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SOLOW DIFFERENTIAL EQUATIONS ON TIME SCALES – A UNIFIED APPROACH TO CONTINUOUS AND DISCRETE SOLOW GROWTH MODEL

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Abstract. In this paper we reformulate the axioms of the well-known Solow macroeconomic growth model by means of the mathematical calculus on time scales. We derive a system of differential equations on a time scale $\mathbb{T}$ which is a generalization of the classical Solow fundamental differential equation for the continuous case as well as its discrete version. We also prove sufficient conditions for the exponential stability of equilibrium points of this system having positive coordinates. Applications of these results to the case of the Cobb-Douglas production function are given.

1. Introduction

Robert Merton Solow published in 1956 and in 1957 (see ([23], [22]) his pioneering works in the field of economic growth theory. He derived a differential equation, called now the Solow fundamental differential equation of growth, for the economic variables $v(t) = \frac{K(t)}{L(t)}$ (the stock capital per worker) and $k(t) = \frac{K(t)}{E(t)L(t)}$, respectively, where $K(t)$ is the capital, $E(t)$ is a measure of the technological progress and $Y(t) = f(K(t), L(t))$, and $Y(t) = f(K(t), E(t)L(t))$, respectively, is the production function. His papers are very useful even today on the explanation of macroeconomic phenomena. Many papers have been published on this topic till now. Some generalizations of the classical Solow model and their mathematical and economical analysis have been published also recently (see e. g. [2, 5, 6, 7, 8, 9, 10]).

In this paper we suggest new growth economical models of the Solow type and derive a dynamic equation for the above defined economic variables $v(t)$ and $k(t)$, respectively, on a time scale $\mathbb{T}-$ an arbitrary subset $\mathbb{T}$ of the set of real numbers $\mathbb{R}$. These models contain the continuous as well as the discrete model as special cases. We formulate axioms of these models under the assumption that if these axioms are satisfied then the economical variable $v(t)$, $k(t)$ are solutions of a first order differential equations of the Solow type on a given time scale $\mathbb{T}$. The derivative in these first order

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differential equations is the $\Delta-$derivative on the time scale $\mathbb{T}$ which is a generalization of the classical derivative on $\mathbb{R}$ and the difference on the integers $\mathbb{Z}$. This means that the models and the differential equations are formulated and analyzed in the framework of the mathematical analysis on time scales. The mathematical analysis on time scales was founded in the Stefan Hilger's work published as his thesis (Würzburg University 1988; the supervisor Bernd Aulbach) and in 1990 in the paper [12] and in [13]. Fundamentals of the mathematical analysis and differential equations on time scales are contained in the books [3], [4]. Very useful information concerning this theory can be found in the paper [1]. We also prove sufficient conditions for the exponential stability of nontrivial equilibria of these differential equations.

The main problems in the study the Solow growth model on time scales lies in the fact that the basic mathematical calculus on time scales differs from the classical one. These problems are natural because this calculus is a unification of the continuous and discrete cases. It is well known that very simple one-dimensional difference equations have chaotic behavior (see e.g. [11, 14, 18, 20, 21]) and solutions of their continuous analogue are behaving deterministically, i. e. non-chaotically. In the papers [5], [6] the dynamics of a discrete neoclassical one-sector growth model is studied. It is proven there that under some hypotheses a strange attractor can appear for such difference equations. Since this discrete model is a special case of the the Solow model on time scales, we cannot anticipate a deterministic behavior of such model in general case. There are some difficulties even with using the power transformation of the nonlinear Solow differential equations on a time scale with the Cobb-Douglas production function to a linear one like in the classical case. It is impossible to use the chain rule for the derivative of composition of two functions in this case like in the continuous case (see [19, 4]). Since the product rule as well as the quotient rule for functions on time scales differ from the classical ones we had to modified the axioms of the growth model on time scales to obtain a model which coincides with the classical continuous model on the time scale $\mathbb{T} = \mathbb{R}$ and the discrete model on the time scale $\mathbb{T} = \mathbb{Z}$. This means that this growth model is described by a hybrid system of equations. Many economical phenomena have the hybrid continuous-discrete behavior and therefore the analysis on time scales is very useful for their description and analysis. One of the first example of an application of this theory to problems of the mathematical economy can be found in the paper [24], where the classical Keynesian-Cross model is formulated and analyzed in the time scales setting.

2. Preliminaries about time scales analysis

In this section we recall some basic definitions and results concerning the calculus on time scales and differential equations on time scales, necessary for introducing the definition of our version of the Solow model and for the derivation of a differential equations describing its dynamic and for an analysis of its stability properties. For more details see [1], [3] and [4].

Throughout we use the notation $\mathbb{T}$ for the time scale which is defined as an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The set is supposed to be endowed with the topology induced by the standard topology on $\mathbb{R}$. The operator $\sigma, \rho : \mathbb{T} \to R$

(σ is called the forward jump operator and ρ is called the backward jump operator) are important in the calculus on time scales. They are defined as follows:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{1}$$

If $\sigma(t) > t$, $t \in \mathbb{T}$,we say t is right-scattered. If $\sigma(t) < t$, $t \in \mathbb{T}$, we say t is left-scattered. If $\sigma(t) = t$, $t \in \mathbb{T}$, we say t is right-dense. If $\sigma(t) < t$, $t \in \mathbb{T}$, we say t is left-dense. The point $t \in \mathbb{T}$ is called isolated if it is right-scattered and also left-scattered. The function $\mu\colon \mathbb{T} \to R$, $\mu(t) = \sigma(t) - t$ is called the forward graininess function of the time scale $\mathbb{T}$.

