{1-\alpha}x_n(t_0)| + \left|\lambda_n t_0^{1-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}(y_n(s))}{\rho_1(s)}ds\right| \\
&\leqslant M + t_0^{1-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\Phi^{-1}\left(s^{\beta-1}[s^{1-\beta}|y_n(s)|]\right)}{\rho_1(s)}ds \\
&\leqslant M + t_0^{1-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\nu\left(s^{\beta-1}\right)}{\rho_1(s)}ds\Phi^{-1}(||y_n||_Y) \\
&\leqslant M + t_0^{1-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}s^k ds\Phi^{-1}(||y_n||_Y) \\
&= M + t_0^{1-\alpha}t_0^{\alpha+k}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^k dw\Phi^{-1}(||y_n||_Y) \\
&\leqslant M + \frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}\Phi^{-1}(||y_n||_Y).
\end{aligned}$$

Then

$$\begin{aligned}
|t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_n(t)| &\leqslant |\Upsilon|\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} + \lambda t^{1+\gamma_1-\alpha}\int_0^t\frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\frac{\Phi^{-1}(y_n(s))}{\rho_1(s)}ds \\
&\leqslant \left[M + \frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}\Phi^{-1}(||y_n||_Y)\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} \\
&\qquad + t^{1+\gamma_1-\alpha}\int_0^t\frac{(t-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}s^k ds\Phi^{-1}(||y_n||_Y) \\
&\leqslant \left[M + \frac{\mathbf{B}(\alpha,k+1)}{\Gamma(\alpha)}\Phi^{-1}(||y_n||_Y)\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} \\
&\qquad + \frac{\mathbf{B}(\alpha-\gamma_1,k+1)}{\Gamma(\alpha-\gamma_1)}\Phi^{-1}(||y_n||_Y).
\end{aligned}$$

One has

$$||x_n||_X \leqslant A_0 + B_0\Phi^{-1}(||y_n||_Y). \tag{37}$$

Case 1. The inequalities (28) hold.

From (34) and (28), (37), we get

$$\begin{aligned}
|t^{1-\beta}y_n(t)| = \Bigg|&\lambda_n t^{1-\beta}\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds \\
&+ \lambda_n\int_0^1 g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} q_1(s)\left|f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)\right|ds \\
&\qquad + \int_0^1 \left|g_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)\right|dt \\
&\leqslant t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} q_1(s)\prod_f (s^{1-\alpha}|x_n(s)|, t^{1+\gamma_1-\alpha}|D_{0^+}^{\gamma_1}x_n(s)|)ds \\
&\qquad + \int_0^1 \phi_g(t)\prod_g (t^{1-\alpha}|x_n(t)|, t^{1+\gamma_1-\alpha}|D_{0^+}^{\gamma_1}x_n(t)|)dt \\
&\leqslant t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^{k_1}(1-s)^{l_1}\prod_f (||x_n||_X, ||x_n||_X)ds \\
&\qquad + \int_0^1 \phi_g(t)\prod_g (||x_n||_X, ||x_n||_X)dt \\
&\leqslant t^{1-\beta}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} s^{k_1}(t-s)^{l_1}\prod_f (||x_n||_X, ||x_n||_X)ds \\
&\qquad + \int_0^1 \phi_g(t)dt\prod_g (||x_n||_X, ||x_n||_X) \\
&= t^{1-\beta}t^{\beta+l_1+k_1}\int_0^1 \frac{(1-w)^{\beta+l_1-1}}{\Gamma(\beta)} w^{k_1}dw\prod_f (||x_n||_X, ||x_n||_X) \\
&\qquad + \int_0^1 \phi_g(t)dt\prod_g (||x_n||_X, ||x_n||_X) \\
&\leqslant \frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f (||x_n||_X, ||x_n||_X) + \int_0^1 \phi_g(t)\prod_g (||x_n||_X, ||x_n||_X) \\
&\leqslant \frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f (A_0+B_0\Phi^{-1}(||y_n||_Y), A_0+B_0\Phi^{-1}(||y_n||_Y)) \\
&\qquad + \int_0^1 \phi_g(t)\prod_g (A_0+B_0\Phi^{-1}(||y_n||_Y), A_0+B_0\Phi^{-1}(||y_n||_Y)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
||y_n||_Y \leqslant{}& \frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f (A_0+B_0\Phi^{-1}(||y_n||_Y), A_0+B_0\Phi^{-1}(||y_n||_Y)) \\
&+ \int_0^1 \phi_g(t)\prod_g (A_0+B_0\Phi^{-1}(||y_n||_Y), A_0+B_0\Phi^{-1}(||y_n||_Y)).
\end{aligned}
$$

Then

$$
1 \leqslant \frac{\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f (A_0+B_0\Phi^{-1}(||y_n||_Y), A_0+B_0\Phi^{-1}(||y_n||_Y))}{||y_n||_Y}
$$

$$+\frac{\int_0^1 \phi_g(t)\prod_g(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y))}{||y_n||_Y}.$$

From $||(x_n,y_n)||\to\infty$ as $n\to\infty$ and (37), we know that $||y_n||_Y\to\infty$ as $n\to\infty$. Let $n\to\infty$. It follows from above inequality that

$$1\leqslant\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}r_f+\int_0^1\phi_g(t)dtr_g,$$

a contradiction to (32). It follows that Ω_1 is bounded.

Case 2. The inequalities (29) hold.

From (35) and (29), (37), we get

$$\begin{aligned}
|t^{1-\beta}y_n(t)| &= \Big|\lambda_n t^{1-\beta}\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds\\
&\quad -\lambda_n\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)ds\\
&\quad +\lambda_n\int_0^1 h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)dt\Big|\\
&\leqslant t^{1-\beta}\int_0^t\frac{(t-s)^{\beta-1}-t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)|f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)|ds\\
&\quad +\int_t^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)|f_1\left(s,x_n(s),D_{0^+}^{\gamma_1}x_n(s)\right)|ds\\
&\quad +\int_0^1|h_1\left(t,x_n(t),D_{0^+}^{\gamma_1}x_n(t)\right)|dt\\
&\leqslant 2\int_0^1\frac{(1-w)^{\beta+l_1-1}}{\Gamma(\beta)}w^{k_1}dw\prod_f(||x||_X,||x||_X)\\
&\quad +\int_0^1\phi_h(t)dt\prod_h(||x||_X,||x||_X)\\
&\leqslant 2\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y))\\
&\quad +\int_0^1\phi_h(t)dt\prod_h(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y)).
\end{aligned}$$

Then

$$\begin{aligned}
||y_n||_Y &\leqslant 2\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y))\\
&\quad +\int_0^1\phi_h(t)dt\prod_h(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y)).
\end{aligned}$$

It follows that

$$1 \leqslant \frac{2\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}\prod_f(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y))}{||y_n||_Y} + \frac{\int_0^1\phi_h(t)dt\prod_h(A_0+B_0\Phi^{-1}(||y_n||_Y),A_0+B_0\Phi^{-1}(||y_n||_Y))}{||y_n||_Y}.$$

Let $n \to \infty$. It follows from above inequality that

$$1 \leqslant 2\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}r_f + \int_0^1 \phi_h(t)dt r_h,$$

a contradiction to (32). It follows that Ω_1 is bounded.

Step 2. Let $\Omega_2 = \{(ct^{\alpha-1},0) \in \mathrm{Ker}L:\ N(ct^{\alpha-1},0) \in \mathrm{Im}L\}$. We prove that Ω_2 is bounded.

For $(ct^{\alpha-1},0) \in \Omega_2$, we have

$$N(ct^{\alpha-1},0) = \left(0, f_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right), \int_0^1 g_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)dt, \int_0^1 h_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)dt\right).$$

So

$$\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}q_1(s)f_1\left(s,cs^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}s^{\alpha-\gamma_1-1}\right)ds = \int_0^1 h\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)dt - \int_0^1 g\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)dt.$$

From (H9), we get that $|c| \leqslant M_0$. This shows Ω_2 is bounded.

Step 3. If (30) in (H9) holds for all $|c| > M_0$, we prove that

$$\Omega_3 = \{(ct^{\alpha-1},0) \in \mathrm{Ker}\,L:\ \lambda \wedge (ct^{\alpha-1},0) + (1-\lambda)QN(ct^{\alpha-1},0) = 0,\ \lambda \in [0,1]\}$$

is bounded, where $\wedge$ is the isomorphism given by $\wedge(ct^{\alpha-1},0) = (0,ct^{1-\beta},0,0)$.

For $(ct^{\alpha-1},0) \in \mathrm{Ker}\,L$, one sees that

$$-\lambda ct^{1-\beta} = (1-\lambda)t^{1-\beta}J_1,$$

where

$$J_1 = \frac{1}{\Gamma(\beta)\int_0^1 (1-s)^{\beta-1} q_1(s) s^{1-\beta} ds} \times \int_0^1 \left[\frac{(1-s)^{\beta-1}}{\Gamma(\beta)} q_1(s) f_1\left(s, cs^{\alpha-1}, c\frac{\Gamma(\alpha)s^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\right) - h_1\left(s, cs^{\alpha-1}, c\frac{\Gamma(\alpha)s^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\right) g_1\left(s, cs^{\alpha-1}, c\frac{\Gamma(\alpha)s^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)}\right)\right] ds.$$

Then

$$-\lambda c^2 = (1-\lambda)cJ_1.$$

If $\lambda = 1$, then $c = 0$. If $\lambda \in [0,1)$, and $|c| > M_0$, we get

$$0 \geqslant -\lambda c^2 = (1-\lambda)cJ_1 > 0,$$

a contradiction. Then $|c| \leqslant M_0$. Then Ω_3 is bounded.

If (31) in (H9) holds for all $|c| > M_0$, let

$$\Omega_3 = \{(ct^{\alpha-1}, 0) \in \text{Ker } L : \lambda \wedge (ct^{\alpha-1}, 0) - (1-\lambda)QN(ct^{\alpha-1}, 0) = 0, \lambda \in [0,1]\},$$

We can prove that Ω_3 is bounded too.

Step 4. We shall show that all conditions of Lemma 2.2 are satisfied.

Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$. By Lemma 3.1, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have

(a). $L(x,y) \neq \lambda N(x,y)$ for $(x,y) \in (D(L) \setminus \text{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b). $N(x,y) \notin \text{Im}L$ for $(x,y) \in \text{Ker}L \cap \partial\Omega$.

(c). $\deg(QN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) \neq 0$. In fact, let $H((x,y),\lambda) = \pm\lambda \wedge (x,y) + (1-\lambda)QN(x,y)$. According the definition of Ω, we know $H((x,y),\lambda) \neq 0$ for $(x,y) \in \partial\Omega \cap \text{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \text{Ker}L, 0) \\ &= \deg(H(\cdot,1), \Omega \cap \text{Ker}L, 0) \\ &= \deg(\wedge, \Omega \cap \text{Ker}L, 0) \neq 0.\end{aligned}$$

Thus by Lemma 2.2, $L(x,y) = N(x,y)$ has at least one solution in $D(L) \cap \overline{\Omega}$. Then x is a solution of BVP(10). The proof is complete.

THEOREM 3.2. *Suppose that (H1)-(H6) and (H8), (H9) hold and*

(H7)′ *there exist nonnegative numbers* $A_f, B_f, C_f, A_g, B_g, C_g, A_h, B_h, C_h$, *and nonnegative functions* $\phi_g, \phi_h \in L^1(0,1)$ *such that*

$$\begin{aligned}&\left|f_1\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_1-1}v\right)\right| \leqslant C_f + B_f\Phi(|u|) + A_f\Phi(|v|), \\ &\left|g_1\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_1-1}v\right)\right| \leqslant \phi_g(t)[C_g + B_g\Phi(|u|) + A_g\Phi(|v|)]\end{aligned} \tag{38}$$

or

$$\begin{aligned}&\left|f_1\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_1-1}v\right)\right|\leqslant C_f+B_f\Phi(|u|)+A_f\Phi(|v|),\\&\left|h_1\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_1-1}v\right)\right|\leqslant \phi_h(t)[C_h+B_h\Phi(|u|)+A_h\Phi(|v|)]\end{aligned}\tag{39}$$

hold for all $(u,v)\in\mathbb{R}^2,\ t\in(0,1)$.

Then BVP(10) has at least one solution if

$$\begin{aligned}&\left[(B_f+A_f)\tfrac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}+(B_g+A_g)\int_0^1\phi_g(t)dt\right]\omega(B_0)<1 \text{ if (38) holds},\\&\left[2(B_f+A_f)\tfrac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}+(B_h+A_h)\int_0^1\phi_h(t)dt\right]\omega(B_0)<1 \text{ if (39) holds},\end{aligned}\tag{40}$$

where

$$A_0=\max\left\{M,\ M\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\right\},\ \ B_0=\max\left\{\nu,\ \frac{\mathbf{B}(\alpha,k+1)+\mathbf{B}(\alpha-\gamma_1,k+1)}{\Gamma(\alpha-\gamma_1)}\right\}.$$

Proof. Let E, Z L and N be defined above. By (H1)-(H6), from Lemma 3.1, L be a Fredholm operator of index zero and N be $L-$compact on each open nonempty set Ω centered at zero. We seek fixed point of the operator equation $L(x,y)=N(x,y)$. To apply Lemma 2.2, we should define an open bounded subset Ω of E centered at zero such that (i), (ii) and (iii) in Lemma 2.2 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1=\{(x,y)\in E\cap D(L)\setminus \mathrm{Ker}L,\ L(x,y)=\lambda N(x,y)$ for some $\lambda\in(0,1)\}$. We prove that Ω_1 is bounded.

