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SOME RANDOM FIXED POINT THEOREMS WITH PPF DEPENDENCE FOR WEAKLY CONTRACTIVE RANDOM OPERATOR

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Abstract. In this paper, the fixed point theorem for weakly contraction proved by Rhoades [Nonlinear Anal. 47 (4) (2001), 2683–2693] will be extended for the random case with PPF dependence. By the constructive method, the result is proved and an application of this theorem for random equation is given.

1. Introduction

Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in analysing probabilistic models in the applied sciences. The study of fixed points of random operators forms a central topic in this area and was initialed by the Prague school of probabilist in the 1950s. The theory of random fixed points received futher attention after the appreance of the survey article by Bharucha-Reid [3] in 1976. Since then, a lot of interesting results about random contraction, random expansion, ... etc have appeared. Studied in [1], Bernfeld et al. proved the deterministic fixed point theorem with PPF dependence for contractive mapping where the domain and the range spaces are not the same. In 2012, Dhage in [4] proved the random fixed point theorem with PPF dependence for contractive random operator. Also in [4], the author received some results for the existence of random differential equation's solution. For the case weakly contraction, in [6], Rhoades proved the fixed point theorem for mappings satisfying weakly contractive condition. In the paper [2], Beg and Abbas proved the fixed point theorem for random operator satisfying random weakly contractive condition.

In this paper, we extend the random fixed point with PPF dependence theorem of contractive random operator for weakly contractive random operator. Then, a result for the existence random solution of a random equation is received.

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2. Preliminaries

Let $(\Omega,\mathscr{F})$ be a measurable space ($\mathscr{F}$-sigma algebra) and X be a separable Banach space. A mapping $\xi:\Omega\to X$ is called a X-valued random variable if ξ is $(\mathscr{F},\mathscr{B}(X))$-measurable, where $\mathscr{B}(X)$ denotes the Borel σ-algebra of X.

We recall the concept of random operators (see, e.g. [2, 3, 7]).

DEFINITION 1. Let X,Y be separable Banach spaces.

1. A mapping $T:\Omega\times X\to Y$ is said to be a random operator if for each $x\in X$, the mapping $T(.,\omega)$ is a Y-valued random variable, where $T(.,x)$ denotes the mappings $\omega\mapsto T(\omega,x)$.
2. The random operator $T:\Omega\times X\to Y$ is said to be measurable if the mapping $T:\Omega\times X\to Y$ is $\mathscr{F}\times\mathscr{B}(X)$-measurable.
3. The random operator $T:\Omega\times X\to Y$ is said to be continuous if for each ω the mapping $T(\omega,.)$ is continuous, where $T(\omega,.)$ denotes the mappings $x\mapsto T(\omega,x)$.

THEOREM 1. [5, Theorem 6.1] *Let $T:\Omega\times X\to Y$ be a continuous random operator. Then, the map $(\omega,x)\mapsto T(\omega,x)$ is jointly measurable.*

Now, we look back at some results in the developing process of the weakly contractive problem. In paper [6], Rhoades proved a fixed point theorem for weakly contractive mapping in the following theorem.

THEOREM 2. [6, Theorem 1] *Let (X,d) be a complete metric space and $T:X\to X$ be a weakly contractive self-mapping in the sense that for all $x,y\in X$*

$$d(Tx,Ty)\leqslant d(x,y)-\varphi(d(x,y)) \tag{2.1}$$

where $\varphi:[0,\infty)\to[0,\infty)$ is continuous, nondecreasing function such that $\varphi(t)=0$ if and only if $t=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. Then T has a unique fixed point in X.

It is easy to see that Theorem 2 also includes the Banach contraction principle as a particular case since one can verify that k-contraction mappings are weakly contractive if we choose the function $\varphi(t)=(1-k)t$, $0\leqslant k<1$.

Next, in [2], Beg and Abbas extended the Rhoades result for a weakly contractive random operator. Also in [2], the Mann iteration and Ishikawa iteration converge to the random fixed point of weakly contractive random operator were proved. We have a look at Beg and Abbas theorem.