If $\mathbb{T}$ has a right-scattered minimum m then $\mathbb{T}_k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}_k = \mathbb{T}$.

Examples of the time scales and the jump operators σ:

(1) $\mathbb{T} = \mathbb{R};\ \sigma(t) = t,\ \mu(t) \equiv 0;$

(2) $\mathbb{T} = \mathbb{Z};\ \sigma(t) = t+1,\ \mu(t) \equiv 1;$

(3) $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z},\ h > 0\};\ \sigma(t) = t,\ \mu(t) = h;$

(4) $\mathbb{T} = \mathbb{N}_0^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}\};\ \sigma(t) = \sqrt{t^2+1},\ \mu(t) = \sqrt{t^2+1};$

(5) $\mathbb{T} = \overline{q^{\mathbb{N}}} = \{q^n : n \in \mathbb{N}\},\ q > 1;\ \sigma(t) = (q-1)t,\ \mu(t) = \frac{t}{q}.$

DEFINITION 2.1. We say that a function $f\colon \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}^k$ provided

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where} \quad s \to t,\ s \in \mathbb{T} \setminus \{\sigma(t)\}$$

exists. The value $f^{\Delta}(t)$ is called the Δ-derivative of the function f at the point t. The function f is called Δ-differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

The assertions of the following lemma are proved in [4].

LEMMA 2.2. *Assume $f, g\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then the following assertions hold:*

(i) *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(ii) *If t is right-dense, then f is Δ-differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

exists as a finite number. In this case

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iii) *If f is Δ-differentiable at t, then*

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t).$$

(iV) *If $\alpha,\beta\in\mathbb{R}$, then $\alpha f+\beta g\colon\mathbb{T}\to\mathbb{R}$ is differentiable at t with*

$$(\alpha f+\beta g)^{\Delta}(t)=\alpha f^{\Delta}(t)+\beta g^{\Delta}(t).$$

(V) **Product rule:**
$(f\cdot g)\colon\mathbb{T}\to\mathbb{R}$ *is differentiable at t with*

$$(f\cdot g)^{\Delta}(t)=f^{\Delta}(t)g(t)+d(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)).$$

(VI) **Quotient rule:**
If $g(t)g(\sigma(t))\neq 0$, then $\frac{f}{g}$ is Δ-differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}$$

Throughout this work we use the notation f^{σ} for the function $f\circ\sigma$.

3. Integrals and basics of the theory of linear differential equations on time scales

In this section we recall some definitions and basic results from the theory of functions and differential equations on time scales (see [3], [4], [17]).

DEFINITION 3.1. We say that the function $f\colon\mathbb{T}\to\mathbb{R}$ is rd-continuous provided
(a) f is continuous at each right-dense point or maximal point of $\mathbb{T}$.
(b) $\lim_{s\to t^-}f(s)$ exists for each left-dense point $t\in\mathbb{T}$ and this limit is finite.

We denote the set of all $rd-$continuous functions by $\mathbb{C}_{rd}$. The set of all C^1_{Δ}-functions, e.g the set of all Δ-differentiable functions $f\colon\mathbb{T}\to\mathbb{R}$ for which the Δ-derivative $f^{\Delta}(t)$ is continuous, is denoted by C^1_{rd}.

DEFINITION 3.2. A function $F\colon\mathbb{T}\to\mathbb{R}$ is called an antiderivative of $f\colon\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t,s\in\mathbb{T}$ and we define the integral of f by

$$\int_s^t f(\tau)\Delta\tau:=F(t)-F(s).$$

We can define in the natural way a scalar homogeneous differential equation on the time scale $\mathbb{T}$ as the equation

$$y^{\Delta}=p(t)y,\qquad t\in\mathbb{T},\tag{2}$$

where $p \in \mathbb{C}_{rd}$. In the cases $\mathbb{T} = \mathbb{R}$ the general solution of the equation (6) has the form $y(t) = e_p(t,0)c := \exp\left(\int_{t_0}^t p(\tau)d\tau\right)c$, where $t_0 \in \mathbb{R}$, $\exp u := e^u$ and c is constant.The general solution of the equation (6) in the case of more general time scales can also be expressed throughout an exponential function. Such a function can be defined under the assumption that the function p is regressive in the sense of the following definition.

DEFINITION 3.3. We say that a function $g\colon \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)g(t) \neq 0 \qquad \text{for all } t \in \mathbb{T}. \tag{3}$$

The set of all regressive functions is denoted by $\mathscr{R} = \mathscr{R}(\mathbb{T})$.