It is easy to see that

$$\begin{aligned}\frac{\Phi(A_0+B_0\Phi^{-1}(||y_n||_Y))}{||y_n||_Y}&=\frac{\Phi\left(\left(\frac{A_0}{\Phi^{-1}(||y_n||_Y)}+B_0\right)\Phi^{-1}(||y_n||_Y)\right)}{||y_n||_Y}\\&\leqslant\frac{\omega\left(\frac{A_0}{\Phi^{-1}(||y_n||_Y)}+B_0\right)\Phi\left(\Phi^{-1}(||y_n||_Y)\right)}{||y_n||_Y}\\&=\omega\left(\frac{A_0}{\Phi^{-1}(||y_n||_Y)}+B_0\right)\\&\to\omega(B_0)\text{ as }n\to\infty.\end{aligned}$$

Choose $\prod_f(s)=C_f+B_f\Phi(s)+A_f\Phi(s)$, $\prod_g(s)=C_g+B_g\Phi(s)+A_g\Phi(s)$, $\prod_h(s)=C_h+B_h\Phi(s)+A_h\Phi(s)$. If (38) holds, we get (37) and

$$\begin{aligned}||y_n||_Y\leqslant C_f\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}+C_g\int_0^1\phi_g(t)dt\\+\left[(B_f+A_f)\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)}+(B_g+A_g)\int_0^1\phi_g(t)dt\right]\\\times\Phi(A_0+B_0\Phi^{-1}(||y_n||_Y)).\end{aligned}$$

It follows that

$$1 \leqslant \frac{C_f \frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)} + C_g \int_0^1 \phi_g(t)dt}{||y_n||_Y} + \left[(B_f+A_f)\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)} + (B_g+A_g)\int_0^1 \phi_g(t)dt\right] \times \frac{\Phi(A_0+B_0\Phi^{-1}(||y_n||_Y))}{||y_n||_Y}.$$

Let $n \to \infty$. We get

$$1 \leqslant \left[(B_f+A_f)\frac{\mathbf{B}(\beta+l_1,k_1+1)}{\Gamma(\beta)} + (B_g+A_g)\int_0^1 \phi_g(t)dt\right]\omega(B_0)$$

which contradicts to (40). Hence Ω_1 is bounded. If (39) holds, we also get that Ω_1 is bounded.

The other part of the proof is just same to that of the proof of Theorem 3.1 and is omitted.

THEOREM 3.3. *Suppose that* $(H1)$-$(H6)$ *and* $(H7)'$ *hold and*

$$\lim_{u\to+\infty}\inf_{t\in(0,1),v\in\mathbb{R}}\left[(1-t)^{\beta-1}q_1(t)f_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) - h_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) + g_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right)\right] > 0,$$

and

$$\lim_{u\to-\infty}\sup_{t\in(0,1),v\in\mathbb{R}}\left[(1-t)^{\beta-1}q_1(t)f_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) - h_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) + g_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right)\right] < 0.$$

Then BVP(10) has at least one solution if (40) holds.

Proof. We need to proof that (H8) and (H9) in Theorem 3.2 hold. From the assumptions, there exist $M>0$ and $r>0$ such that

$$(1-t)^{\beta-1}q_1(t)f_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) - h_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) + g_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) > r$$

holds for all $t\in(0,1)$ and $u>M$, $v\in\mathbb{R}$ and

$$(1-t)^{\beta-1}q_1(t)f_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) - h_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) + g_1\left(t,t^{\alpha-1}u,t^{1+\gamma_1-\alpha}v\right) < -r$$

holds for all $t\in(0,1)$ and $u<-M$, $v\in\mathbb{R}$.

If $t^{1-\alpha}x(t)>M$ for all $t\in(0,1)$, then

$$(1-t)^{\beta-1}q_1(t)f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)-h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)+g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)>r.$$

So

$$\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)-h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)+g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\Big]dt>0.$$

If $t^{1-\alpha}x(t)<-M$ for all $t\in(0,1)$, then

$$(1-t)^{\beta-1}q_1(t)f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)-h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)+g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)<-r.$$

So

$$\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)-h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)+g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\Big]dt<0.$$

Since $x\in E$, we have $t\to t^{1-\alpha}x(t)$ is continuous on $[0,1]$. Then $t^{1-\alpha}|x(t)|>M$ for all $t\in[0,1]$ implies

$$\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)-h_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)+g_1\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\Big]dt\neq 0.$$

Similarly, we have that either

$$c\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)-h_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)+g_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)\Big]dt>0 \quad (41)$$

holds for all $|c|>M_0$ or

$$c\int_0^1\Big[(1-t)^{\beta-1}q_1(t)f_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)-h_1\left(t,ct^{\alpha-1},c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)$$

$$+g_1\left(t, ct^{\alpha-1}, c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}t^{\alpha-\gamma_1-1}\right)\Bigg]dt < 0, \quad (42)$$

holds for all $|c| > M_0$.

Hence (H8) and (H9) hold. Then the proof follows from Theorem 3.2.

REMARK 3.3. Let $\alpha = \frac{1}{2} = \beta$ and $\gamma_1 = \frac{1}{4}$. Consider the functions

$$f_1(t,x,y) = t^{\frac{5}{2}}x^5 + t^{\frac{3}{4}}\sin y,$$

$$g_1(t,x,y) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\left(t^{\frac{3}{2}}x^3 + t^{\frac{3}{4}}y\right),$$

$$h_1(t,x,y) = g_1(t,x,y),$$

$$q_1(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}}.$$

It is easy to see that f_1, g_1 and h_1 satisfy (H1), (H2) and (H3) respectively. q_1 satisfies (H4). Furthermore, we have the followings:

• for $\phi_g(t) = \phi_h(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}$ and bi-nondecreasing functions $\prod_f(u,v) = u^5 + v$ and $\prod_g(u,v) = \prod_h(u,v) = u^3 + v$, it holds that

$$\left|f_1\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_1-1}v\right)\right| \leqslant \prod_f(|u|,|v|),$$

$$\left|g_1\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_1-1}v\right)\right| \leqslant \phi_g(t)\prod_g(|u|,|v|),$$

$$\left|h_1\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_1-1}v\right)\right| \leqslant \phi_h(t)\prod_h(|u|,|v|)$$

hold for all $(u,v) \in \mathbb{R}^2$, $t \in (0,1)$;

• there exists a constant $M > 0$ such that for $x \in X$, if $t^{1-\alpha}x(t) > M$ for all $t \in (0,1)$ then

$$\begin{aligned}
&\int_0^1 \Big[(1-t)^{\beta-1}q_1(t)f_1\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right) - h_1\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right) \\
&\qquad\qquad + g_1\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right)\Big]dt \\
&= \int_0^1 (1-t)^{\beta-1}q_1(t)f_1\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right)dt \\
&= \int_0^1 (1-t)^{-\frac{1}{2}}t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}}\left(t^{\frac{5}{2}}x(t)^5 + t^{\frac{3}{4}}\sin D_{0^+}^{\gamma_1}x(t)\right)dt \\
&> \int_0^1 (1-t)^{-\frac{1}{2}}t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}}\left(t^{\frac{5}{2}}t^{-\frac{1}{2}}M^5 - t^{\frac{3}{4}}\right)dt > 0.
\end{aligned}$$

If $t^{1-\alpha}x(t) < -M$ for all $t \in (0,1)$ then similarly we get

$$\int_0^1 \Big[(1-t)^{\beta-1} q_1(t) f_1\left(t, x(t), D_{0^+}^{\gamma_1} x(t)\right) \\ - h_1\left(t, x(t), D_{0^+}^{\gamma_1} x(t)\right) + g_1\left(t, x(t), D_{0^+}^{\gamma_1} x(t)\right)\Big] dt < 0.$$

So (H8) holds.

• Since

$$c\int_0^1 \left[(1-t)^{\beta-1} q_1(t) f_1\left(t, ct^{\alpha-1}, c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} t^{\alpha-\gamma_1-1}\right)\right. \\ \left. - h_1\left(t, ct^{\alpha-1}, c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} t^{\alpha-\gamma_1-1}\right) + g_1\left(t, ct^{\alpha-1}, c\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} t^{\alpha-\gamma_1-1}\right)\right] dt \\ = c\int_0^1 (1-t)^{-\frac{1}{2}} t^{-\frac{1}{4}} (1-t)^{-\frac{1}{4}} \left(c^5 t^{-\frac{5}{2}} t^{\frac{5}{2}} + \sin c \frac{\Gamma(1/2)}{\Gamma(1/4)} t^{-\frac{3}{4}}\right) dt > 0$$

holds for all sufficiently large $|c|$. Then (H9) holds.

4. Solvability of BVP(11)

Suppose that

(G1) $f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following items:

- $t \to f_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)$ is continuous on $[0,1]$ for each $(x,y) \in \mathbb{R}^2$,
- $(x,y) \to f_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t \in [0,1]$,
- for each $r > 0$, there exists a nonnegative $M_r \geqslant 0$ such that

$$\left|f_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)\right| \leqslant M_r, t \in (0,1), |x|, |y| \leqslant r.$$

(G2) $g_2 : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following items:

- $t \to g_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)$ is continuous on $[0,1]$ for each $(x,y) \in \mathbb{R}^2$,
- $(x,y) \to g_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t \in [0,1]$,
- for each $r > 0$, there exists a nonnegative function $\phi_r \in L^1(0,1)$ such that

$$\left|g_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)\right| \leqslant \phi_r(t), \ \ t \in (0,1), |x|, |y| \leqslant r.$$

(G3) $h_2 : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following items:

- $t \to h_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)$ is continuous on $[0,1]$ for each $(x,y) \in \mathbb{R}^2$,
- $(x,y) \to f_3\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t \in [0,1]$,
- for each $r > 0$, there exists a nonnegative function $\psi_r \in L^1(0,1)$ such that

$$\left|h_2\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)\right| \leqslant \psi_r(t), \ \ t \in (0,1), |x|, |y| \leqslant r.$$

(G4) q_2 satisfies that there exist numbers $k_2 > -\alpha$ and $l_2 \in (-\beta, 0)$ with $\beta + k_2 + l_2 > 0$ such that $q_2(t) \leqslant t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$ and $q_2(t) \not\equiv 0$ on $(0,1)$

satisfying $\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds \neq 0$, where

$$G(\eta,s) = \begin{cases} \frac{\eta^{\alpha+\beta-1}(1-s)^{\alpha+\beta-1}-(\eta-s)^{\alpha+\beta-1}}{\eta^{\alpha+\beta-1}\Gamma(\alpha+\beta)}, \ 0 \leqslant s < \eta, \\ \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}, \ \eta \leqslant s < 1. \end{cases}$$

Denote

$$Q =: \frac{\int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha-1}ds}{\int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds},$$

$$Q_0 = \frac{\int_0^1 (1-t)^{\alpha-1}q_2(t)\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)dsdt}{\int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds},$$

$$Q_1 =: \frac{\int_0^1 (1-t)^{\alpha-1}q_2(t)\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)s^{1-\beta}dsdt}{\int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds}.$$

REMARK 4.1. If $\alpha+\beta \geqslant 1$, then $\eta^{\alpha+\beta-1}(1-s)^{\alpha+\beta-1} - (\eta-s)^{\alpha+\beta-1} \geqslant 0$. So $G(\eta,s) \geqslant 0$ for all $s \in (0,1)$. Then $\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds > 0$. If $q_2(t) = t^{k_2}(1-t)^{l_2}$ with $k_2+l_2+1>0$, then

$$\begin{aligned}
&\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds \\
&\quad = \int_0^1 (1-s)^{\alpha+\beta-1}s^{k_2}(1-s)^{l_2}s^{1-\beta}ds - \frac{1}{\eta^{\alpha+\beta-1}}\int_0^\eta (\eta-s)^{\alpha+\beta-1}s^{k_2}(1-s)^{l_2}s^{1-\beta}ds \\
&\quad \geqslant \mathbf{B}(\alpha+\beta+l_2, k_2+2-\beta) - \frac{1}{\eta^{\alpha+\beta-1}}\int_0^\eta (\eta-s)^{\alpha+\beta-1}s^{k_2+1-\beta}(\eta-s)^{l_2}ds \\
&\quad = \mathbf{B}(\alpha+\beta+l_2, k_2+2-\beta) - \frac{1}{\eta^{\alpha+\beta-1}}\eta^{\alpha+1+l_2+k_2}\int_0^1 (1-w)^{\alpha+\beta+l_2-1}w^{k_2+1-\beta}dw \\
&\quad = \mathbf{B}(\alpha+\beta+l_2, k_2+2-\beta) - \eta^{2-\beta+k_2+l_2}\mathbf{B}(\alpha+\beta+l_2, k_2+2-\beta) > 0.
\end{aligned}$$

REMARK 4.2. Q, Q_0 and Q_1 are well defined. In fact, we have

$$\begin{aligned}
0 &< \int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds \\
&\leqslant \int_0^1 (1-s)^{\alpha-1}s^{k_2}(1-s)^{l_2}s^{\alpha+\beta-1}ds = \mathbf{B}(\alpha+l_2, k_2+\alpha+\beta) < \infty,
\end{aligned}$$

and

$$\begin{aligned}
Q_0 &\leqslant \frac{\int_0^1 (1-t)^{\alpha-1}t^{k_2}(1-t)^{l_2}\int_0^t (t-s)^{\alpha+\beta-1}s^{k_2}(1-s)^{l_2}dsdt}{\int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds} \\
&\leqslant \frac{\int_0^1 (1-t)^{\alpha-1}t^{k_2}(1-t)^{l_2}\int_0^t (t-s)^{\alpha+\beta-1}s^{k_2}(t-s)^{l_2}dsdt}{\int_0^1 (1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds}
\end{aligned}$$

$$= \frac{\int_0^1 (1-t)^{\alpha-1} t^{k_2} (1-t)^{l_2} t^{\alpha+\beta+l_2+k_2} \int_0^1 (1-w)^{\alpha+\beta+l_2-1} w^{k_2} dw dt}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds}$$
$$= \frac{\mathbf{B}(\alpha+l_2, 2k_2+\alpha+\beta+l_2+1)\mathbf{B}(\alpha+\beta+l_2, k_2+1)}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds} < \infty.$$

Similarly we have $Q < \infty$ and $Q_1 < \infty$.