THEOREM 3. [2, Theorem 5.2] *Let F be a closed and convex subset of a complete separable metric space X, and $T:\Omega\times F\to F$ be a weakly contractive random operator in the sense that for all $\omega\in\Omega$, all $x,y\in X$*

$$d(T(\omega,x),T(\omega,y))\leqslant d(x,y)-\varphi(d(x,y)) \tag{2.2}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then T has a unique fixed point in X. Then T has a random fixed point.

Let $I = [a,b]$ be a closed, bounded interval for some $a,b \in \mathbb{R}$, $a < b$. The set $X_0 = C(I,X)$ is denoted for the set of all continuous X-valued function equipped with the supremum norm $||.||_{X_0}$ defined by

$$\|x\|_{X_0} = \sup_{t\in I} \|x(t)\|_{X_0}. \tag{2.3}$$

For a fixed $c \in I$, we denote

$$\mathscr{R}_c = \{\phi \in X_0 : ||\phi||_{X_0} = ||\phi(c)||_X\}. \tag{2.4}$$

and $\mathscr{R}_c$ is called Razumkhin or Minimal class of function in X_0.

The class $\mathscr{R}_c$ is algebraically closed with respect to difference if for all $\phi, \xi \in \mathscr{R}_c$ then $\phi - \xi \in \mathscr{R}_c$. The class $\mathscr{R}_c$ is topologicaly closed if it is closed w.r.t the topology on X_0 generated by the norm $||.||_{X_0}$

DEFINITION 2. [4] Let $T : \Omega \times X_0 \to X$ be a random operator. A measurable function $\xi^* : \Omega \to X_0$ is call a PPF dependent random fixed point of $T(\omega)$ if

$$T(\omega, \xi^*(\omega)) = \xi^*(c, \omega) \tag{2.5}$$

for some $c \in I$.

DEFINITION 3. A random operator $T : \Omega \times X_0 \to X$ is call random weak contraction if for each $\omega \in \Omega$,

$$\|T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))\|_X \leqslant \|\xi(\omega) - \eta(\omega)\|_{X_0} - \varphi\left(\|\xi(\omega) - \eta(\omega)\|_{X_0}\right) \tag{2.6}$$

for all $\xi(\omega), \eta(\omega) \in X_0$, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

In [4], Dhage proved the following theorem about PPF dependent random fixed point of the random operator satisfying random contractive condition.

THEOREM 4. [4, Theorem 3.1] *Let $(\Omega, \mathscr{A})$ be a measurable space and let X be a separable Banach space. If the random operator $T : \Omega \times X_0 \to X$ is a random contraction, then the following statements hold in X.*
(a) If $\mathscr{R}_c$ is algebraically with respect to difference, then for a given $\xi_0 \in X_0$ and for a given $c \in I$, every sequence $\{\xi_n(\omega)\}$ of measurable function satisfying

$$T(\omega, \xi_n(\omega)) = \xi_{n+1}(c, \omega) \tag{2.7}$$

and $||\xi_n(\omega) - \xi_{n+1}(\omega)||_{X_0} = ||\xi_n(c,\omega) - \xi_{n+1}(c,\omega)||_X$ converges to a PPF dependent random fixed point of the random operator $T(\omega)$, i.e. there is a measurable function $\xi^* : \Omega \to X_0$ such that for each $\omega \in \Omega$,

$$T(\omega, \xi^*(\omega)) = T(\omega)\xi^*(\omega) = \xi^*(\omega).$$

(b) Given $\xi_0, \eta_0 \in X_0$*, let* $\{\xi_n(\omega)\}$ *and* $\eta_n(\omega)$ *be the sequences of itereates of measurable functions corresponding to* ξ_0 *and* η_0 *contructed as in (a). Then,*

$$\left\|\xi_n(\omega) - \eta_n(\omega)\right\|_{X_0} \leqslant \frac{1}{1-\lambda(\omega)}\left[\|\xi_0 - \xi_1(\omega)\|_{X_0} + \|\eta_0 - \eta_1(\omega)\|_{X_0}\right] + \|\xi_0 - \eta_0\|_{X_0}.$$

If, in particular, $\xi_0 = \eta_0$*, and* $\{\xi_n(\omega)\} \neq \{\eta_n(\omega)\}$*, then*

$$\|\xi_n(\omega) - \eta_n(\omega)\|_{X_0} \leqslant \frac{2}{1-\lambda(\omega)}\|\xi_0 - \xi_1(\omega)\|_{X_0}.$$