The exponential function on a time scale $\mathbb{T}$ is defined in the monograph [4] as follows:

DEFINITION 3.4. For $p \in \mathscr{R}(\mathbb{T})$ the exponential function is defined as

$$e_p(t,s) := \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \qquad t,s \in \mathbb{T}, \tag{4}$$

where

$$\xi_{\mu(\tau)}(p(\tau)) = \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)p(\tau)) \qquad \text{for } \mu(\tau) \neq 0,$$
$$\text{and } \xi_{\mu(\tau)}(p(\tau)) = p(\tau) \qquad \text{for } \mu(\tau) = 0,$$

where

$$\xi\colon \mathbb{C} \to \mathbb{Z} := \{z \in \mathbb{C} : -\frac{\pi}{h} < \Im z < \frac{\pi}{h}\},\ \xi_h(u) := \frac{1}{h}\mathrm{Log}(1+uh),\ h > 0,$$

Log is the principal logarithm function and for $h = 0$, $\xi_0(u) \equiv u$ for all $u \in \mathbb{C}$.

By [4, Theorem 2.33] the exponential function $e_p(\cdot,t_0)$ is a solution of the initial value problem

$$y^\Delta = p(t)y,\ y(t_0) = 1, \qquad t \in \mathbb{T}. \tag{5}$$

This means that the general solution of the Δ-differential equation (6) is of the form

$$y(t) = e_p(\cdot,t_0)c, \tag{6}$$

where c is a constant.

Examples of exponential functions:
1. If $\mathbb{T} = \mathbb{R}$ then $\mu(t) \equiv 0$ and

$$\exp_p(t,0) = \exp\left(\int_0^t p(s)ds\right), \quad t \in \mathbb{T}.$$

2. If $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$, then $\mu(t) \equiv h$ and $p(t) \equiv \alpha$, α is a real number then

$$\exp_\alpha(hn,0) = \exp\left(\operatorname{Log}(1+\mu(nh)\alpha)\right)^{\frac{hn}{\mu(hn)}} = (1+h\alpha)^n, \quad n \in \mathbb{Z}. \tag{7}$$

Obviously, This exponential function is the solution of the difference equation

$$\frac{x(t_{i+1}) - x(t_i)}{h} = \alpha x(t_i), \quad t_i = hi, \ i \in \mathbb{Z},$$

under the condition $1 + h\alpha \neq 0$, i. e., the constant function $p(t) \equiv \alpha$ is regressive. From the formula (7) it follows that if $1 + h\alpha < 1$, i. e., if $\alpha < 0$ then $\lim_{n\to\infty} \exp_\alpha(hn) = 0$. This means that the point $x_0 = 0$ is exponentially stable.

4. Nonlinear dynamical systems on time scales and their stability

In this section we shall consider the autonomous differential equation

$$x^\Delta = f(x), \quad c \in D \subset \mathbb{R}, \quad t \in \mathbb{T}, \tag{8}$$

where $f : D \to \mathbb{R}$ is an open set, $\mathbb{T}$ is a time scale on $\mathbb{R}$ with $0 \in \mathbb{T}$ and $\mathbb{T}^+ := \mathbb{T} \cup \mathbb{R}^+ (\mathbb{R}^+ = (0,\infty))$ is unbounded.

Local existence and uniqueness result for nonlinear differential equations on time scales corresponding to the classical Picard-Lindelöf existence theorem can be found in the paper [24]. Its proof is based on the Banach fixed point theorem. The same method is also applied in the paper [16] in the proof of an existence result for Volterra integral equations on time scales. The proof of the Picard-Lindelöf theorem based on the technique of successive approximations with application of the Weierstrass test for the uniform convergence of series can be found in the paper [25]. We assume that all solutions of the equation (8) are global, i.e., they exist on $\mathbb{T}$. A sufficient condition for the global existence of any solution of the equation (8) is proved in [17].

DEFINITION 4.1. We call $\bar{x} \in D$ an equilibrium solutions, or an equilibrium point, of the equation (8) if $f(\bar{x}) = 0$.

DEFINITION 4.2. We say that the equilibrium point $\bar{x}$ of the equation (8) is exponentially stable if there exist $\delta > 0$, $\lambda > 0$, $\beta > 0$ with $-\lambda > 0$ such that if $x(t)$ is a solution of the equation (8) with $\|x(0) - \bar{x}\| < \delta$ then

$$\|x(t) - \bar{x}\| \leqslant \beta \exp_{-\lambda}(t,0)\|x(0) - \bar{x}\|, \qquad t \in \mathbb{T}^+. \tag{9}$$

DEFINITION 4.3. An $n \times n$ matrix valued function $A(t)$, $t \in \mathbb{T}$ is called regressive provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, where $\mu(t)$ is the forward graininess function on $\mathbb{T}$.

Now we can formulate **the Hoffacker-Jackson's stability theorem** (see [15, Theorem 1.1]).