Let $C(0,1]$ or $C[0,1]$ be the set of all continuous functions on $(0,1]$ or $[0,1]$. We use the Banach spaces

$$E = \Big\{ x \in C(0,1] : \ D_{0^+}^{\gamma_2} x \in C(0,1], \\ \text{there exist the limits } \lim_{t\to 0} t^{1-\alpha} x(t), \lim_{t\to 0} t^{1+\gamma_2-\alpha} D_{0^+}^{\gamma_2} x(t) \Big\}$$

with the norm

$$||x|| = \max\left\{ \sup_{t\in(0,1]} t^{1-\alpha}|x(t)|, \lim_{t\in(0,1]} t^{1+\gamma_2-\alpha} D_{0^+}^{\gamma_2} x(t) \right\}, x \in E,$$

and $Z = C[0,1] \times \mathbb{R}^2$ with the norm $||(u,a,b)|| = \max\left\{ \max_{t\in[0,1]} |u(t)|, |a|, |b| \right\}$.

Define

$$D(L) = \left\{ x \in E : t \to \frac{D_{0^+}^{\beta}[D_{0^+}^{\alpha} x(t)]}{q_2(t)} \text{ is continuous on } [0,1] \right\}$$

and $L : E \cap D(L) \to Z$ by

$$Lx(t) = \left(\frac{D_{0^+}^{\beta}[D_{0^+}^{\alpha} x(t)]}{q_2(t)}, \lim_{t\to 0} t^{1-\alpha} x(t), x(1) - \frac{1}{\eta^{\alpha+\beta-1}} x(\eta) \right) \text{ for } x \in E.$$

Define $N : E \to Z$ by

$$Nx(t) = \left(f_2\left(t, x(t), D_{0^+}^{\gamma_2} x(t)\right), \int_0^1 g_2(s, x(s), D_{0^+}^{\gamma_2} x(s)) ds, \int_0^1 h_2(s, x(s), D_{0^+}^{\gamma_2} x(s)) ds \right)$$

for $x \in E$. Then BVP(11) can be written as $Lx = Nx, \ x \in E \bigcap D(L)$.

LEMMA 4.1. *Suppose that (G1)-(G4) hold. Then* L *is a Fredholm operator of index zero and* $N : X \to Y$ *is* $L-$*compact.*

Proof. First, for $x \in E \bigcap D(L)$, then $t \to \frac{D_{0^+}^{\beta}[D_{0^+}^{\alpha} x(t)]}{q_2(t)}$ is continuous on $[0,1]$ and the limits $\lim_{t\to 0} t^{1-\alpha} x(t), x(1) - \frac{1}{\eta^{\alpha+\beta-1}} x(\eta)$ exist. So $Lx \in Z$. Then $L : E \bigcap D(L) \to Z$ is well defined.

To prove that L is a Fredholm operator of index zero, we should do the following three steps.

Step (i) Prove that $\mathrm{Ker}L = \{ct^{\alpha+\beta-1} \in E,\ c \in \mathbb{R}\}$.

We know that $x \in \mathrm{Ker}L$ if and only if

$$\begin{cases} \frac{D_{0^+}^{\beta}[D_{0^+}^{\alpha}x(t)]}{q_2(t)} = 0, \\ \lim_{t\to 0} t^{1-\alpha}x(t) = 0, \\ x(1) - \frac{1}{\eta^{\alpha+\beta-1}}x(\eta) = 0. \end{cases}$$

So $x \in \mathrm{Ker}L$ if and only if there exist numbers $\Upsilon, \Psi \in \mathbb{R}$ such that

$$\begin{cases} x(t) = \Upsilon t^{\alpha+\beta-1} + \Psi t^{\alpha-1}, t \in (0,1], \\ \lim_{t\to 0} t^{1-\alpha}x(t) = 0, \\ x(1) - \frac{1}{\eta^{\alpha+\beta-1}}x(\eta) = 0. \end{cases}$$

Hence $x \in \mathrm{Ker}L$ if and only if $x(t) = ct^{\alpha+\beta-1}$ for some $c \in \mathbb{R}$. Thus $\mathrm{Ker}L = \{ct^{\alpha+\beta-1} \in E,\ c \in \mathbb{R}\}$.

Step (ii) Prove that

$$\mathrm{Im}L = \left\{ (u,a,b) \in Z :\ \int_0^1 G(\eta,s)q_2(s)u(s)u(s)ds = b - \left(1 - \frac{1}{\eta^{\beta}}\right) a \right\}. \tag{43}$$

For $(u,a,b) \in Z$, we know that $(u,a,b) \in \mathrm{Im}L$ if and only if there exist $x \in E \bigcap D(L)$ such that

$$\begin{cases} \frac{D_{0^+}^{\beta}[D_{0^+}^{\alpha}x(t)]}{q_2(t)} = u(t), \\ \lim_{t\to 0} t^{1-\alpha}x(t) = a, \\ x(1) - \frac{1}{\eta^{\alpha+\beta-1}}x(\eta) = b. \end{cases}$$

It follows that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)u(s)ds + \Upsilon t^{\alpha+\beta-1} + \Psi t^{\alpha-1} \text{ with } \Upsilon, \Psi \in \mathbb{R},$$
$$\lim_{t\to 0} t^{1-\alpha}x(t) = a,\ x(1) - \frac{1}{\eta^{\alpha+\beta-1}}x(\eta) = b.$$

It is easy to see that

$$\begin{aligned} t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)u(s)ds\right| &\leqslant t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} s^{k_2}(1-s)^{l_2}ds||u||_\infty \\ &\leqslant t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\beta)} s^{k_2}(t-s)^{l_2}ds||u||_\infty \\ &= t^{1+\beta+l_2+k_2}\int_0^1 \frac{(1-w)^{\alpha+\beta+l_2-1}}{\Gamma(\alpha+\beta)} w^{k_2}dw||u||_\infty \\ &\to 0 \text{ as } t \to 0. \end{aligned}$$

Then $\lim_{t\to 0} t^{1-\alpha}x(t) = a$ implies that $\Psi = a$. So $(u,a,b) \in \mathrm{Im}L$ if and only if

$$\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)u(s)ds - \frac{1}{\eta^{\alpha+\beta-1}} \int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)u(s)ds = b - \left(1 - \frac{1}{\eta^\beta}\right) a.$$

This ends the proof.

Step (iii) Prove that $\mathrm{Im}L$ is closed in X and $\dim \mathrm{Ker}L =$ co $\dim \mathrm{Im}L < +\infty$.

From (41) $\mathrm{Im}L$ is closed in Z. It follows from $\mathrm{Ker}L = \{ct^{\alpha+\beta-1} \in E,\ c \in \mathbb{R}\}$ that $\dim \mathrm{Ker}L = 1$. Define the projector $P:\ E \to E$ by

$$P(x)(t) = \frac{\int_0^1 (1-s)^{\alpha-1} q_2(s)x(s)ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s)s^{\alpha+\beta-1}ds} t^{\alpha+\beta-1} \text{ for } x \in E. \tag{44}$$

It is easy to prove that

$$\mathrm{Im}\, P \subseteq \mathrm{Ker}\, L. \tag{45}$$

For $ct^{\alpha+\beta-1} \in \mathrm{Ker}L$, choose $x_c(t) = ct^{\alpha+\beta-1}$. It follows that

$$\frac{D_{0+}^\beta [D_{0+}^\alpha x_c(t)]}{q_2(t)} = \frac{D_{0+}^\beta [D_{0+}^\alpha ct^{\alpha+\beta-1}]}{q_2(t)} = 0, t \in [0,1]$$

One can show that $x_c \in E \bigcap D(L)$ and

$$P(x_c)(t) = \frac{\int_0^1 (1-s)^{\alpha-1} q_2(s)cs^{\alpha+\beta-1}ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s)s^{\alpha+\beta-1}ds} t^{\alpha+\beta-1} = ct^{\alpha+\beta-1}.$$

So

$$\mathrm{Im}\, P \supseteq \mathrm{Ker}\, L. \tag{46}$$

Hence $\mathrm{Im}\, P = \mathrm{Ker}\, L$.

For $(x,y) \in E$, we have

$$P(x - P(x)) = P\left(x - \frac{\int_0^1 (1-s)^{\alpha-1} q_2(s)x(s)ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s)s^{\alpha+\beta-1}ds} t^{\alpha+\beta-1}\right)$$

$$= \frac{\int_0^1 (1-s)^{\alpha-1} q_2(s)\left(x(s) - \frac{\int_0^1 (1-s)^{\alpha-1} q_2(s)x(s)ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s)s^{\alpha+\beta-1}ds} s^{\alpha+\beta-1}\right)ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s)s^{\alpha+\beta-1}ds} t^{\alpha+\beta-1} = 0.$$

So $X = \mathrm{Ker}L + \mathrm{Ker}P$. For $(x,y) \in \mathrm{Ker}L \cap \mathrm{Ker}P$, we get $x(t) = 0$. Hence $X = \mathrm{Ker}L \oplus \mathrm{Ker}P$.

Define the projector $Q:\ Z \to Z$ by

$$Q(u,a,b)(t) = \left(\frac{\int_0^1 G(\eta,s)q_2(s)u(s)ds - \left[b - \left(1 - \frac{1}{\eta^\beta}\right)a\right]}{\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds} t^{1-\beta},\ 0,\ 0\right) \tag{47}$$

for $(u,a,b) \in Z$.

It is easy to show that $\mathrm{Im}L \supseteq \mathrm{Ker}Q$.

For $(u,a,b) \in \mathrm{Im}L$, we have

$$\int_0^1 G(\eta,s)q_2(s)u(s)ds = b - \left(1 - \frac{1}{\eta^\beta}\right)a.$$

Then $Q(u,a,b) = (0,0,0)$. It follows that $\mathrm{Im}L \subseteq \mathrm{Ker}Q$. Hence $\mathrm{Im}L = \mathrm{Ker}Q$.

For $(u,a,b) \in Z$, we have

$$(u,a,b) - Q(u,a,b) = (u,a,b) - \left(c_2 t^{1-\beta},0,0\right) = \left(u - c_2 t^{1-\beta},a,b\right),$$

where the constant c_2 is defined by:

$$c_2 = \frac{\int_0^1 G(\eta,s)q_2(s)u(s)ds - \left[b - \left(1 - \frac{1}{\eta^\beta}\right)a\right]}{\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds}.$$

It is easy to see that

$$\int_0^1 G(\eta,t)q_2(t)\left(u(t) - c_2 t^{1-\beta}\right)dt = b - \left(1 - \frac{1}{\eta^\beta}\right)a.$$

Hence $(u,a,b) - Q(u,a,b) \in \mathrm{Im}L$. Thus $Y = \mathrm{Im}Q + \mathrm{Im}L$. It is easy to show that $\mathrm{Im}Q \cap \mathrm{Im}L = (0,0,0)$. Hence $Y = \mathrm{Im}Q \oplus \mathrm{Im}L$.

From above discussion, we see that $\dim \mathrm{Ker}L =$ co $\dim \mathrm{Im}L = \dim \mathrm{Im}Q = 1 < +\infty$. So L is a Fredholm operator of index zero.

Step (iv) We prove that N is $L-$compact. This is divided into three steps.

Substep (iv1) We prove that N is continuous. Let $x_n \in E$ with $x_n \to x_0$ as $n \to \infty$. We will show that $N(x_n) \to N(x_0)$ as $n \to \infty$.