(c) Finally, if $\mathscr{R}_c$ *is topologicaly closed, then for a given* $\xi_0 \in X_0$*, every sequence* $\{\xi_n(\omega)\}$ *of itereates of* T *contructed as in (a), converges to a unique PPF dependent random fixed point* $\xi^*(\omega)$ *of* $T(\omega)$*, i.e. there is a unique measurable function* $\xi^* : \Omega \to X_0$ *such that* $T(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$ *for all* $\omega \in \Omega$*.*

In this paper, first of all we will extend the results of Dhage [4] and Rhoades [6] for the mappings with weakly contractive condition. Then, some results are obtained.

THEOREM 5. *Let* X *be a separable Banach space and* $T : \Omega \times X_0 \to X$ *is a weakly contractive random operator, then the following statements hold in* X*.*

(a) If $\mathscr{R}_c$ *is algebraically with respect to difference, then for a given* $\xi_0 \in X_0$ *and for a given* $c \in I$*, every sequence* $\{\xi_n(\omega)\}$ *of measurable function satisfying*

$$T(\omega, \xi_n(\omega)) = \xi_{n+1}(c, \omega) \tag{2.8}$$

and $||\xi_n(\omega) - \xi_{n+1}(\omega)||_{X_0} = ||\xi_n(c,\omega) - \xi_{n+1}(c,\omega)||_X$ *converges to a PPF dependent random fixed point of the random operator* $T(\omega)$*, i.e. there is a measurable function* $\xi^* : \Omega \to X_0$ *such that for each* $\omega \in \Omega$*,*

$$T(\omega, \xi^*(\omega)) = T(\omega)\xi^*(\omega) = \xi^*(c, \omega).$$

(b) Assume that there exists

$$\inf_{t>0} \frac{\varphi(t)}{t} = 1 - \lambda > 0 \quad \lambda \in (0,1). \tag{2.9}$$

Given $\xi_0, \eta_0 \in E_0$*, let* $\{\xi_n(\omega)\}$ *and* $\{\eta_n(\omega)\}$ *be the sequences of itereates of measurable functions corresponding to* ξ_0 *and* η_0 *contructed as in (a). Then,*

$$\left\|\xi_n(\omega) - \eta_n(\omega)\right\|_{X_0} \leqslant \frac{1}{1-\lambda(\omega)}\left[\|\xi_0 - \xi_1(\omega)\|_{X_0} + \|\eta_0 - \eta_1(\omega)\|_{X_0}\right] + \|\xi_0 - \eta_0\|_{X_0}.$$

If, in particular, $\xi_0 = \eta_0$*, and* $\{\xi_n(\omega)\} \neq \{\eta_n(\omega)\}$*, then*

$$\|\xi_n(\omega) - \eta_n(\omega)\|_{X_0} \leqslant \frac{2}{1-\lambda(\omega)}\|\xi_0 - \xi_1(\omega)\|_{X_0}.$$

(c) Finally, if $\mathscr{R}_c$ is topologicaly closed, then for a given $\xi_0 \in X_0$, every sequence $\{\xi_n(\omega)\}$ of itereates of T contructed as in (a), converges to a unique PPF dependent random fixed point $\xi^*(\omega)$ of $T(\omega)$, i.e. there is a unique measurable function $\xi^* : \Omega \to X_0$ such that $T(\omega, \xi^*(\omega)) = \xi^*(c, \omega)$ for all $\omega \in \Omega$.

Next, we will prove the existence of a random solution with PPF dependent of the following random equation (2.10). This random equation type was consider in [8].