THEOREM 4.4. *Let* $\mathbb{T} \subset \mathbb{R}$ *be a time scale with* $\mathbb{T}^+ := \mathbb{T} \bigcap [0,\infty)$ *unbounded and* $\mu^* := \limsup_{t\to\infty} \mu(t) < \infty$. *Let* $f \in C^1(D,R^n)$ *and* $\bar{x} \in D$ *be an equilibrium point of the differential equation* (8). *Assume that the Jacobi matrix* $A = \frac{\partial f(\bar{x})}{\partial x}$ *is regressive and having eigenvalues all within the Hilger imaginary circle*

$$I_{\mu^*} := \{z \in C_{\mu^*} : \left|z + \frac{1}{\mu^*}\right| = \frac{1}{\mu^*}\}, \text{ if } \mu* \neq 0,$$

where $C_{\mu^*} = \{z \in \mathbb{C} : z \neq -\frac{1}{\mu^*}\}$ *is the Hilger complex plane and*

$$I_0 = \{z_1 + iz_2 \in \mathbb{C} : z_1 < 0\}, \text{ if } \mu* = 0.$$

Then the equilibrium solution $\bar{x}$ *is exponentially stable.*

5. Classical Solow model without technological progress

In the classical Solow model it is assumed that the production function $Y(t)$ is a function of the capital $K(t)$ and the labor $L(t)$, i. e., $Y(t) = f(K(t),L(t))$, where f is a continuously differentiable function. The dynamic equation

$$\dot{k}(t) = s\phi(k) - [\lambda + \delta]k(t), \tag{10}$$

where $\dot{k}(t) := \frac{d\phi(t)}{dt}$, $k(t) := \frac{K(t)}{L(t)}$, $\phi(k) := f(k,1)$ is derived under the following assumptions:

(i) A part $sY(t)$, $0 < s < 1$ of the production $Y(t)$ is invested to the capital and the measure of growth of labor is $\lambda \in \mathbb{R}$, where the dynamic relation for the development of $K(t)$, $L(t)$ and $Y(t)$ are given by the following differential equation:

$$\dot{K}(t) = sY(t) - \delta K(t), \tag{11}$$

where $\delta K(t)$, $\delta > 0$ is the amount of depreciation of the capital.

The dynamic of the labor $L(t)$ is assumed to be described by the differential equation:

$$\dot{L}(t) = \lambda L(t). \tag{12}$$

(ii) The function $f(x,y)$ is assumed to be homogeneous:

$$f(\gamma x, \gamma y) = \gamma f(x,y), \qquad x,y,\gamma \in \mathbb{R}. \tag{13}$$

6. Solow Δ-model without technological progress

We will derive a generalization of the fundamental Solow differential equation on the time scale $\mathbb{T}$ on the basis of the calculus on $\mathbb{T}$ with the Δ-derivative. For this purpose we need to modify the above mentioned Solow axioms as follows:

(a)

$$Y(t) = f(K(t),L(t)), \qquad t \in \mathbb{T}. \tag{14}$$

(b) The change in the capital stock is described by the Δ-differential equation:

$$K^{\Delta}(t) = sY(t) - \delta K(t), t \in \mathbb{T}, \qquad \delta \in (0,1). \tag{15}$$

(c)

$$L^{\Delta}(t) = \frac{\lambda - bL(t)}{1 - \lambda\mu(t)} L(t), \qquad \lambda\mu \neq 1, \tag{16}$$

where $\lambda \in \mathbb{R}$ and $\mu(t)$ is the forward graininess function of the time scale $\mathbb{T}$, $b > 0$. The equation (3.4) means that the labor force grows at rate $\frac{L^{\Delta}(t)}{L(t)} = \frac{\lambda - bL(t)}{1-\lambda\mu(t)}$.
(d)

$$f(\gamma x, \gamma y) = \gamma f(x,y), \qquad x, y, \gamma \in \mathbb{R}. \tag{17}$$

Obviously, if $\mathbb{T} = \mathbb{R}$ then $\mu(t) \equiv 0$ and the axioms (a)-(d) coincide with the axioms of the classical Solow model. We need the following lemma.

LEMMA 6.1.

$$Ł^{\sigma} = \frac{1 - \mu bL}{1 - \lambda\mu} L, \tag{18}$$

i.e.,

$$L(\sigma(t)) = \frac{1 - \mu(t)bL(t)}{1 - \lambda\mu(t)} L(t), \qquad t \in \mathbb{T}, \tag{19}$$

where $L = L(t)$, $\mu = \mu(t)$.

Proof. Using Lemma 2.2 and the axiom (c) we obtain

$$\begin{aligned} L(\sigma(t)) &= L(t) + \mu(t)L^{\Delta}(t) = L(t) + \mu(t)\frac{1}{1-\lambda\mu(t)}[\lambda - bL(t)]L(t) \\ &= \frac{1 - \mu(t)bL(t)}{1 - \lambda\mu(t)} L(t). \end{aligned} \tag{20}$$

THEOREM 6.2. *If the axioms (a)-(d) are satisfied and* $v(t) = \frac{K(t)}{L(t)}$, $t \in \mathbb{T}$ *then* $(v(t),\ L(t))$, *where* $v(t) := \frac{K(t)}{L(t)}$ *is a solution of the following system of* Δ*-differential equations:*

$$v^{\Delta} = \frac{1}{1 - \mu bL}\Big((1-\lambda\mu)s\Phi(v) - [(1-\lambda\mu)\delta + \lambda - bL]v\Big), \tag{21}$$

$$L^{\Delta} = \frac{\lambda - bL}{1 - \lambda\mu} L. \tag{22}$$

where $\Phi(v) = f(v,1)$, *and* $\mu = \mu(t)$ *is the graininess of the time scale* $\mathbb{T}$.