In fact, we have $||x_n|| \leqslant r < +\infty$ and

$$\sup_{t\in(0,1]} t^{1-\alpha}|x_n(t) - x_0(t)| \to 0, \ \sup_{t\in(0,1]} t^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(t) - D_{0^+}^{\gamma_2}x_0(t)| \to 0, n \to \infty. \quad (48)$$

By

$$N(x_n)(t) = \left(f_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right),\right.$$
$$\left.\int_0^1 g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds, \int_0^1 h_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds\right),$$

similarly to the proof of Substep (iv1) in Lemma 3.1, we get that

$$||N(x_n) - N(x_0)|| \to 0, n \to \infty.$$

It follows that N is continuous.

Let Ω be a bounded open subset of E. We have that there exists $r>0$ such that $||x||\leqslant r$ for all $x\in\overline{\Omega}$. Since (G1)-(G3) hold, then there exists $M_r\geqslant 0$, $\phi_r,\psi_r\in L^1(0,1)$ such that

$$\begin{aligned}&\left|f_2\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\right|\leqslant M_r, t\in(0,1), x\in\overline{\Omega},\\&\left|g_2\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\right|\leqslant \phi_r(t), t\in(0,1), x\in\overline{\Omega},\\&\left|h_2\left(t,x(t),D_{0^+}^{\gamma_1}x(t)\right)\right|\leqslant \psi_r(t), t\in(0,1), x\in\overline{\Omega}.\end{aligned}\tag{49}$$

Substep (iv2) Prove that $QN(\overline{\Omega})$ is bounded.

Furthermore, one has

$$\begin{aligned}QN(x)(t)=Q\bigg(&f_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right),\int_0^1 g_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)dt,\\&\int_0^1 h_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)dt\bigg)=\left(J_2t^{1-\beta},0,0\right).\end{aligned}\tag{50}$$

where the constant J_2 is defined by

$$\begin{aligned}J_2=\frac{1}{\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds}\int_0^1\bigg[&G(\eta,t)q_2(t)f_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)\\&-h_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)+\left(1-\frac{1}{\eta^\beta}\right)g_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)\bigg]dt\end{aligned}$$

It is easy to see from (47) and (48) that

$$\begin{aligned}|J_2t^{1-\beta}|\leqslant{}&\frac{1}{\left|\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds\right|}\bigg[M_r\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}s^{k_2}(1-s)^{l_2}ds\\&+\frac{M_r}{\eta^{\alpha+\beta-1}}\int_0^\eta\frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}s^{k_2}(1-s)^{l_2}ds+\int_0^1\psi_r(t)dt\\&+\left|1-\frac{1}{\eta^\beta}\right|\int_0^1\phi_r(t)dt\bigg]\\\leqslant{}&\frac{1}{\left|\int_0^1 G(\eta,s)q_2(s)s^{1-\beta}ds\right|}\bigg[M_r\frac{\mathbf{B}(\alpha+\beta+l_2,k_2+1)}{\Gamma(\alpha+\beta)}\\&+M_r\eta^{1+l_2}\frac{\mathbf{B}(\alpha+\beta+l_2,k_2+1)}{\Gamma(\alpha+\beta)}+\int_0^1\psi_r(t)dt+\left|1-\frac{1}{\eta^\beta}\right|\int_0^1\phi_r(t)dt\bigg]\\<{}&+\infty.\end{aligned}$$

It follows $QN(\overline{\Omega})$ is bounded.

Substep (iv3) Prove that $K_P(I-Q)N:\overline{\Omega}\to E$ is compact, i.e., prove that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact.

Let $P : X \to X$ and $Q : Y \to Y$ be defined by (42) and (45). For $(u,a,b) \in \mathrm{Im}L$, let

$$K_P(u,a,b)(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)u(s)ds - \int_0^1 (1-t)^{\alpha-1} q_2(t) \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)u(s)dsdt \times \frac{t^{\alpha+\beta-1}}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds} - a\frac{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha-1} ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds} t^{\alpha+\beta-1} + at^{\alpha-1}. \tag{51}$$

for $(u,a,b) \in \mathrm{Im}L$.

One sees $K_P(u,a,b) \in E$ and $K_p(u,a,b) \in \mathrm{Ker}P$. Then $K_P : \mathrm{Im}L \to D(L) \cap \mathrm{Ker}P$ is well defined.

Furthermore, for $(u,a,b) \in \mathrm{Im}L$, we have $(LK_P)(u,a,b)(t) = (u,a,b)$. On the other hand, for $x \in \mathrm{Ker}\, P \cap E$, we have $K_P L(x)(t)x(t)$. Then K_P is the inverse of $L : D(L) \bigcap \mathrm{Ker}P \to \mathrm{Im}L$. The isomorphism $\wedge :\ \mathrm{Ker}L \to Y/\mathrm{Im}L$ is given by

$$\wedge(ct^{\alpha+\beta-1}) = (ct^{1-\beta}, 0, 0), \ \ c \in \mathbb{R}.$$

Let J_2 be a constant defined as in previous subset. Then

$$\begin{aligned}
&K_p(I-Q)N(x,y)(t) \\
&\quad =: x_1(t) \\
&\quad = K_p(I-Q)\left(f_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right), \int_0^1 g_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right)dt,\right. \\
&\qquad\quad \left.\int_0^1 h_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right)dt\right) \\
&\quad = K_P\left(f_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right), \int_0^1 g_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right)dt, \int_0^1 h_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right)dt\right) \\
&\qquad\quad - K_P\left(J_2 t^{1-\beta},0,0\right) \\
&\quad = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s) f_2\left(s,x(s),D_{0+}^{\gamma_2}x(s)\right)ds - \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s)s^{1-\beta}ds \times J_2 \\
&\qquad\quad - t^{\alpha+\beta-1}\frac{\int_0^1 (1-t)^{\alpha-1} q_2(t) \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s) f_2\left(s,x(s),D_{0+}^{\gamma_2}x(s)\right)dsdt}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds} \\
&\qquad\quad + t^{\alpha+\beta-1}\frac{\int_0^1 (1-t)^{\alpha-1} q_2(t) \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} q_2(s) s^{1-\beta}dsdt}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds} \times J_2 \\
&\qquad\quad - t^{\alpha+\beta-1}\frac{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha-1} ds}{\int_0^1 (1-s)^{\alpha-1} q_2(s) s^{\alpha+\beta-1} ds} \int_0^1 g_2\left(t,x(t),D_{0+}^{\gamma_2}x(t)\right)dt
\end{aligned}$$

$$+t^{\alpha-1}\int_0^1 g_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)dt.$$

It follows that

$$\begin{aligned}
&D_{0^+}^{\gamma_2}x_1(t)\\
&=\int_0^t \frac{(t-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)f_2\left(s,x(s),D_{0^+}^{\gamma_2}x(s)\right)ds\\
&-\int_0^t \frac{(t-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)s^{1-\beta}ds\times J_2\\
&-t^{\alpha+\beta-\gamma_2-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)}\\
&\quad\times\frac{\int_0^1(1-t)^{\alpha-1}q_2(t)\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)f_2\left(s,x(s),D_{0^+}^{\gamma_2}x(s)\right)dsdt}{\int_0^1(1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds}\\
&+t^{\alpha+\beta-\gamma_2-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)}\frac{\int_0^1(1-t)^{\alpha-1}q_2(t)\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)s^{1-\beta}dsdt}{\int_0^1(1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds}\times J_2\\
&-t^{\alpha+\beta-\gamma_2-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)}\frac{\int_0^1(1-s)^{\alpha-1}q_2(s)s^{\alpha-1}ds}{\int_0^1(1-s)^{\alpha-1}q_2(s)s^{\alpha+\beta-1}ds}\int_0^1 g_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)dt\\
&+t^{\alpha-\gamma_2-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\int_0^1 g_2\left(t,x(t),D_{0^+}^{\gamma_2}x(t)\right)dt.
\end{aligned}$$

To complete this step, we must prove that $K_P(I-Q)N(\overline{\Omega})$ is bounded and equi-continuous on each subinterval $[e,f]\subseteq(0,1]$ and equi-convergent at $t=0$.

For easily reading, we give the following estimation:

$$|J_2|\leqslant\frac{M_r[1+\eta^{1+l_2}](\alpha+\beta+l_2,k_2+1)+\int_0^1\phi_r(t)dt+\left|1-\frac{1}{\eta^\beta}\right|\int_0^1\phi_r(t)dt}{\left|\int_0^1 G(\eta,w)q_2(w)w^{1-\beta}dw\right|}=:\overline{M}_r.$$

Firstly, use (47), we can prove that both $t\to t^{1-\alpha}|x_1(t)|$ and $t\to t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}|x_1(t)|$ are bounded on $[0,1]$. So $K_P(I-Q)N(\overline{\Omega})$ is bounded.

Second, for each $[e,f]\subseteq(0,1]$, and $t_1,t_2\in[e,f]$ with $t_2>t_1$, we can prove that

$$\begin{aligned}
&|t_1^{1-\alpha}x_1(t_1)-t_2^{1-\alpha}x_1(t_2)|\\
&\leqslant\Bigg|t_1^{1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)f_2\left(s,x(s),D_{0^+}^{\gamma_2}x(s)\right)ds\\
&\qquad-t_2^{1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)f_2\left(s,x(s),D_{0^+}^{\gamma_2}x(s)\right)ds\Bigg|
\end{aligned}$$

$$+\left|t_1^{1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)s^{1-\beta}ds\right.$$
$$\left.-t_2^{1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)s^{1-\beta}ds\right|\overline{M}_r$$
$$+|t_1^{\beta}-t_2^{\beta}|\left[Q_0M_r+Q_1\overline{M}_r+Q\int_0^1\phi_r(t)dt\right],$$

and

$$|t_1^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_1)-t_2^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_2)|$$
$$\leqslant\left|t_1^{1+\gamma_1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)f_2\left(s,x(s),D_{0^+}^{\gamma_2}x(s)\right)ds\right.$$
$$\left.-t_2^{1+\gamma_1-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)f_2\left(s,x(s),D_{0^+}^{\gamma_2}x(s)\right)ds\right|$$
$$+\left|t_1^{1+\gamma_1-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)s^{1-\beta}ds\right.$$
$$\left.-t_2^{1+\gamma_1-\alpha}\int_0^{t_2}\frac{(t_1-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)s^{1-\beta}ds\right|\overline{M}_r$$
$$+|t_1^{\beta}-t_2^{\beta}|\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)}\left[Q_0M_r+Q_1\overline{M}_r+Q\int_0^1\phi_r(t)dt\right].$$

Similarly to the proof of Substep (iv3) in the proof of Lemma 3.1, we can prove that

$$|t_1^{1-\alpha}x_1(t_1)-t_2^{1-\alpha}x_1(t_2)|\to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t_1\to t_2,$$

$$|t_1^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_1)-t_2^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t_2)|\to 0 \text{ uniformly as } t_1\to t_2.$$

So $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f]\subseteq(0,1]$.

Third, we can prove that

$$\left|t^{1-\alpha}x_1(t)-\int_0^1 g_2(t,x(t),D_{0^+}^{\gamma_2}x(t))dt\right|\to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t\to 0.$$

Similarly we can show that

$$\left|t^{1+\gamma_1-\alpha}D_{0^+}^{\gamma_1}x_1(t)-\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_2)}\int_0^1 g_2(t,x(t),D_{0^+}^{\gamma_2}x(t))dt\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}\right|$$
$$\to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t\to 0.$$

Hence $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent at $t=0$. Then N is $L-$compact. The proofs are completed.

Suppose that

(G5) there exist nonnegative functions $\phi_g, \phi_h \in L^1(0,1)$ and bi-nondecreasing functions $\prod_f, \prod_g : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} \left| f_2\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_2-1}v\right)\right| &\leqslant \prod\nolimits_f(|u|,|v|), \\ \left| g_2\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_2-1}v\right)\right| &\leqslant \phi_g(t)\prod\nolimits_g(|u|,|v|) \end{aligned} \tag{52}$$

hold for all $(u,v) \in \mathbb{R}^2$, $t \in (0,1)$.

(G6) there exists a constant $M > 0$ such that $t^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x(t)| > M$ for all $t \in (\eta,1)$ implies that

$$\int_0^1 \left[G(\eta,t)q_2(t)f_2\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right) - h_2\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right) + \left(1 - \frac{1}{\eta^\beta}\right) g_2\left(t, x(t), D_{0^+}^{\gamma_1}x(t)\right)\right] dt \neq 0.$$

(G7) there exists a constant $M_0 > 0$ such that

$$\begin{aligned} c\int_0^1 \Bigg[G(\eta,t)q_2(t)f_2&\left(t, ct^{\alpha+\beta-1}, \frac{c\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_1)}t^{\alpha+\beta-\gamma_1-1}\right) \\ &- h_2\left(t, ct^{\alpha+\beta-1}, \frac{c\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_1)}t^{\alpha+\beta-\gamma_1-1}\right) \\ &+ \left(1-\frac{1}{\eta^\beta}\right) g_2\left(t, ct^{\alpha+\beta-1}, \frac{c\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_1)}t^{\alpha+\beta-\gamma_1-1}\right)\Bigg] dt > 0 \end{aligned} \tag{53}$$

holds for all $|c| > M_0$ or

$$\begin{aligned} c\int_0^1 \Bigg[G(\eta,t)q_2(t)f_2&\left(t, ct^{\alpha+\beta-1}, \frac{c\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_1)}t^{\alpha+\beta-\gamma_1-1}\right) \\ &- h_2\left(t, ct^{\alpha+\beta-1}, \frac{c\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_1)}t^{\alpha+\beta-\gamma_1-1}\right) \\ &+ \left(1-\frac{1}{\eta^\beta}\right) g_2\left(t, ct^{\alpha+\beta-1}, \frac{c\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_1)}t^{\alpha+\beta-\gamma_1-1}\right)\Bigg] dt < 0. \end{aligned} \tag{54}$$

holds for all $|c| > M_0$.