THEOREM 6. *Let X be a separable Banach space, $T : \Omega \times X_0 \to X$ be a weakly contractive random operator and $S : \Omega \times X \to X$ is a nonexpansive random operator by the meaning $||S(\omega, x) - S(\omega, y)|| \leqslant ||x - y||$ for all $x, y \in X$ and all $\omega \in \Omega$. Assume that $\mathscr{R}_c$ is topologicaly and algebraically closed with respect to difference, then for a given $c \in I$, and for all $t \in (0,1)$, there exists a random solution with PPF dependent of the equation*

$$\xi_t(c, \omega) = tT(\omega, \xi_t(\omega)) + (1-t)S(\omega, \xi_t(c, \omega)) \tag{2.10}$$

for each $\omega \in \Omega$, i.e. for a given $c \in I$, there is a measurable function $\zeta : \Omega \to X_0$ such that

$$\zeta_t(c, \omega) = tT(\omega, \zeta_t(\omega)) + (1-t)S(\omega, \zeta_t(c, \omega)) \tag{2.11}$$

for all $\omega \in \Omega$.

Apparently, as far as we know, the conditional inequality appearing in almost all contractive fixed-point theorems has the same left hand side, that is to ensure the existence of fixed-points using estimates for $\|T(\omega, \xi(\omega)) - T(\omega, \eta(\omega))\|_X$ from above. By an other way, the left hand side condition will be changed and we receive the following theorem.

THEOREM 7. *Let X be a separable Banach space and $T : \Omega \times X_0 \to X$ is a weakly contractive random operator. Assume that T is continuous and satisfies for all random variables $\xi, \eta : \Omega \to X_0$*

$$\begin{aligned} \min\Big\{ \|\xi(c,\omega) - T(\omega, \xi(\omega))\|_X, \|\eta(c,\omega) - T(\omega, \eta(\omega))\|_X, \\ \big\|T\big(\omega, \xi(\omega)\big) - T(\omega, \eta(\omega))\big\|_X \Big\} \\ \leqslant ||\xi(\omega) - \eta(\omega)||_{X_0} - \varphi\big(||\xi(\omega) - \eta(\omega)||_{X_0}\big) \end{aligned} \tag{2.12}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. If $\mathscr{R}_c$ is algebraically with respect to difference, then for a given $\xi_0 \in X_0$ and for a given $c \in I$, every sequence $\{\xi_n(\omega)\}$ of measurable function satisfying

$$T(\omega, \xi_n(\omega)) = \xi_{n+1}(c, \omega) \tag{2.13}$$

and $||\xi_n(\omega) - \xi_{n+1}(\omega)||_{X_0} = ||\xi_n(c,\omega) - \xi_{n+1}(c,\omega)||_X$ converges to a PPF dependent random fixed point of the random operator $T(\omega)$, i.e. there is a measurable function $\xi^* : \Omega \to X_0$ such that for each $\omega \in \Omega$,

$$T(\omega, \xi^*(\omega)) = T(\omega)\xi^*(\omega) = \xi^*(c, \omega).$$

Before closing this section, we note that there is not many resuls where the condition is given in the left hand side like in (2.12).

3. Proofs

This section is devoted to proofs of Theorems 5, 6 and 7.

Proof. [Proof of Theorem 5] (a) Let ξ_0 be arbitrary continuous X-valued function on I. Then, there exists a random variable $\xi_1 : \Omega \to X_0$ such that $T(\omega, \xi_0) = x_1(\omega)$. Chose $\xi_1 : \Omega \to E_0$ such that

$$\xi_1(c, \omega) = x_1(\omega)$$

and

$$||\xi_1(c, \omega) - f_0(c)||_X = ||\xi_1(\omega) - \xi_0||_{X_0}.$$

By induction, we can define a sequence $\{\xi_n(\omega)\}$ of measurable function from Ω to X_0 so that $T(\omega, \xi_n(\omega)) = \xi_{n+1}(c, \omega)$ and

$$||\xi_{n+1}(c, \omega) - \xi_n(c, \omega)||_X = ||\xi_{n+1}(\omega) - \xi_n(\omega)||_{X_0} \text{ for all } \omega \in \Omega.$$

Step 1. Proving $\lim \|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{X_0} = 0$.