Proof. Using Lemma 2.2, axioms (a)-(d) and Lemma 6.1 we can proceed as follows:

$$\begin{aligned} v^{\Delta} &= \left(\frac{K}{L}\right)^{\Delta} = \frac{K^{\Delta}\cdot L(t) - K\cdot L^{\Delta}}{L\cdot L^{\sigma}} = \frac{K^{\Delta}}{L^{\sigma}} - \frac{L^{\Delta}}{L^{\sigma}}\cdot\frac{K}{L} \\ &= \frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{1}{L}\cdot\left(sf(K,L)-\delta K\right) - \frac{1-\lambda\mu}{1-\mu bL}\frac{1}{L}\cdot\frac{\lambda-bL}{1-\lambda\mu}\cdot L\cdot\frac{K}{L} \\ &= \frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{1}{L}\cdot[sLf\left(\frac{K}{L},1\right)-\delta K] - \frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{1}{L}\cdot\frac{\lambda-bL}{1-\lambda\mu}\cdot L\cdot\frac{K}{L} \\ &= \frac{1-\lambda\mu}{1-\mu bL}\cdot sf\left(\frac{K}{L},1\right) - \delta\frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{K}{L} - \frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{\lambda-bL}{1-\lambda\mu}\cdot\frac{K}{L}. \end{aligned}$$

This means that the couple of functions $(v(t),\ L(t))$ is a solution of the system (21), (22).

7. Application 1: Solow Δ-model without technological progress and with the neoclassical Cobb-Douglas production function

Let us consider the Solow model without technological progress given by the Axioms (a)-(d) with the Cobb-Douglas production function:

$$Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}, \tag{23}$$

where A is a positive constant and $\alpha\in(0,1)$. Then the generalized system of Solow fundamental Δ-differential equations for $(v(t),L(t))$, where $v(t)=\frac{K(t)}{L(t)}$, is

$$v^{\Delta} = \frac{1}{1-\mu bL}\left(s(1-\lambda\mu)Av^{\alpha} - [(1-\lambda\mu)\delta+\lambda-bL]v\right), \qquad t\in\mathbb{T}, \tag{24}$$

$$L^{\Delta} = \frac{\lambda-bL}{1-\lambda\mu}L. \tag{25}$$

The problem of the stability of nontrivial equilibrium point of this equation is more complicated than for the classical case. The classical Solow differential equation. i.e. the differential equation (10) with $\mu(t)\equiv 0$ and with the production function (23) is solvable in a closed form (see, e.g., [8]) by using the power transformation $y=v^{\alpha-1}$. However even in the case $\mu(t)\equiv\mu^*$, where $\mu^*\neq 0$ is a constant, this transformation does not work. The reason is the following theorem concerning the chain rule for functions on time scales proven in the paper [19] (see also [4]).

THEOREM 7.1. *Let $V:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose that $x:\mathbb{T}\to\mathbb{R}$ is a Δ-differentiable function on a time scale $\mathbb{T}$. Then $V\circ x$ is Δ-differentiable on $\mathbb{T}$ and*

$$[V(x(t))]^{\Delta} = \left(V'(x(t)+h\mu(t)x^{\Delta}(t))dh\right)\cdot c^{\Delta}(t), \qquad t\in\mathbb{T}. \tag{26}$$

As a consequence of the Hoffacker-Jackson's stability theorem (see [15, Theorem 1.1]) we obtain the following theorem.

THEOREM 7.2. *Let $\mathbb{T}\subset\mathbb{R}$ be a time scale with $\mathbb{T}^+ := \mathbb{T}\cup[0,\infty)$ unbounded with the constant graininess μ^* and let $\lambda < 1$. Then the following assertions hold:*

(1) *The system* (24), (25) *has the unique equilibrium point*

$$\bar{x} = \left(\bar{v},\bar{L}\right) = \left(\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}, \frac{\lambda}{b}\right)$$

with positive coordinates;

(2) *The Jacobi matrix $\frac{\partial F(\bar{x})}{\partial x}$ of the right-hand side $F(x) = \big(F_1(v,L),F_2(v,L)\big)$, $x=(v,L)$ of the system at the point $\bar{x}$ has the eigenvalues*

$$\lambda_1 = -\delta(1-s\alpha) < 0, \qquad \lambda_2 = -\frac{\lambda}{1-\lambda\mu^*} < 0;$$

(3) *Let the eigenvalues λ_1,λ_2 lie within the Hilger circle I_{μ^*}, i.e.,*

$$-\frac{2}{\mu^*} < \lambda_1 < 0,\ \ -\frac{2}{\mu^*} < \lambda_2 < 0,\ \textit{if}\ \mu* \neq 0$$

and

$$\lambda_1 = -\delta(1-s\alpha) < 0,\ \ \lambda_2 = -\lambda < 0,\ \textit{if}\ \mu* = 0.$$

Then the equilibrium $(\bar{v},\bar{L})$ is exponentially stable.