THEOREM 4.1. *Suppose that $(G1)$-$(G7)$ hold. Then BVP(11) has at least one solution if*

$$A_0 r_f + B_0 r_g < 1, \tag{55}$$

where

$$A_0 = \max\left\{ \frac{\mathbf{B}(\alpha+\beta+l_2, k_2+1)}{\Gamma(\alpha+\beta)} + \frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2, k_2+1)}{\eta^\beta\Gamma(\alpha+\beta)} \right.$$

$$, \frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\Gamma(\alpha+\beta-\gamma_2)}+\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\eta^\beta\Gamma(\alpha+\beta-\gamma_2)}\Big\},$$

$$B_0=\max\Big\{1+\frac{\Gamma(\alpha)}{\eta^\beta|\Gamma(\alpha-\gamma_2)|}\frac{\Gamma(\alpha+\beta-\gamma_2)}{\Gamma(\alpha+\beta)}, \frac{\Gamma(\alpha)}{\eta^\beta|\Gamma(\alpha-\gamma_2)|}+\frac{|\Gamma(\alpha+\beta-1)|}{|\Gamma(\alpha+\beta-\gamma_2-1)|}\Big\}\int_0^1\phi_g(s)ds,$$

$$r_f=\lim_{v\to+\infty}\frac{\Pi_f(v,v)}{v},\ \ r_g=\lim_{v\to+\infty}\frac{\Pi_g(v,v)}{v}.$$

Proof. Let E, Z L and N be defined above. By (G1)-(G4), from Lemma 4.1, L be a Fredholm operator of index zero and N be $L-$compact on each open nonempty set Ω centered at zero. We seek fixed point of the operator equation $L(x)=N(x)$. To apply Lemma 2.2, we should define an open bounded subset Ω of E centered at zero such that (i), (ii) and (iii) in Lemma 2.2 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1=\{x\in E\cap D(L)\setminus \mathrm{Ker}L,\ L(x)=\lambda N(x)$ for some $\lambda\in(0,1)\}$. We prove that Ω_1 is bounded.

In fact, if Ω_1 is unbounded, then there exists two sequences $\{x_n\in E\bigcap D(L)\setminus \mathrm{Ker}L\}$ and $\{\lambda_n\in[0,1]\}$ such that $L(x_n)=\lambda_nN(x_n)$ and $N(x_n)\in \mathrm{Im}L$ and $||x_n||\to\infty$ as $n\to\infty$. Then

$$\begin{cases}\frac{D_{0^+}^\beta[D_{0^+}^\alpha x_n(t)]}{q_2(t)}=\lambda_nq_2(s)f_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right),\ \ t\in(0,1),\\ \lim\limits_{t\to0}t^{1-\alpha}x_n(t)=\lambda_n\int_0^1g_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right)dt,\\ x_n(1)-\frac{1}{\eta^{\alpha+\beta-1}}x_n(t)=\lambda_n\int_0^1h_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right)dt.\end{cases}$$

and

$$\int_0^1\Big[G(\eta,t)q_2(s)f_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right)-h_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right)+\left(1-\frac{1}{\eta^\beta}\right)g_2\left(t,x_n(t),D_{0^+}^{\gamma_2}x_n(t)\right)\Big]dt=0. \quad (56)$$

Then there exists number $\Upsilon\in\mathbb{R}$ such that

$$x_n(t)=\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)f_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds+\Upsilon t^{\alpha+\beta-1}+t^{\alpha-1}\int_0^1g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds,$$

and

$$D_{0^+}^{\gamma_2}x_n(t)=\int_0^t\frac{(t-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)f_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds \\ +\Upsilon t^{\alpha+\beta-\gamma_2-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)} \\ +t^{\alpha-\gamma_2-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_2)}\int_0^1 g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds.$$

It follows from (G6) that there exists $t_0\in(\eta,1)$ such that $|t_0^{1+\gamma_2-\alpha}D_{0^+}^{\gamma_2}x_n(t_0)|\leqslant M$. Then

$$\begin{aligned}
&|\Upsilon|t_0^{\beta}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)}\\
&=\Bigg|t_0^{1+\gamma_2-\alpha}D_{0^+}^{\gamma_2}x_n(t_0)\\
&\quad-t_0^{1+\gamma_2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)f_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds\\
&\quad+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_2)}\int_0^1 g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))ds\Bigg|\\
&\leqslant M+t_0^{1+\gamma_2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)|f_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))|ds\\
&\quad+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1|g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))|ds\\
&\leqslant M+t_0^{1+\gamma_2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}s^{k_2}(1-s)^{l_2}\Pi_f(s^{1-\alpha}|x_n(s)|,s^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(s)|)ds\\
&\quad+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1\phi_g(s)\Pi_g(s^{1-\alpha}|x_n(s)|,s^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(s)|)ds\\
&\leqslant M+t_0^{1+\gamma_2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}s^{k_2}(t_0-s)^{l_2}ds\Pi_f(||x_n||,||x_n||)\\
&\quad+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1\phi_g(s)ds\Pi_g(||x_n||,||x_n||)\\
&=M+t_0^{1+\gamma_2-\alpha}t_0^{\alpha+\beta+l_2-\gamma_2+k_2}\int_0^1\frac{(1-w)^{\alpha+\beta+l_2-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}w^{k_2}dw\Pi_f(||x_n||,||x_n||)\\
&\quad+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1\phi_g(s)ds\Pi_g(||x_n||,||x_n||)\\
&\leqslant M+\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\Gamma(\alpha+\beta-\gamma_2)}\Pi_f(||x_n||,||x_n||)\\
&\quad+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1\phi_g(s)ds\Pi_g(||x_n||,||x_n||)
\end{aligned}$$

Then we have

$$
\begin{aligned}
t^{1-\alpha}|x_n(t)| &\leqslant t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}q_2(s)|f_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))|ds \\
&\quad +|\Upsilon|t^{\beta}+\int_0^1 |g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))|ds \\
&\leqslant t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}s^{k_2}(1-s)^{l_2}\Pi_f(s^{1-\alpha}|x_n(s)|,s^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(s)|)ds \\
&\quad +|\Upsilon|t_0^{\beta}\frac{t^{\beta}}{t_0^{\beta}}+\int_0^1 \Pi_g(s)\Pi_g(s^{1-\alpha}|x_n(s)|,s^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(s)|)ds \\
&\leqslant \frac{\mathbf{B}(\alpha+\beta+l_2,k_2+1)}{\Gamma(\alpha+\beta)}\Pi_f(||x_n||,||x_n||) \\
&\quad +\eta^{-\beta}\left(M+\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\Gamma(\alpha+\beta-\gamma_2)}\Pi_f(||x_n||,||x_n||)\right. \\
&\qquad \left.+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||)\right)\frac{\Gamma(\alpha+\beta-\gamma_2)}{\Gamma(\alpha+\beta)} \\
&\quad +\int_0^1 \Pi_g(s)ds\Pi_g(||x_n||,||x_n||).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
&t^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(t)| \\
&\quad\leqslant t^{1+\gamma_2-\alpha}\int_0^t \frac{(t-s)^{\alpha+\beta-\gamma_2-1}}{\Gamma(\alpha+\beta-\gamma_2)}q_2(s)|f_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))|ds \\
&\qquad +|\Upsilon|t_0^{\beta}\frac{t^{\beta}}{t_0^{\beta}}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta-\gamma_2)}+\frac{|\Gamma(\alpha+\beta-1)|}{|\Gamma(\alpha+\beta-\gamma_2-1)|}\int_0^1 |g_2(s,x_n(s),D_{0^+}^{\gamma_2}x_n(s))|ds \\
&\quad\leqslant \frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\Gamma(\alpha+\beta-\gamma_2)}\Pi_f(||x_n||,||x_n||) \\
&\qquad +\frac{M+\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\Gamma(\alpha+\beta-\gamma_2)}\Pi_f(||x_n||,||x_n||)+\frac{\Gamma(\alpha)}{|\Gamma(\alpha-\gamma_2)|}\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||)}{\eta^{\beta}} \\
&\qquad +\frac{|\Gamma(\alpha+\beta-1)|}{|\Gamma(\alpha+\beta-\gamma_2-1)|}\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sup_{t\in(0,1]} t^{1-\alpha}|x_n(t)| &\leqslant \frac{M}{\eta^{\beta}}\frac{\Gamma(\alpha+\beta-\gamma_2)}{\Gamma(\alpha+\beta)} \\
&\quad +\left[\frac{\mathbf{B}(\alpha+\beta+l_2,k_2+1)}{\Gamma(\alpha+\beta)}+\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\eta^{\beta}\Gamma(\alpha+\beta)}\right]\Pi_f(||x_n||,||x_n||) \\
&\qquad +\left[1+\frac{\Gamma(\alpha)}{\eta^{\beta}|\Gamma(\alpha-\gamma_2)|}\frac{\Gamma(\alpha+\beta-\gamma_2)}{\Gamma(\alpha+\beta)}\right]\int_0^1 \Pi_g(s)ds\Pi_g(||x_n||,||x_n||),
\end{aligned}
$$

and

$$\sup_{t\in(0,1]} t^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(t)| \leqslant \frac{M}{\eta^\beta}$$
$$+\left[\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\Gamma(\alpha+\beta-\gamma_2)}+\frac{\mathbf{B}(\alpha+\beta+l_2-\gamma_2,k_2+1)}{\eta^\beta\Gamma(\alpha+\beta-\gamma_2)}\right]\Pi_f(||x_n||,||x_n||)$$
$$+\left[\frac{\Gamma(\alpha)}{\eta^\beta|\Gamma(\alpha-\gamma_2)|}+\frac{|\Gamma(\alpha+\beta-1)|}{|\Gamma(\alpha+\beta-\gamma_2-1)|}\right]\int_0^1\phi_g(s)ds\Pi_g(||x_n||,||x_n||).$$

Then

$$||x_n|| \leqslant \frac{M}{\eta^\beta}\frac{\Gamma(\alpha+\beta-\gamma_2)}{\Gamma(\alpha+\beta)}+\frac{M}{\eta^\beta}+A_0\Pi_f(||x_n||,||x_n||)+B_0\Pi_g(||x_n||,||x_n||). \tag{57}$$

It follows that

$$1 \leqslant \frac{\frac{M}{\eta^\beta}\frac{\Gamma(\alpha+\beta-\gamma_2)}{\Gamma(\alpha+\beta)}+\frac{M}{\eta^\beta}}{||x_n||}+A_0\frac{\Pi_f(||x_n||,||x_n||)}{||x_n||}+B_0\frac{\Pi_g(||x_n||,||x_n||)}{||x_n||}.$$

From $||x_n|| \to \infty$ as $n\to\infty$, let $n\to\infty$. It follows from above inequality that $1 \leqslant A_0r_f+B_0r_g$, a contradiction to (55). It follows that Ω_1 is bounded.

The remainder of the proof is similar to that of the proof of Theorem 3.1 and is omitted.

THEOREM 4.2. *Suppose that* $(G1)$-$(G4)$ *and* $(G6)$, $(G7)$ *and*

(G5)′ *there exist nonnegative functions* $\phi_g,\phi_h \in L^1(0,1)$ *and nonnegative numbers* A_f,B_f,C_f,A_g,B_g,C_g *and* A_h,B_h,C_h *such that*

$$\begin{aligned}&\left|f_2\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_2-1}y\right)\right| \leqslant C_f+B_f|x|+A_f|y|,\\ &\left|g_2\left(t,t^{\alpha-1}x,t^{\alpha-\gamma_2-1}y\right)\right| \leqslant \phi_g(t)[C_g+B_g|x|+A_g|y|]\end{aligned} \tag{58}$$

hold for all $(x,y)\in\mathbb{R}^2$, $t\in(0,1)$.

Then BVP(11) has at least one solution if

$$A_0(B_f+A_f)+B_0(B_g+A_g)<1, \text{ where } A_0,B_0 \text{ are defined in Theorem 4.1.} \tag{59}$$

Proof. Let $\Pi_f(u,v)=C_f+B_fu+A_fv$ and $\Pi_g(u,v)=C_g+B_gu+A_gv$. The proof is similar to that of the proof of Theorem 3.2 and is omitted.