Since T is a weakly contractive random operator, from (3.3) we have for each $\omega \in \Omega$,

$$\begin{aligned} \|\xi_{n+1}(\omega) - \xi_n(\omega)\|_{X_0} &= \|\xi_{n+1}(c, \omega) - \xi_n(c, \omega)\|_X \\ &= \|T(\omega, \xi_n(\omega)) - T(\omega, \xi_{n-1}(\omega))\|_X \\ &\leqslant \|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{X_0} - \phi\left(\|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{X_0}\right). \end{aligned} \tag{3.1}$$

So, we have $\{\|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{X_0}\}$ is nonegative, nonincreasing sequence, and then it has a limit $L \geqslant 0$. Let $n \to \infty$, from (3.1) we receive $L \leqslant L - \varphi(L)$, equivalence $\varphi(L) = 0$. So $\{\|\xi_n(\omega) - \xi_{n-1}(\omega)\|_{X_0}\}$ is a nonegative nonincreasing sequence which has the limit 0.

Step 2. Proving that $\{\xi_n(\omega)\}$ is a Cauchy sequence.

Suppose that $\{\xi_n(\omega)\}$ is not a Cauchy sequence for $\omega \in \Omega$. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{\xi_{m_k}(\omega)\}$ and $\{\xi_{n_k}(\omega)\}$ with $n_k > m_k > k$ and n_k corresponding to m_k, in such way that is the smallest integer with $n_k > m_k$ such that

$$\left\|\xi_{n_k}(\omega) - \xi_{m_k}(\omega)\right\|_{X_0} \geqslant \varepsilon, \quad \left\|\xi_{n_k-1}(\omega) - \xi_{m_k}(\omega)\right\|_{X_0} < \varepsilon. \tag{3.2}$$

Then, we have

$$\begin{aligned} \varepsilon &\leqslant \left\|\xi_{n_k}(\omega) - \xi_{m_k}(\omega)\right\|_{X_0} \\ &\leqslant \left\|\xi_{n_k}(\omega) - \xi_{n_k-1}(\omega)\right\|_{X_0} + \left\|\xi_{n_k-1}(\omega) - \xi_{m_k}(\omega)\right\|_{X_0} \\ &< \left\|\xi_{n_k}(\omega) - \xi_{n_k-1}(\omega)\right\|_{X_0} + \varepsilon. \end{aligned}$$

Let $k\to\infty$, we have $\lim\left\|\xi_{n_k}(\omega)-\xi_{m_k}(\omega)\right\|=\varepsilon$. Again, from

$$\begin{aligned}\varepsilon &\leqslant \left\|\xi_{n_k}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}\\ &\leqslant \left\|\xi_{n_k}(\omega)-\xi_{n_k+1}(\omega)\right\|_{X_0}+\left\|\xi_{n_k+1}(\omega)-\xi_{m_k+1}(\omega)\right\|_{X_0}\\ &\qquad +\left\|\xi_{m_k+1}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}\\ &\leqslant 2\left\|\xi_{n_k}(\omega)-\xi_{n_k+1}(\omega)\right\|_{X_0}+\left\|\xi_{n_k}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}\\ &\qquad +2\left\|\xi_{m_k+1}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}.\end{aligned}$$

Let $k\to\infty$, we have $\lim\left\|\xi_{n_{k+1}}(\omega)-\xi_{m_{k+1}}(\omega)\right\|_{X_0}=\varepsilon$. Next, from

$$\begin{aligned}\varepsilon &\leqslant \left\|\xi_{n_k}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}\\ &\leqslant \left\|\xi_{n_k}(\omega)-\xi_{m_k+1}(\omega)\right\|_{X_0}+\left\|\xi_{m_k+1}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}\\ &\leqslant \left\|\xi_{n_k}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}+2\left\|\xi_{m_k+1}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}\end{aligned}$$

and let $k\to\infty$, we have $\lim\left\|\xi_{m_k+1}(\omega)-\xi_{n_k}(\omega)\right\|_{X_0}=\varepsilon$. By the similar arguments, we receive $\lim\left\|\xi_{n_k+1}(\omega)-\xi_{m_k}(\omega)\right\|_{X_0}=\varepsilon$. Then from (3.3), we have

$$\begin{aligned}&\left\|\xi_{n_k+1}(\omega)-\xi_{n_k}(\omega)\right\|_{X_0}\\ &\quad=\left\|T\left(\omega,\xi_{n_k}(\omega)\right)-T\left(\omega,\xi_{n_k-1}(\omega)\right)\right\|_{X_0}\\ &\quad\leqslant\left\|\xi_{n_k}(\omega)-\xi_{n_k-1}(\omega)\right\|_{X_0}-\varphi\left(\left\|\xi_{n_k}(\omega)-\xi_{n_k-1}(\omega)\right\|_{X_0}\right).\end{aligned}$$

Taking the limit $k\to\infty$, we obtain

$$\varepsilon\leqslant\varepsilon-\varphi(\varepsilon)$$

which is a contraction because $\varphi(\varepsilon)>0$. Therefore, $\{\xi_n(\Omega)\}$ is a Cauchy sequence and $\lim_n\xi_n(\omega)=\xi^*(\omega)$ for all $\omega\in\Omega$.