Proof. The equations for the equilibrium point $\bar{x} = (\bar{v},\bar{L})$ with nonzero coordinates have the form

$$\begin{aligned}(1-\lambda\mu)sAv^{\alpha-1} - [(1-\lambda\mu)\delta+\lambda-bL] = 0,\\ \lambda - bL = 0.\end{aligned}$$

One can easily check that $\bar{x} = (\bar{v},\bar{L})$ given in the assertion (1) is the only solution of these equations. From the structure of the system it follows that the Jacobi matrix $\frac{\partial F(\bar{x})}{\partial x}$ is upper triangular and therefore its eigenvalues are

$$\begin{aligned}\lambda_1 = \frac{\partial F_1(\bar{x})}{\partial v} &= \frac{1}{1-\lambda\mu}\left((1-\lambda\mu)sA\alpha\left(\frac{sA}{\delta}\right)^{\frac{\alpha-1}{1-\alpha}} - (1-\lambda\mu)\delta\right)\\ &= -\delta(1-s\alpha) < 0;\end{aligned}$$

and

$$\lambda_2 = \frac{\partial F_2(\bar{x})}{\partial L} = \frac{1}{1-\lambda\mu}\left(-b.\frac{\lambda}{b}\right) = -\frac{\lambda}{1-\lambda\mu} < 0.$$

Since the eigenvalues λ_1, λ_2 are real they lie within the Hilger circle I_{μ^*} if and only if $-\frac{2}{\mu^*} < \lambda_1 < 0$, $-\frac{2}{\mu^*} < \lambda_2 < 0$. The assertion of the theorem follows from Theorem 4.4.

8. Classical Solow model with technological progress

First let us recall some basic facts concerning the classical Solow models with so called **Harward-neutral technology** where the production function is of the form $Y(t) = f(K(t), E(t)L(t))$ and the dynamics of the measure $E(t)$ of the technological progress is described by the differential equation (29).

Let us consider the third model. If the dynamic equation for the capital $K(t)$ is described by the equation

$$\dot{K}(t) = sY(t) - \delta K(t), \qquad s \in (0,1), \qquad \delta \in \mathbb{R}, \tag{27}$$

and the equation for the labor $L(t)$ is

$$\dot{L}(t) = \lambda L(t), \tag{28}$$

and the equation for the measure of the technological progress $E(t)$ is

$$\dot{E}(t) = \gamma E(t), \qquad \gamma > 0, \tag{29}$$

then under the assumption of the homogeneity of the function $f(K,L)$ the following Solow differential equation for $k(t) = \frac{K(t)}{E(t)L(t)}$ can be derived:

$$\dot{k}(t) = s\phi(k(t)) + [\gamma + \delta + \lambda]k(t). \tag{30}$$

If $\phi(k) = Ak^{\alpha}$, $0 < \alpha < 1$ the one can show by using the transformation $y = k^{\alpha}$ (see e. g. [8] or [26]) that the solution of $k(t)$ with $k(0) = k_0$ of this equation is in the closed form

$$k(t) = \left(\frac{sA}{\gamma+\delta+\lambda} + \left(k_0^{1-\alpha} - \frac{sA}{\gamma+\delta+\lambda}\right)e^{-(1-\alpha)(\gamma+\delta+\lambda)t}\right)^{\frac{1}{1-\alpha}} \tag{31}$$

and

$$\lim_{t\to\infty} k(t) = \bar{k} = \frac{sA}{\gamma+\delta+\lambda} \quad \text{– the equilibrium of the equation (30).}$$

9. Solow Δ-model with technological progress with Harward-neutral technology

Let us formulate axioms of a Δ-Solow model with technological progress under the Harward-neutral technology. We suppose that the axioms (a)-(d) of the Δ-Solow model without technological progress be satisfied and we add to these axioms and axiom for the measure $E(t)$ of the technological progress formulated as follows:

(e)

$$E^{\Delta}(t) = \frac{\gamma - cE(t)}{1 - \gamma\mu(t)}E(t), \qquad t \in \mathbb{T}. \tag{32}$$

LEMMA 9.1.

$$E(\sigma(t)) = \frac{1}{1-\gamma\mu(t)}E(t), \qquad t \in \mathbb{T}. \tag{33}$$

The proof of this lemma is the same as the proof of Lemma 6.1.

THEOREM 9.2. *If the axioms (a)–(e) are satisfied then the couple of functions $\big(k(t), L(t)\big)$, where $k(t) := \frac{K(t)}{E(t)L(t)}$, is a solution of the system of Δ-differential equations*

$$k^\Delta = \frac{1}{(1-\mu cE)(1-\mu bL)} \cdot \Big((1-\gamma\mu)(1-\lambda\mu)s\Phi(k) - \Psi(L,E,\lambda,\gamma,\delta,\mu)k\Big), \tag{34}$$

$$L^\Delta = \frac{\lambda - bL}{1-\lambda\mu}L, \tag{35}$$

$$E^\Delta = \frac{\gamma - cE}{1-\lambda\mu}E, \tag{36}$$

where $\Phi(v) = f(v,1)$ and $\mu = \mu(t)$ is the graininess of the time scale $\mathbb{T}$ and

$$\Psi(L,E,\lambda,\gamma,\delta,\mu) = (1-\gamma\mu)(1-\lambda\mu)\delta + (1-\lambda\mu)(\gamma - cE) + \lambda - bL. \tag{37}$$