THEOREM 4.3. *Suppose that* $(G1)$-$(G4)$ *and* $(H5)'$ *hold and*

$$\lim_{v\to+\infty}\inf_{t\in(\eta,1),u\in\mathbb{R}}\left[G(\eta,t)q_2(t)f_2\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_2-1}v\right)\right.$$
$$\left.-h_2\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_2-1}v\right)+g_2\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_2-1}v\right)\right]>0,$$

and

$$\lim_{v\to-\infty}\sup_{t\in(\eta,1),u\in\mathbb{R}}\Big[G(\eta,t)q_2(t)f_2\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_2-1}v\right)$$
$$-h_2\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_2-1}v\right)+g_2\left(t,t^{\alpha-1}u,t^{\alpha-\gamma_2-1}v\right)\Big]<0.$$

Then BVP(11) has at least one solution if (59) holds.

Proof. We need to proof that (G6) and (G7) in Theorem 4.2 hold. The proof is similar to that of the proof of Theorem 3.3. Then the proof follows from Theorem 4.2.

REMARK 4.3. We can give examples for the functions f_2, g_2 and $h_2 : (0,1)\times\mathbb{R}^2\to\mathbb{R}$ which satisfy the assumed hypotheses (G1)-(G3), (G5)-(G7). The details are omitted.

5. Solvability of BVP(12)

Suppose that

(I1) $f_3 : [0,1]\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items:

- $t\to f_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)$ is continuous on $[0,1]$ for each $(x,y)\in\mathbb{R}^2$,
- $(x,y)\to f_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t\in[0,1]$,
- for each $r>0$, there exists a nonnegative $M_r\geqslant 0$ such that

$$\left|f_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)\right|\leqslant M_r, t\in(0,1),|x|,|y|\leqslant r.$$

(I2) $g_3 : (0,1)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items:

- $t\to g_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)$ is continuous on $(0,1)$ for each $(x,y)\in\mathbb{R}^2$,
- $(x,y)\to g_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t\in(0,1)$,
- for each $r>0$, there exists a nonnegative function $\phi_r\in L^1(0,1)$ such that

$$\left|g_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)\right|\leqslant\phi_r(t),\ \ t\in(0,1),|x|,|y|\leqslant r.$$

(I3) $h_3 : (0,1)\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following items:

- $t\to h_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)$ is continuous on $(0,1)$ for each $(x,y)\in\mathbb{R}^2$,
- $(x,y)\to h_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t\in(0,1)$,
- for each $r>0$, there exists a nonnegative function $\psi_r\in L^1(0,1)$ such that

$$\left|h_3\left(t,t^{\delta-2}x,t^{\delta-\gamma_3-2}y\right)\right|\leqslant\psi_r(t),\ \ t\in(0,1),|x|,|y|\leqslant r.$$

(I4) q_3 satisfies that there exist numbers $k_1>-\alpha$ and $l_1\in(-\beta,0)$ with $2+k_1+l_1>0$ such that $q_3(t)\leqslant t^{k_1}(1-t)^{l_1}$ for all $t\in(0,1)$ and $q(t)\not\equiv 0$ on $(0,1)$.

Let $C(0,1]$ or $C[0,1]$ be the set of all continuous functions on $(0,1]$ or $[0,1]$. We use the Banach spaces

$$E = \Big\{x \in C(0,1]:\ D_{0^+}^{\gamma_3}x \in C(0,1], \text{ there exist the limits } \lim_{t\to 0} t^{2-\delta}x(t),\ \lim_{t\to 0} t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x(t)\Big\}$$

with the norm

$$||x||_\infty = \max\left\{\sup_{t\in(0,1]} t^{2-\alpha}|x(t)|,\ \lim_{t\in(0,1]} t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x(t)\right\},\ \ x\in E,$$

and

$$Z = \left\{(u,a,b)\in C[0,1]\times\mathbb{R}^2\right\}$$

with the norm

$$||(u,a,b)|| = \max\left\{\sup_{t\in[0,1]} |u(t)|,|a|,|b|\right\} \text{ for } (u,a,b)\in Z.$$

Define

$$D(L) = \left\{x\in E : t\to \frac{D_{0^+}^{\delta}x(t)}{q_3(t)} \in C[0,1]\right\}$$

and $L: E\cap D(L)\to Z$ by

$$L(x)(t) = \left(\frac{D_{0^+}^{\delta}x(t)}{q_3(t)}, \lim_{t\to 0} t^{2-\delta}x(t) - x(1),\ \lim_{t\to 0} t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x(t) - D_{0^+}^{\gamma_3}x(1)\right)$$

for $x\in E$. Define $N: E\to Z$ by

$$N(x)(t) = \left(f_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right), \int_0^1 g_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt, \int_0^1 h_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt\right)$$

for $x\in E$. Then BVP(12) can be written as $L(x) = N(x),\ \ x\in E$.

LEMMA 5.1. *Suppose that (I1)-(I4) hold. Then L is a Fredholm operator of index zero and $N:\ X\to Y$ is $L-$compact.*

Proof. First, for $x\in E\bigcap D(L)$, we see that $L(x)\in Z$. To prove that L is a Fredholm operator of index zero, we should do the following three steps.

Step (i) Prove that $\text{Ker}L = \{ct^{\delta-2}\in E,\ c\in\mathbb{R}\}$.

We know that $(x,y)\in\text{Ker}L$ if and only if

$$\begin{cases} \frac{D_{0^+}^{\delta}x(t)}{q_3(t)} = 0, \\ \lim_{t\to 0} t^{2-\delta}x(t) - x(1) = 0, \\ \lim_{t\to 0} t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x(t) - D_{0^+}^{\gamma_3}x(1) = 0. \end{cases}$$

Hence $x\in \mathrm{Ker}L$ if and only if $x(t)=ct^{\delta-2}$ for some $c\in\mathbb{R}$. Thus $\mathrm{Ker}L=\{ct^{\delta-2}\in E,\ c\in\mathbb{R}\}$.

Step (ii) Prove that

$$\mathrm{Im}L=\left\{\ (u,a,b)\in Z,\ \int_0^1\left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}-\frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right]q_3(s)u(s)ds+b-a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}=0\ \right\}.$$

For $(u,a,b)\in Z$, we know that $(u,a,b)\in \mathrm{Im}L$ if and only if there exist $x\in E\bigcap D(L)$ such that

$$\begin{cases}\frac{D_{0+}^{\delta}x(t)}{q_3(t)}=u(t),\\ \lim\limits_{t\to 0}t^{2-\delta}x(t)-x(1)=a,\\ \lim_{t\to 1}t^{2+\gamma_3-\delta}D_{0+}^{\gamma_3}x(t)-D_{0+}^{\gamma_3}x(1)=b.\end{cases}$$

It follows that there exist numbers $\Upsilon,\Psi\in\mathbb{R}$ such that

$$x(t)=\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds+\Upsilon t^{\delta-1}+\Psi t^{\delta-2}\text{ with }\Upsilon\in\mathbb{R}.$$

Since

$$\begin{aligned}t^{2-\delta}&\left|\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds\right|\\ &\leqslant t^{2-\delta}\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}s^{k_3}(1-s)^{l_3}||u||_\infty ds\\ &=t^{2-\delta}t^{\delta+k_3+l_3}\int_0^1\frac{(1-w)^{\delta+l_3-1}}{\Gamma(\delta)}w^{k_3}dw||u||_\infty\to 0\text{ as }t\to 0,\end{aligned}$$

together with $\lim\limits_{t\to 0}t^{2-\delta}x(t)-x(1)=a$, we get that

$$\Psi-\left(\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds+\Upsilon+\Psi\right)=a.$$

Then $\Upsilon=-a-\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds$. So

$$\begin{aligned}D_{0+}^{\gamma_3}x(t)=&\int_0^t\frac{(t-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}q_3(s)u(s)ds\\ &-\left(a+\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds\right)\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}t^{\delta-\gamma_3-1}\\ &+\Psi\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)}t^{\delta-\gamma_3-2}.\end{aligned}$$

It is easy to show that

$$t^{2+\gamma_3-\delta}\left|\int_0^t\frac{(t-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}q_3(s)u(s)ds\right|$$

$$\leqslant t^{2+\gamma_3-\delta}\int_0^t \frac{(t-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}s^{k_3}(1-s)^{l_3}||u||_\infty ds$$
$$= t^{2+\gamma_3-\delta}t^{\delta-\gamma_3+k_3+l_3}\int_0^1 \frac{(1-w)^{\delta-\gamma_3+l_3-1}}{\Gamma(\delta)}w^{k_3}dw||u||_\infty \to 0 \text{ as } t\to 0.$$

From $\lim_{t\to 0} t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x(t)-D_{0^+}^{\gamma_3}x(1)=b$, we get

$$-\left[\int_0^1 \frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}q_3(s)u(s)ds-\left(a+\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds\right)\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\right]=b. \tag{60}$$

Hence $(u,a,b)\in \mathrm{Im}L$ if and only if

$$\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}-\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}\right]q_3(s)u(s)ds+b-a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}=0.$$

Then the proof is completed.

Step (iii) Prove that $\mathrm{Im}L$ is closed in X and $\dim \mathrm{Ker}L=\operatorname{co}\dim \mathrm{Im}L<+\infty$.

From Step (ii), $\mathrm{Im}L$ is closed in Z. It follows from $\mathrm{Ker}L=\{ct^{\delta-2}\in E,\ c\in\mathbb{R}\}$ that $\dim \mathrm{Ker}L=1$. Define the projector $P:\ E\to E$ by

$$P(x)(t)=\frac{\int_0^1(1-s)^{\delta-1}q_3(s)x(s)ds}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}t^{\delta-2} \text{ for } x\in E. \tag{61}$$

It is easy to prove that $\mathrm{Im}\,P\subseteq \mathrm{Ker}\,L$.

For $ct^{\delta-2}\in \mathrm{Ker}L$, choose $x_c(t)=ct^{\delta-2}$. One can show that $x_c\in E\bigcap D(L)$ and

$$P(x_c)(t)=\frac{\int_0^1(1-s)^{\delta-1}q_3(s)cs^{\delta-2}ds}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}t^{\delta-2}=ct^{\delta-2}.$$

So $\mathrm{Im}\,P\supseteq \mathrm{Ker}\,L$. Hence $\mathrm{Im}\,P=\mathrm{Ker}\,L$.

For $x\in E$, we have

$$P(x-P(x))=P\left(x-\frac{\int_0^1(1-s)^{\delta-1}q_3(s)x(s)ds}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}t^{\delta-2}\right)$$
$$=\frac{\int_0^1(1-s)^{\delta-1}q_3(s)\left(x(s)-\frac{\int_0^1(1-s)^{\delta-1}q_3(s)x(s)ds}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}s^{\delta-2}\right)ds}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}t^{\delta-2}=0.$$

So $X=\mathrm{Ker}L+\mathrm{Ker}P$. For $x\in \mathrm{Ker}L\cap\mathrm{Ker}P$, we get $x(t)=0$. Hence $X=\mathrm{Ker}L\oplus\mathrm{Ker}P$.

Define the projector $Q:\ Z\to Z$ by

$$Q(u,a,b)(t)=\left(\frac{\int_0^1\left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}-\frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right]q_3(s)u(s)ds+b-a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}}{\int_0^1\left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}-\frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right]q_3(s)s^{2-\delta}ds}t^{2-\delta},\ 0,\ 0\right) \tag{62}$$

for $(u,a,b)\in Z$.

It is easy to show that $\mathrm{Ker}Q \subseteq \mathrm{Im}L$
For $(u,a,b) \in \mathrm{Im}L$, we have

$$\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)u(s)ds + b - a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} = 0.$$

Then $Q(u,a,b) = (0,0,0)$. It follows that $\mathrm{Im}L \subseteq \mathrm{Ker}Q$. Hence $\mathrm{Im}L = \mathrm{Ker}Q$.
For $(u,a,b) \in Z$, we have

$$\begin{aligned}(u,a,b) - Q(u,a,b) &= (u,a,b) \\ &\quad - \left(\frac{\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)u(s)ds + b - a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}}{\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)s^{2-\delta}ds} t^{2-\delta},\ 0,\ 0\right) \\ &= \left(u - \frac{\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)u(s)ds + b - a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}}{\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)s^{2-\delta}ds} t^{2-\delta},\ a,\ b\right).\end{aligned}$$

It is easy to see that

$$\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)u(s)ds + b - a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} = 0.$$

Hence $(u,a,b) - Q(u,a,b) \in \mathrm{Im}L$. Thus $Y = \mathrm{Im}Q + \mathrm{Im}L$. It is easy to show that $\mathrm{Im}Q \cap \mathrm{Im}L = (0,0,0)$. Hence $Y = \mathrm{Im}Q \oplus \mathrm{Im}L$.

From above discussion, we see that $\dim \mathrm{Ker}L =$ co $\dim \mathrm{Im}L = 1 < +\infty$. So L is a Fredholm operator of index zero.

Finally, we prove that N is $L-$compact. This is divided into three steps.

Step (i) We prove that N is continuous. Let $x_n \in E \cap D(L)$ with $x_n \to x_0$ as $n \to \infty$. We will show that $N(x_n) \to N(x_0)$ as $n \to \infty$.