Step 3. Proving that ξ^* is a random fixed point with PPF dependence of the random operator $T(\omega)$ in X_0.

From (2.3), $\{\xi_n(c,\omega)\}$ is a Cauchy sequence as well as by the completeness of X, $\{\xi_n(c,\omega)\}$ converges. Because $T(\omega)$ is continuous, we have for all $\omega\in\Omega$

$$\begin{aligned}T\left(\omega,\xi^*(\omega)\right)&=T\left(\omega,\lim_n\xi_n(\omega)\right)\\ &=\lim_n T\left(\omega,\xi_n(\omega)\right)\\ &=\lim_n\xi_{n+1}(c,\omega)\\ &=\xi^*(c,\omega).\end{aligned}$$

Therefore, $T(\omega)$ has a random fixed point ξ^* with PPF dependence on X_0.

(b) From (2.9), we have $\varphi(t)\geqslant(1-\lambda)t$, for all $t>0$ and then $\varphi(t)\geqslant(1-\lambda)t$, for all $t\geqslant0$. So, (3.3) implies

$$\|T(\omega,x)-T(\omega,y)\|_X \leqslant \lambda \|x-y\|_{X_0} \tag{3.3}$$

for all $\omega \in \Omega$, $x,y \in X_0$. By the same arguments as [4, Theorem 3.1], we receive the rest of (b).

(c) From the topological closeness of $\mathscr{R}_c$, we have $\xi^* \in \mathscr{R}_c$, where $\xi^*(\omega)$ is the limit of $\{\xi_n(\omega)\}$ constructed in (a). Assume that η^* is another random fixed point of T in $\mathscr{R}_c$ with PPF dependence. So,

$$\begin{aligned}\|\xi^*(\omega)-\eta^*(\omega)\|_{X_0} &= \|\xi^*(c,\omega)-\eta^*(c,\omega)\|_X \\ &= \|T(\omega,\xi^*(\omega))-T(\omega,\eta^*(\omega))\|_X \\ &\leqslant \|\xi^*(\omega)-\eta^*(\omega)\|_{X_0} - \varphi\left(\|\xi^*(\omega)-\eta^*(\omega)\|_{X_0}\right).\end{aligned}$$

So,

$$\varphi\left(\|\xi^*(\omega)-\eta^*(\omega)\|_{X_0}\right)=0$$

which implies $\xi^*(\omega)=\eta^*(\omega)$ for all $\omega \in \Omega$.

We now turn to prove Theorem 6.

Proof. [Proof of Theorem 6] Assume that $\eta : \Omega \to X$ is a given measurable function. We define a mapping $\Phi : \Omega \times X_0 \to X$ by

$$\Phi_{\eta(\omega)}(\omega,\xi(\omega)) = tT(\omega,\xi(\omega)) + (1-t)S(\omega,\eta(\omega)). \tag{3.4}$$

Then

$$\begin{aligned}&\left\|\Phi_{\eta(\omega)}(\omega,\xi_1(\omega)) - \Phi_{\eta(\omega)}(\omega,\xi_2(\omega))\right\|_X \\ &\quad = t\|T(\omega,\xi_1(\omega))-T(\omega,\xi_2(\omega))\|_X \\ &\quad \leqslant t\|\xi_1(\omega)-\xi_2(\omega)\|_{X_0} - t\varphi\left(\|\xi_1(\omega)-\xi_2(\omega)\|_{X_0}\right) \\ &\quad \leqslant \|\xi_1(\omega)-\xi_2(\omega)\|_{X_0} - \psi\left(\|\xi_1(\omega)-\xi_2(\omega)\|_{X_0}\right)\end{aligned}$$

for all $\omega \in \Omega$, where ξ_1,ξ_2 are X_0-valued measurable functions and $\psi(s)=t\varphi(s)$, $s \in [0,\infty)$. It implies that Φ is a weakly contractive random operator on X_0. Hence, by Theorem 5, there exists a unique measurable function $\xi^* : \Omega \to X_0$ which is a PPF dependent random fixed point of $\Phi_{\eta(\omega)}$. It implies the fact that

$$tT(\omega,\xi^*(\omega)) + (1-t)S(\omega,\eta(\omega)) = \xi^*(c,\omega). \tag{3.5}$$