Proof. If $k(t) = \frac{K(t)}{E(t)L(t)}$ then using [Lemma 2.2, (V), (VI)] we obtain

$$\begin{aligned}
k^\Delta &= \frac{K^\Delta \cdot E \cdot L - K(E\cdot L)^\Delta}{(E\cdot L)(E\cdot L)^\sigma} = \frac{K^\Delta}{E^\sigma \cdot L^\sigma} - \frac{(E\cdot L)^\Delta}{E^\sigma \cdot L^\sigma}\cdot\frac{K}{E\cdot L} \\
&= \frac{1-\gamma\mu}{1-\mu cE}\cdot\frac{1}{E}\cdot\frac{1-\lambda\mu}{1-\mu bL}\frac{1}{L}\cdot[sf(K,E\cdot L) - \delta K] \\
&\quad - \frac{1-\gamma\mu}{1-\mu cE}\cdot\frac{1}{E}\cdot\frac{1-\lambda\mu}{1-\mu bL}\frac{1}{L}\cdot(E^\Delta\cdot L + E^\sigma\cdot L^\Delta)\cdot\frac{K}{E\cdot L} \\
&= \frac{1-\gamma\mu}{1-\mu cE}\cdot\frac{1-\lambda\mu}{1-\mu bL}\cdot sf\left(\frac{K}{E\cdot L},1\right) \\
&\quad - \frac{1-\gamma\mu}{1-\mu cE}\cdot\frac{1-\lambda\mu}{1-\mu bL}\cdot\delta\cdot\frac{K}{E\cdot L} - \frac{1-\gamma\mu}{1-\mu cE}\cdot\frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{\gamma - cE}{1-\gamma\mu}\frac{K}{E\cdot L} \\
&\quad - \frac{1-\gamma\mu}{1-\mu cE}\cdot\frac{1}{E}\cdot\frac{1-\lambda\mu}{1-\mu bL}\cdot\frac{1}{1-\gamma\mu}\cdot E\cdot\frac{\lambda - bL}{1-\lambda\mu}L\cdot\frac{K}{E\cdot L} \\
&= \frac{1}{(1-\mu cE)(1-\mu bL)}\Big((1-\gamma\mu)(1-\lambda\mu)sf(k,1) \\
&\quad - [(1-\gamma\mu)(1-\lambda\mu)\delta + (1-\gamma\mu)(\gamma - cE) + \lambda - bL]k\Big).
\end{aligned}$$

This equality yields the assertion of Theorem 9.2.

10. Application 2: Solow Δ-model with technological progress and with the neoclassical Cobb-Douglas production function

Let us consider the Δ-model with technological progress given by the axioms (a)-(e) with the Cobb-Douglas production function $Y(t) = AK(t)^{\alpha}L(t)^{1-\alpha}$, $A > 0$, $0 < \alpha < 1$. Then the generalized system of the Solow Δ-differential equations for $(k(t), L(t), E(t))$, where $k(t) = \frac{K(t)}{E(t)\cdot L(t)}$ is

$$k^{\Delta} = \frac{1}{(1-\mu cE)(1-\mu bL)}\Big((1-\gamma\mu)(1-\lambda\mu)sAk^{\alpha} - \Psi(L,E,\lambda,\gamma,\delta,\mu)k\Big), \tag{38}$$

$$L^{\Delta} = \frac{\lambda - bL}{1-\lambda\mu}L, \tag{39}$$

$$E^{\Delta}(t) = \frac{\gamma - cE(t)}{1-\gamma\mu(t)}E(t), \qquad t \in \mathbb{T}. \tag{40}$$

where $\Phi(k) = sAk^{\alpha}$ and the function $\Psi(L,E,\lambda,\gamma,\delta,\mu)$ is defined in Theorem 9.2.

THEOREM 10.1. *Let $\mathbb{T} \subset \mathbb{R}$ be a time scale with $\mathbb{T}^{+} := \mathbb{T} \cap [0,\infty)$ unbounded with the constant graininess μ^* and let $\lambda < 1$, $0 < \gamma < 1$. Then the following assertions hold:*
(1) *The system* (38), (39), (40) *has the unique equilibrium point*

$$\bar{x} = \Big(\bar{v}, \bar{L}, \bar{E}\Big) = \left(\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}, \frac{\lambda}{b}, \frac{\gamma}{c}\right)$$

with positive coordinates;
(2) *The Jacobi matrix $\frac{\partial F(\bar{x})}{\partial x}$ of the right-hand side*

$$G(x) = \Big(G_1(v,L,E), G_2(v,L,E), G_3(v,L,E)\Big), \quad x = (v,L,E),$$

of the system at the point $\bar{x}$ has the eigenvalues

$$\lambda_1 = -\delta(1-s\alpha) < 0, \quad \lambda_2 = -\frac{\lambda}{1-\lambda\mu^*} < 0, \quad \lambda_3 = -\frac{\gamma}{1-\gamma\mu^*} < 0;$$

(3) *Let the eigenvalues λ_1, λ_2, λ_3 lie within the Hilger circle I_{μ^*}, i.e.,*

$$-\frac{2}{\mu^*} < \lambda_1 < 0, \ -\frac{2}{\mu^*} < \lambda_2 < 0, \ -\frac{2}{\mu^*} < \lambda_3 < 0, \ \textit{if } \mu* \neq 0$$

and

$$\lambda_1 = -\delta(1-s\alpha) < 0, \ \lambda_2 = -\lambda < 0, \ \lambda_3 = -\gamma < 0, \ \textit{if } \mu* = 0.$$

Then the equilibrium $\bar{x} = (\bar{v}, \bar{L}, \bar{E})$ is exponentially stable.