In fact, there exists $r > 0$ such that $||x_n|| \leqslant r < +\infty$ and

$$\sup_{t\in(0,1]} t^{2-\delta}|x_n(t) - x_0(t)| \to 0, \quad \sup_{t\in(0,1]} t^{2+\gamma_3-\delta}|D_{0^+}^{\gamma_3}x_n(t) - D_{0^+}^{\gamma_3}x_0(t)| \to 0,\ n \to \infty. \tag{63}$$

By

$$\begin{aligned}N(x_n)(t) = \Big(f_3\left(t, x_n(t), D_{0^+}^{\gamma_3}x_n(t)\right), \int_0^1 g_3\left(t, x_n(t), D_{0^+}^{\gamma_3}x_n(t)\right)dt, \\ \int_0^1 h_3\left(t, x_n(t), D_{0^+}^{\gamma_3}x_n(t)\right)dt\Big) \text{ for } x \in E,\end{aligned}$$

similarly to the proof of Substep (iv1) in Lemma 3.1, we get that

$$||N(x_n) - N(x_0)|| \to 0, n \to \infty.$$

It follows that N is continuous.

Let Ω be a bounded open subset of E. We have that there exists $r>0$ such that $||x||\leqslant r$ for all $x\in\overline{\Omega}$. Since (I1)-(I3) hold, then there exists $M_r\geqslant 0$, $\phi_r,\psi_r\in L^1(0,1)$ such that

$$\begin{aligned}&\left|f_3\left(t,x(t),D_{0+}^{\gamma_1}x(t)\right)\right|\leqslant M_r,t\in(0,1),x\in\overline{\Omega},\\&\left|g_3\left(t,x(t),D_{0+}^{\gamma_1}x(t)\right)\right|\leqslant \phi_r(t),t\in(0,1),x\in\overline{\Omega},\\&\left|h_3\left(t,x(t),D_{0+}^{\gamma_1}x(t)\right)\right|\leqslant \psi_r(t),t\in(0,1),x\in\overline{\Omega}.\end{aligned}\tag{64}$$

Substep (iv2) Prove that $QN(\overline{\Omega})$ is bounded.

Denote

$$H(t)=\frac{(1-t)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}-\frac{(1-t)^{\delta-1}}{\Gamma(\delta-\gamma_3)}.$$

One has

$$\begin{aligned}QN(x)(t)=Q\bigg(&f_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right),\int_0^1 g_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)dt,\\&\int_0^1 h_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)dt\bigg)=\left(J_3t^{2-\delta},0,0\right),\end{aligned}\tag{65}$$

where the constant J_3 is defined by

$$\begin{aligned}J_3=&\left(\int_0^1\left[H(t)q_3(t)f_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)+h_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)\right.\right.\\&\left.\left.-g_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\right]dt\right)\\&\times\frac{1}{\int_0^1\left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}-\frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right]q_3(s)s^{2-\delta}ds}.\end{aligned}$$

It is easy to see from (64) that $QN(\overline{\Omega})$ is bounded.

Substep (iv3) Prove that $K_P(I-Q)N:\overline{\Omega}\to E$ is compact, i.e., prove that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact.

Let $P:X\to X$ and $Q:Y\to Y$ be defined by (61) and (62). For $(u,a,b)\in \mathrm{Im}L$, let

$$\begin{aligned}K_P(u,a,b)(t)=&\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds\\&-t^{\delta-1}\left(a+\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds\right)\\&-t^{\delta-2}\frac{\int_0^1(1-t)^{\delta-1}q_3(t)\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)dsdt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\end{aligned}$$

$$+t^{\delta-2}\left(a+\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)u(s)ds\right)$$
$$\times\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}. \qquad (66)$$

for $(u,a,b)\in \mathrm{Im}L$.

One sees $K_P(u,a,b)\in E$ and $K_p(u,a,b)\in \mathrm{Ker}P$. Then $K_P:\ \mathrm{Im}L\to D(L)\cap\mathrm{Ker}P$ is well defined.

Furthermore, for $(u,a,b)\in \mathrm{Im}L$, we have $(LK_P)(u,a,b)(t)=(u,a,b)$. On the other hand, for $x\in \mathrm{Ker}\,P\cap E$, we have $K_PL(x)(t)x(t)$. Then K_P is the inverse of $L:$ $D(L)\bigcap\mathrm{Ker}P\to\mathrm{Im}L$. The isomorphism $\wedge:\ \mathrm{Ker}L\to Y/\mathrm{Im}L$ is given by

$$\wedge(ct^{\delta-2})=(ct^{2-\delta},0,0),\ \ c\in\mathbb{R}.$$

Then

$$\begin{aligned}
&K_p(I-Q)N(x,y)(t)\\
&\quad=: x_1(t)\\
&\quad=K_p(I-Q)\Bigg(f_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right),\int_0^1 g_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt,\\
&\qquad\qquad\int_0^1 h_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt\Bigg)\\
&\quad=K_P\left(f_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right),\int_0^1 g_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt,\int_0^1 h_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt\right)\\
&\qquad-K_P\left(J_3t^{2-\delta},0,0\right)\\
&\quad=\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)ds-\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}ds\times J_3\\
&\qquad-t^{\delta-1}\int_0^1 g_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt\\
&\qquad+t^{\delta-2}\int_0^1 g_3\left(t,x(t),D_{0^+}^{\gamma_3}x(t)\right)dt\,\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\\
&\qquad-t^{\delta-1}\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)ds\\
&\qquad+t^{\delta-1}\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}ds\times J_3\\
&\qquad-t^{\delta-2}\frac{\int_0^1(1-t)^{\delta-1}q_3(t)\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)dsdt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\\
&\qquad+t^{\delta-2}\frac{\int_0^1(1-t)^{\delta-1}q_3(t)\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}dsdt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\times J_3
\end{aligned}$$

$$+t^{\delta-2}\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0+}^{\gamma_3}x(s)\right)ds\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}$$
$$-t^{\delta-2}\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}ds\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\times J_3.$$

It follows that

$$\begin{aligned}
&D_{0+}^{\gamma_2}x_1(t)\\
&=\int_0^t \frac{(t-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}q_3(s)f_3\left(s,x(s),D_{0+}^{\gamma_3}x(s)\right)ds\\
&\quad-\int_0^t \frac{(t-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)}q_3(s)s^{2-\delta}ds\times J_3\\
&\quad-t^{\delta-\gamma_3-1}\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\int_0^1 g_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)dt\\
&\quad+t^{\delta-2}\int_0^1 g_3\left(t,x(t),D_{0+}^{\gamma_3}x(t)\right)dt\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\\
&\quad-t^{\delta-\gamma_3-1}\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0+}^{\gamma_3}x(s)\right)ds\\
&\quad+t^{\delta-\gamma_3-1}\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}ds\times J_3\\
&\quad-t^{\delta-\gamma_3-2}\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)}\\
&\qquad\qquad\times\frac{\int_0^1(1-t)^{\delta-1}q_3(t)\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0+}^{\gamma_3}x(s)\right)dsdt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\\
&\quad+t^{\delta-\gamma_3-2}\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)}\frac{\int_0^1(1-t)^{\delta-1}q_3(t)\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}dsdt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\times J_3\\
&\quad+t^{\delta-\gamma_3-2}\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)}\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0+}^{\gamma_3}x(s)\right)ds\\
&\qquad\qquad\times\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\\
&\quad-t^{\delta-\gamma_3-2}\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)}\int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}ds\\
&\qquad\qquad\times\frac{\int_0^1(1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1(1-s)^{\delta-1}q_3(s)s^{\delta-2}ds}\times J_3.
\end{aligned}$$

To complete this step, we must prove that $K_P(I-Q)N(\overline{\Omega})$ is bounded and equi-continuous on each subinterval $[e,f]\subseteq(0,1]$ and equi-convergent at $t=0$.

For easily reading, we give the following estimation:

$$|J_3| \leqslant \frac{\int_0^1 \left[H(t)q_3(t)M_r + h_3\psi_r(t) + \phi_r(t)\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\right] dt}{\int_0^1 \left[\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right] q_3(s)s^{2-\delta}ds} =: \overline{M}_r.$$

Firstly, use (64), we can prove that both $t \to t^{2-\delta}|x_1(t)|$ and $t \to t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}|x_1(t)|$ are bounded on $[0,1]$. So $K_P(I-Q)N(\overline{\Omega})$ is bounded.

Second, for each $[e,f] \subseteq (0,1]$, and $t_1,t_2 \in [e,f]$ with $t_2 > t_1$, we can prove that

$$|t_1^{2-\delta}x_1(t_1) - t_2^{2-\delta}x_1(t_2)| \to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t_1 \to t_2,$$

$$|t_1^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x_1(t_1) - t_2^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x_1(t_2)| \to 0 \text{ uniformly as } t_1 \to t_2.$$

So $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f] \subseteq (0,1]$.

Third, we can prove that

$$\begin{aligned}
&\Bigg|t^{2-\delta}x_1(t) - \Bigg(- \frac{\int_0^1 (1-t)^{\delta-1}q_3(t)\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)dsdt}{\int_0^1 (1-s)^{\delta-1}q_3(s)s^{\delta-2}ds} \\
&\quad + \frac{\int_0^1 (1-t)^{\delta-1}q_3(t)\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}dsdt}{\int_0^1 (1-s)^{\delta-1}q_3(s)s^{\delta-2}ds} \times J_3 \\
&\quad + \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)ds \frac{\int_0^1 (1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1 (1-s)^{\delta-1}q_3(s)s^{\delta-2}ds} \\
&\quad - \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}ds \frac{\int_0^1 (1-t)^{\delta-1}q_3(t)t^{\delta-1}dt}{\int_0^1 (1-s)^{\delta-1}q_3(s)s^{\delta-2}ds} \times J_3 \Bigg)\Bigg| \\
&\qquad \to 0 \text{ uniformly on } \overline{\Omega} \text{ as } t \to 0.
\end{aligned}$$

Similarly we can show that

$$\begin{aligned}
&\Bigg|t^{2+\gamma_3-\delta}D_{0^+}^{\gamma_3}x_1(t) \\
&\quad - \Bigg(- \frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} \frac{\int_0^1 (1-t)^{\delta-1}q_3(t)\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)dsdt}{\int_0^1 (1-s)^{\delta-1}q_3(s)s^{\delta-2}ds} \\
&\quad + \frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} \frac{\int_0^1 (1-t)^{\delta-1}q_3(t)\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)s^{2-\delta}dsdt}{\int_0^1 (1-s)^{\delta-1}q_3(s)s^{\delta-2}ds} \times J_3 \\
&\quad + \frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3\left(s,x(s),D_{0^+}^{\gamma_3}x(s)\right)ds
\end{aligned}$$

$$\times \frac{\int_0^1 (1-t)^{\delta-1} q_3(t) t^{\delta-1} dt}{\int_0^1 (1-s)^{\delta-1} q_3(s) s^{\delta-2} ds}$$

$$- \frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} \int_0^1 \frac{(1-s)^{\delta-1}}{\Gamma(\delta)} q_3(s) s^{2-\delta} ds \frac{\int_0^1 (1-t)^{\delta-1} q_3(t) t^{\delta-1} dt}{\int_0^1 (1-s)^{\delta-1} q_3(s) s^{\delta-2} ds} \times J_3 \Big) \Big|$$

$\to 0$ uniformly on $\overline{\Omega}$ as $t \to 0$.

Hence $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent at $t=0$. Then N is $L-$compact. The proofs are completed.

Suppose that

(I5) there exist nonnegative functions $\phi_g, \phi_h \in L^1(0,1)$ and bi-nondecreasing functions $\prod_f, \prod_g : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} &\left|f_3\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_2-1}v\right)\right| \leqslant \prod_f(|u|,|v|), \\ &\left|g_3\left(t, t^{\alpha-1}u, t^{\alpha-\gamma_2-1}v\right)\right| \leqslant \phi_g(t) \prod_g(|u|,|v|). \end{aligned} \tag{67}$$

(I6) there exists a constant $M > 0$ such that $t^{2-\delta}|x(t)| > M$ for all $t \in (0,1)$ implies that

$$\int_0^1 \Bigg[\left(\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)} \right) q_3(s) f_3(s, x(s), D_{0^+}^{\gamma_3} x(s)) - a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} g_3(s, x(s), D_{0^+}^{\gamma_3} x(s)) + h_3(s, x(s), D_{0^+}^{\gamma_3}(s)) \Bigg] ds \neq 0.$$

(I7) there exists a constant $M_0 > 0$ such that

$$\begin{aligned} c\int_0^1 \Bigg[\Big(&\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)} \Big) q_3(s) f_3\left(s, cs^{\delta-2}, c\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} s^{\delta-\gamma_3-2}\right) \\ &- a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} g_3\left(s, cs^{\delta-2}, c\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} s^{\delta-\gamma_3-2}\right) \\ &+ h_3\left(s, cs^{\delta-2}, c\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} s^{\delta-\gamma_3-2}\right) \Bigg] dsdt > 0 \end{aligned} \tag{68}$$

holds for all $|c| > M_0$ or

$$\begin{aligned} c\int_0^1 \Bigg[\Big(&\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)} \Big) q_3(s) f_3\left(s, cs^{\delta-2}, c\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} s^{\delta-\gamma_3-2}\right) \\ &- a\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} g_3\left(s, cs^{\delta-2}, c\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} s^{\delta-\gamma_3-2}\right) \\ &+ h_3\left(s, cs^{\delta-2}, c\frac{\Gamma(\delta-1)}{\Gamma(\delta-\gamma_3-1)} s^{\delta-\gamma_3-2}\right) \Bigg] ds < 0, \end{aligned} \tag{69}$$

holds for all $|c| > M_0$.