Define a random operator $\Psi : \Omega \times X \to X$ by

$$\Psi(\omega,\eta(\omega)) = \xi^*(c,\omega) \tag{3.6}$$

for all $\omega \in \Omega$. Now we prove that Ψ has a random fixed point. Assume that $\eta_1,\eta_2 : \Omega \to X$ be two measurable functions. So, there exists measurable functions $\xi_1^*,\xi_2^* : \Omega \to X_0$ such that

$$\Psi(\omega,\eta_1(\omega)) = \xi_1^*(c,\omega) \text{ and } \Psi(\omega,\eta_2(\omega)) = \xi_2^*(c,\omega) \tag{3.7}$$

for all $\omega \in \Omega$. Then, we have

$$\begin{aligned}
&\big\|\Psi\big(\omega,\eta_1(\omega)\big)-\Psi(\omega,\eta_2(\omega))\big\|_X\\
&\quad=\big\|\xi_1^*(\omega)-\xi_2^*(\omega)\big\|_X\\
&\quad\leqslant t\Big\|T(\omega,\xi_1^*(\omega))-T(\omega,\xi_2^*(\omega))\big\|_X+(1-t)\Big\|S(\omega,\eta_1(c,\omega))\\
&\qquad\qquad -S(\omega,\eta_2(c,\omega))\Big\|_X\\
&\quad\leqslant t\big\|\xi_1^*(\omega)-\xi_2^*(\omega)\big\|_{X_0}-t\varphi(\big\|\xi_1^*(\omega)-\xi_2^*(\omega)\big\|_{X_0})\\
&\qquad\qquad +(1-t)\big\|\eta_1(\omega)-\eta_2(\omega)\big\|_X\\
&\quad=t\big\|\xi_1^*(\omega)-\xi_2^*(\omega)\big\|_{X_0}-t\varphi(\big\|\xi_1^*(\omega)-\xi_2^*(\omega)\big\|_{X_0})\\
&\qquad\qquad +(1-t)\big\|\eta_1(\omega)-\eta_2(\omega)\big\|_X\\
&\quad=t\big\|\Psi(\omega,\eta_1(\omega))-\Psi(\omega,\eta_2(\omega))\big\|_X\\
&\qquad\qquad -t\varphi(\big\|\Psi(\omega,\eta_1(\omega))-\Psi(\omega,\eta_2(\omega))\big\|_X)\\
&\qquad\qquad +(1-t)\big\|\eta_1(\omega)-\eta_2(\omega)\big\|_X.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big\|\Psi(\omega,\eta_1(\omega))-\Psi\big(\omega,\eta_2(\omega)\big)\big\|_X\\
&\qquad\leqslant\|\eta_1(\omega)-\eta_2(\omega)\|_X-\frac{t}{1-t}\varphi\left(\|\Psi(\omega,\eta_1(\omega))-\Psi(\omega,\eta_2(\omega))\|_X\right).
\end{aligned}$$

By the similar arguments as in Theorem 3, Ψ has a unique random fixed point $\zeta_t:\Omega\to X_0$ such that $\Psi(\omega,\zeta_z(c,\omega))=\zeta_t(c,\omega)$. It is easy to see that

$$tT(\omega,\zeta_t(\omega))+(1-t)S(\omega,\zeta_t(c,\omega))=\zeta_t(c,\omega). \tag{3.8}$$

This complete the proof.

We now prove Theorem 7.