Proof. The equations for the equilibrium point $\bar{x} = (\bar{k}, \bar{L}, \bar{E})$ with nonzero coordinates have the form

$$(1-\gamma\mu)(1-\lambda\mu)sA\bar{x}^{\alpha-1} - [(1-\gamma\mu)(1-\lambda\mu)\delta + (1-\lambda\mu)(\gamma - cE) + \lambda - bL] = 0,$$
$$\bar{L} = \frac{\lambda}{b}, \qquad \bar{E} = \frac{\gamma}{c}, \qquad b \neq 0 \qquad c \neq 0.$$

Obviously

$$\bar{x} = (\bar{k}, \bar{L}, \bar{E}) = \left(\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}, -\frac{\lambda}{b}, -\frac{\gamma}{c}\right)$$

and the eigenvalues of the Jacobi matrix $\frac{\partial G}{\partial x}(\bar{x})$ is upper triangular and therefore its eigenvalues are

$$\lambda_1 = s\alpha A\bar{k}^{\alpha-1} - \delta = -\delta(1-s\alpha) < 0,$$
$$\lambda_2 = -\frac{\lambda}{1-\lambda\mu}, \qquad \lambda_3 = -\frac{\gamma}{1-\gamma\mu}.$$

The assertion (3) of Theorem 10.1 follows from Theorem 4.4.

EXAMPLE 1. If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \bigcup_{i=1}^{\infty} a_i$ where $\{a_i\}_{i=1}^{\infty}$ is a strictly increasing sequence then the graininess $\mu(t) \equiv 0$. Therefore if the assumptions of Theorem 4.4 are satisfied with $\mu^* = 0$ then the equilibrium $\bar{x}$ of the system (38), (39), (40) is exponentially stable.

EXAMPLE 2. The graininess of the time scale $\mathbb{T}$ is $\mu(t) \equiv 1$ and therefore if the assumptions of Theorem 4.4 are satisfied with $\mu^* = 1$ then the equilibrium $\bar{x}$ of the system (38), (39), (40) is exponentially stable.

EXAMPLE 3. If $\mathbb{T} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$ then

$$\mu(t) = \begin{cases} 0 & \text{for } 2k \leqslant t < 2k+1 \\ 1 & \text{for } t = 2k+1 \end{cases} \tag{41}$$

Let us write $\mathbb{T}$ as $\mathbb{T} = T_1 \bigcup T_2$, where $T_1 = \bigcup_{k=0}^{\infty}[2k, 2k+1)$ and $T_2 = \bigcup_{k=0}^{\infty}\{2k+1\}$. If $0 < \lambda < 1$, $0 < \gamma < 1$ then one can check that the assumptions of Theorem 4.4 are satisfied for both $\mu^* = 0$ and also for $\mu^* = 1$ and therefore by this theorem the equilibrium point $\bar{x}$ is asymptotically stable on T_1 and also on T_2, i.e., there exist numbers $\delta_i > 0$, $\beta_i > 0$, $\eta_i > 0$, $i = 1,2$ such that if $x(t) = \big(k(t), L(t), E(t)\big)$ is a solution of the system (38), (39), (40) with $\|x(0) - \bar{x}\| < \delta_i$, $i = 1,2$ then

$$\|x(t) - \bar{x}\| \leqslant \beta_i \exp_{-\eta_i}(t,0)\|x(0) - \bar{x}\|, \qquad t \in \mathbb{T}_i, \;\; i = 1,2.$$

We have proved that the system (38), (39), (40) is exponentially stable on the time scale $\mathbb{T}_1$ and on $\mathbb{T}_1$ and therefore this system is asymptotically stable, i.e., for any $\varepsilon > 0$ there is an $\delta > 0$ ($\delta = \min\{\delta_1, \delta_2\} > 0$) such that if $\|x(0) - \bar{x}\| < \delta$ then $\|x(t) - \bar{x}\| \leqslant \varepsilon$ for all $t \in [0, \infty) \cap \mathbb{T}$ and moreover $\lim_{t\to\infty} \|x(t) - \bar{x}\| = 0$.

11. Conclusion

In the present paper we formulate axioms for the system of Solow-like differential equations on time scales and under the assumption that the graininess function is constant we prove stability results for equilibria with positive coordinates. Applications to the model with Cobb-Douglas production function are also given. The problem is much more difficult in the case of more general graininess function and more general nonlinearity. If the equations for the labor and measure of the technological progress are time-dependent then this system is time-dependent in the triangular or cascade form and it can have complex chaotic dynamic. In the papers [6, 7, 10] such type of discrete growth models are studied and results on the existence of attractors and strange attractors, or chaotic attractors, respectively, are obtained. The same problems for such type of equations however on general time scales remain open.

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