THEOREM 5.1. *Suppose that* $(I1)$-$(I6)$ *hold. Then BVP(12) has at least one solution if*

$$A_0 r_f + B_0 r_g < 1, \tag{70}$$

where

$$A_0 = \max\left\{4\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}, \frac{\mathbf{B}(\delta-\gamma_3+l_2,k_2+1)}{\Gamma(\delta-\gamma_3)} + 2\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta-\gamma_3)} + \frac{|\Gamma(\delta-1)|}{|\Gamma(\delta-\gamma_3-1)|}\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}\right\},$$

$$B_0 = \max\left\{2, \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} + \frac{|\Gamma(\delta-1)|}{|\Gamma(\delta-\gamma_3-1)|}\right\}\int_0^1 \phi_g(s)ds,$$

$$r_f = \lim_{v\to+\infty}\frac{\Pi_f(v,v)}{v},\ \ r_g = \lim_{v\to+\infty}\frac{\Pi_g(v,v)}{v}.$$

Proof. Let E, Z L and N be defined above. By (I1)-(I6), from Lemma 5.1, L be a Fredholm operator of index zero and N be $L-$compact on each open nonempty set Ω centered at zero. We seek fixed point of the operator equation $L(x) = N(x)$. To apply Lemma 2.2, we should define an open bounded subset Ω of E centered at zero such that (i), (ii) and (iii) in Lemma 2.2 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1 = \{x \in E\cap D(L)\setminus \mathrm{Ker}L,\ L(x) = \lambda N(x) \text{ for some } \lambda\in(0,1)\}$. We prove that Ω_1 is bounded.

In fact, if Ω_1 is unbounded, then there exists two sequences $\{x_n \in E\bigcap D(L)\setminus \mathrm{Ker}L\}$ and $\{\lambda_n\in[0,1]\}$ such that $L(x_n) = \lambda_n N(x_n)$ and $N(x_n)\in \mathrm{Im}L$ and $||x_n||\to\infty$ as $n\to\infty$. Then

$$\begin{cases} \frac{D_{0+}^{\delta}x_n(t)}{q_3(t)} = \lambda_n f_3\left(t,x_n(t),D_{0+}^{\gamma_3}x_n(t)\right), t\in(0,1),\\ \lim\limits_{t\to 0} t^{2-\delta}x_n(t) - x_n(1) = \lambda_n\int_0^1 g_3\left(t,x_n(t),D_{0+}^{\gamma_3}x_n(t)\right)dt,\\ \lim\limits_{t\to 1} t^{2+\gamma_3-\delta}D_{0+}^{\gamma_3}x_n(t) - D_{0+}^{\gamma_3}x_n(1) = \lambda_n\int_0^1 h_3\left(t,x_n(t),D_{0+}^{\gamma_3}x_n(t)\right)dt.\end{cases}$$

Hence there exists a constant $\Upsilon\in\mathbb{R}$ such that

$$\begin{aligned} x_n(t) = &\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3(s,x_n(s),D_{0+}^{\gamma_3}x_n(s))ds\\ &-\left(\int_0^1 g_3\left(t,x_n(t),D_{0+}^{\gamma_3}x_n(t)\right)dt\right.\\ &\left.+\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3(s,x_n(s),D_{0+}^{\gamma_3}x_n(s))ds\right)t^{\delta-1} + \Upsilon t^{\delta-2}.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1 \Bigg[&\left(\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)}\right)q_3(s)f_3(s,x_n(s),D_{0+}^{\gamma_3}x_n(s))\\ &+h_3\left(s,x_n(s),D_{0+}^{\gamma_3}x_n(s)\right) - \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}g_3\left(s,x_n(s),D_{0+}^{\gamma_3}x_n(s)\right)\Bigg]ds = 0.\end{aligned}$$

It follows from (I6) that there exists $t_0 \in (0,1)$ such that $|t_0^{2-\delta}x_n(t_0)| \leqslant M$. By (36), we have

$$t_0^{2-\delta}x_n(t_0) = \lambda_n t_0^{2-\delta}\int_0^{t_0}\frac{(t_0-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))ds \\ -\left(\int_0^1 g_3\left(t,x_n(t),D_{0^+}^{\gamma_3}x_n(t)\right)dt \right.\\ \left. +\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)f_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))ds\right)t+\Upsilon.$$

Then

$$|\Upsilon| \leqslant |t_0^{2-\delta}x_n(t_0)| + t_0^{2-\delta}\int_0^{t_0}\frac{(t_0-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)|f_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))|ds \\ +\int_0^1 |g_3\left(t,x_n(t),D_{0^+}^{\gamma_3}x_n(t)\right)|dt \\ +\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)|f_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))|ds \\ \leqslant M + t_0^{2-\delta}\int_0^{t_0}\frac{(t_0-s)^{\delta-1}}{\Gamma(\delta)}s^{k_2}(1-s)^{l_2}\Pi_f(s^{2-\delta}|x_n(s)|,s^{2+\gamma_3-\delta}|D_{0^+}^{\gamma_3}x_n(s)|)ds \\ +\int_0^1\phi_g(s)\Pi_g(s^{2-\delta}|x_n(s)|,s^{2+\gamma_3-\delta}|D_{0^+}^{\gamma_3}x_n(s)|)ds \\ +\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}s^{k_2}(1-s)^{l_2}\Pi_f(s^{2-\delta}|x_n(s)|,s^{2+\gamma_3-\delta}|D_{0^+}^{\gamma_3}x_n(s)|)ds \\ \leqslant M + 2\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}\Pi_f(||x_n||,||x_n||) + \int_0^1\phi_g(s)ds\Pi_g(||x_n||,||x_n||).$$

Then we have

$$t^{2-\delta}|x_n(t)| \leqslant t^{2-\delta}\int_0^t\frac{(t-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)|f_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))|ds + |\Upsilon| \\ +\left(\int_0^1|g_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))|ds \right.\\ \left. +\int_0^1\frac{(1-s)^{\delta-1}}{\Gamma(\delta)}q_3(s)|f_3(s,x_n(s),D_{0^+}^{\gamma_3}x_n(s))|ds\right) \\ \leqslant M + 4\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}\Pi_f(||x_n||,||x_n||) + 2\int_0^1\phi_g(s)ds\Pi_g(||x_n||,||x_n||).$$

Similarly we get

$$t^{2+\gamma_3-\delta}|D_{0^+}^{\gamma_3}x_n(t)| \\ \leqslant \frac{\mathbf{B}(\delta-\gamma_3+l_2,k_2+1)}{\Gamma(\delta-\gamma_3)}\Pi_f(||x_n||,||x_n||) + |\Upsilon|\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}$$

$$
\begin{aligned}
&+\frac{|\Gamma(\delta-1)|}{|\Gamma(\delta-\gamma_3-1)|}\left(\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||)\right.\\
&\qquad\qquad\left.+\frac{\mathbf{B}(\delta,k_2+1)}{\Gamma(\delta)}\Pi_f(||x_n||,||x_n||)\right)\\
\leqslant\ &\frac{\mathbf{B}(\delta-\gamma_3+l_2,k_2+1)}{\Gamma(\delta-\gamma_3)}\Pi_f(||x_n||,||x_n||)\\
&+\left[M+2\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}\Pi_f(||x_n||,||x_n||)\right.\\
&\qquad\qquad\left.+\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||)\right]\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}\\
&+\frac{|\Gamma(\delta-1)|}{|\Gamma(\delta-\gamma_3-1)|}\left(\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||)+\frac{\mathbf{B}(\delta,k_2+1)}{\Gamma(\delta)}\Pi_f(||x_n||,||x_n||)\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sup_{t\in(0,1]} t^{2-\delta}|x_n(t)| \leqslant M+4\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}\Pi_f(||x_n||,||x_n||)&\\
+2\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||)&,
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{t\in(0,1]} t^{1+\gamma_2-\alpha}|D_{0^+}^{\gamma_2}x_n(t)| \leqslant\ & M\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}+\left[\frac{\mathbf{B}(\delta-\gamma_3+l_2,k_2+1)}{\Gamma(\delta-\gamma_3)}\right.\\
&\left.+2\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta-\gamma_3)}+\frac{|\Gamma(\delta-1)|}{|\Gamma(\delta-\gamma_3-1)|}\frac{\mathbf{B}(\delta+l_2,k_2+1)}{\Gamma(\delta)}\right]\Pi_f(||x_n||,||x_n||)\\
&+\left[\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}+\frac{|\Gamma(\delta-1)|}{|\Gamma(\delta-\gamma_3-1)|}\right]\int_0^1 \phi_g(s)ds\Pi_g(||x_n||,||x_n||).
\end{aligned}
$$

Then

$$
||x_n|| \leqslant M+M\tfrac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}+A_0\Pi_f(||x_n||,||x_n||)+B_0\Pi_g(||x_n||,||x_n||). \tag{71}
$$

It follows that

$$
1 \leqslant \frac{M+M\frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)}}{||x_n||}+A_0\frac{\Pi_f(||x_n||,||x_n||)}{||x_n||}+B_0\frac{\Pi_g(||x_n||,||x_n||)}{||x_n||}.
$$

From $||x_n|| \to \infty$ as $n\to\infty$, let $n\to\infty$. It follows from above inequality that $1 \leqslant A_0r_f+B_0r_g$, a contradiction to (70). It follows that Ω_1 is bounded.

It follows that Ω_1 is bounded.

The remainder of the proof is similar to that of the proof of Theorem 3.1 and is omitted.

THEOREM 5.2. *Suppose that* $(I1)$-$(I4)$ *and* $(I6)$, $(I7)$ *and*

(I5)$'$ *there exist nonnegative functions* $\phi_g \in L^1(0,1)$ *and nonnegative numbers* $A_f, B_f, C_f, A_g, B_g, C_g$ *such that*

$$\begin{aligned} &\left| f_3\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)\right| \leqslant C_f + B_f|x| + A_f|y|, \\ &\left| g_3\left(t, t^{\alpha-1}x, t^{\alpha-\gamma_2-1}y\right)\right| \leqslant \phi_g(t)[C_g + B_g|x| + A_g|y|], \end{aligned} \tag{72}$$

holds for all $(x,y) \in \mathbb{R}^2$, $t \in (0,1)$.

Then BVP(12) has at least one solution if

$$A_0(B_f + A_f) + B_0(B_g + A_g) < 1 \text{ where } A_0, B_0 \text{ is defined in Theorem } 5.1. \tag{73}$$

Proof. The proof is similar to that of the proof of Theorem 3.2 and is omitted.

THEOREM 5.3. *Suppose that* $(I1)$-$(I4)$ *and* $(I5)'$ *hold and*

$$\begin{aligned} \lim_{v\to+\infty} \inf_{t\in(0,1),u\in\mathbb{R}} \Bigg[&\left(\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)} \right) q_3(s) f_3(s, s^{\delta-2}u, s^{\alpha-\gamma_3-2}v) \\ &- \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} g_3(s, s^{\delta-2}u, s^{\alpha-\gamma_3-2}v) + h_3(s, s^{\delta-2}u, s^{\alpha-\gamma_3-2}v) \Bigg] > 0, \end{aligned}$$

and

$$\begin{aligned} \lim_{v\to-\infty} \sup_{t\in(0,1),u\in\mathbb{R}} \Bigg[&\left(\frac{(1-s)^{\delta-\gamma_3-1}}{\Gamma(\delta-\gamma_3)} - \frac{(1-s)^{\delta-1}}{\Gamma(\delta-\gamma_3)} \right) q_3(s) f_3(s, s^{\delta-2}u, s^{\alpha-\gamma_3-2}v) \\ &- \frac{\Gamma(\delta)}{\Gamma(\delta-\gamma_3)} g_3(s, s^{\delta-2}u, s^{\alpha-\gamma_3-2}v) + h_3(s, s^{\delta-2}u, s^{\alpha-\gamma_3-2}v) \Bigg] < 0. \end{aligned}$$

Then BVP(12) has at least one solution if (73) *holds.*

Proof. We need to proof that (I6) and (I7) in Theorem 5.2 hold. The proof is similar to that of the proof of Theorem 3.3. Then the proof follows from Theorem 5.2.

REMARK 5.1. We can give examples for the functions f_2, g_2 and $h_2 : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ which satisfy the assumed hypotheses (I1)-(G7). The details are omitted.

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Yuji Liu and Xiaohui Yang
Department of Mathematics
Guangdong University of Business Studies
Guangzhou 510320
P.R.China
e-mail: liuyuji888@sohu.com