Proof. [Proof of Theorem 7] Let ξ_0 be arbitrary continuous X-valued function on I. Then, there exists a random variable $\xi_1:\Omega\to X_0$ such that $T(\omega,\xi_0)=x_1(\omega)$. Chose $\xi_1:\Omega\to X_0$ such that $\xi_1(c,\omega)=x_1(\omega)$ and

$$||\xi_1(c,\omega)-f_0(c)||_X=||\xi_1(\omega)-\xi_0||_{X_0}.$$

By induction, we can define a sequence $\{\xi_n(\omega)\}$ of measurable function from Ω to X_0 so that $T(\omega,\xi_n(\omega))=\xi_{n+1}(c,\omega)$ and

$$||\xi_{n+1}(c,\omega)-\xi_n(c,\omega)||_X=||\xi_{n+1}(\omega)-\xi_n(\omega)||_{X_0}\ \text{ for all }\ \omega\in\Omega.$$

Now we prove that $\{\xi_n(\omega)\}$ is a Cauchy sequence. From (2.12), we have for all $n\geqslant 0$

$$\begin{aligned}
\min\big\{&\|\xi_{n-1}(c,\omega)-\xi_n(c,\omega)\|_X,\|\xi_n(c,\omega)-\xi_{n+1}(c,\omega)\|_X,\\
&\qquad\qquad\|\xi_n(c,\omega)-\xi_{n+1}(c,\omega)\|_X\big\}\\
&\qquad\leqslant\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0}-\varphi\left(\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0}\right).
\end{aligned}$$

So,

$$\min\left\{\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0},\|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{X_0}\right\}$$
$$\leqslant\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0}-\varphi\left(\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0}\right).$$

If there exists $n_0 \geqslant 1$ such that

$$\min\left\{\left\|\xi_{n_0-1}(\omega)-\xi_{n_0}(\omega)\right\|_{X_0},\left\|\xi_{n_0}(\omega)-\xi_{n_0+1}(\omega)\right\|_{X_0}\right\}$$
$$=\left\|\xi_{n_0-1}(\omega)-\xi_{n_0}(\omega)\right\|_{X_0}$$

then $\varphi\left(\left\|\xi_{n_0-1}(\omega)-\xi_{n_0}(\omega)\right\|_{X_0}\right)=0$ and

$$\left\|\xi_{n_0-1}(\omega)-\xi_{n_0}(\omega)\right\|_{X_0}=\left\|c,\xi_{n_0-1}(\omega)-c,\xi_{n_0}(\omega)\right\|_X=0.$$

Hence, $T(\omega,\xi_{n_0-1})=\xi_{n_0-1}(c,\omega)$ and ξ_{n_0} is a PPF dependence fixed point of T. On the contrary, we have for all $n \geqslant 1$

$$\|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{X_0}\leqslant\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0}-\varphi\left(\|\xi_{n-1}(\omega)-\xi_n(\omega)\|_{X_0}\right).$$

So, we have that $\|\xi_n(\omega)-\xi_{n+1}(\omega)\|_{X_0}$ is a nonnegative, nonincreasing sequence. By the similar arguments as in Theorems 5, $\{\xi_n(\omega)\}$ is a Cauchy sequence and it has the limit $\xi^*(\omega)$.

Next, we prove that ξ^* is a random fixed point with PPF dependence of the random operator $T(\omega)$ in X_0. From (2.3), $\{\xi_n(c,\omega)\}$ is a Cauchy sequence. By the completeness of X, $\{\xi_n(c,\omega)\}$ converges. Because $T(\omega)$ is continuous, we have for all $\omega\in\Omega$

$$\begin{aligned}T(\omega,\xi^*(\omega)) &= T(\omega,\lim_n\xi_n(\omega))\\ &=\lim_n T(\omega,\xi_n(\omega))\\ &=\lim_n\xi_{n+1}(c,\omega)\\ &=\xi^*(c,\omega).\end{aligned}$$

Therefore, $T(\omega)$ has a random fixed point ξ^* with PPF dependence on X_0.

4. Conclusions

In this paper, some weak conditions to ensure the existence of random fixed point is investigated. The random fixed point theorems if useful to prove the existence of solution of random equation, random differential equation (see e.g. [4]), ...

Another question which is not considered much is to study conditional assumption having fixed left hand side. Theorem 7 is just a first step to answer this question. Perhaps, some interesting results with these type assumptions could be obtained in this research direction.

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