Differential
Equations
& Applications
Volume 5, Number 4 (2013), 501–518
doi:10.7153/dea-05-29

# RANDOM SET–VALUED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH THE SECOND TYPE HUKUHARA DERIVATIVE

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(*Communicated by Marek T. Malinowski*)

*Abstract.* This paper concerns with the initial value problem for random set-valued functional differential equations with the second type Hukuhara derivative (RSFDEs). By using the techniques of successive approximations, the existence and uniqueness of solutions are established. Two kinds of boundedness of the solution are also established. In addition, the problem at least one solution under some conditions is proven and two examples illustrate the results.

## 1. Introduction

Functional differential equation (FDE) show the fact that the velocity of the system depends not only on the state of the system at a given instant but depends upon the history of the trajectory until this instant. The class of differential equations with delay encompasses a large variety of differential equations. FDE plays an important role in an increasing number of system models in biology, engineering, physics and other sciences. There exists an extensive literature dealing with functional differential equations and their applications. We refer to the monograph [14], and references therein.

The study of set differential equations in a semilinear metric space has gained much attention [20]. Many interesting results in this direction one can find e.g. in [1, 2, 3, 4, 5, 6, 9, 10, 15, 19, 21, 29, 30, 31, 32, 42, 44, 49, 48, 46, 11]. The set differential equations have a significant influence in fuzzy differential equations (see e.g. [18, 21, 23, 7, 25, 16, 43, 8, 17]), random fuzzy differential equations (see e.g. [26, 27, 28, 33]), set-valued and fuzzy stochastic differential equations (see e.g. [12, 13, 34, 35, 36, 37, 38, 39, 40, 41]). A solution to set differential equations with Hukuhara derivative defined as in [20], is the Hukuhara differentiable mapping. This implies that the diameter of the solution values is a nondecreasing function of time (see e.g. [20]). From the application's point of view this property can be sometimes inconvenient because it means practically that uncertainty, contained in a model of a physical system which is described by set differential equations, can only grow as times goes by. Hence the successive values of modelled phenomenon are covered by nondecreasing

*Mathematics subject classification* (2010): 34F05, 34G20, 34A12.
*Keywords and phrases*: Random set-valued variable, set-valued functional differential equations, successive approximation, at least one solution, second type Hukuhara derivative.
This research is supported by Ton Duc Thang University, Ho Chi Minh city, Vietnam.


Paper DEA-05-29

sets of tolerance. In [47], authors introduced a concept of generalized Hukuhara differentiability of interval-valued mapping, which allows them to obtain the solutions of interval differential equations with decreasing diameter of solutions values. The papers [29, 30, 31, 32] present some investigations of the interval differential equations and set differential equations with the second type Hukuhara derivative. In [18], the authors proved the global existence of solutions for interval-valued integro-differential equations with initial conditions under generalized H-differentiability. In [2], the authors established the existence of solutions and some properties of set solutions for a class of set functional differential equations in a separable Banach space.

In this paper, inspired and motivated by Malinowski [26, 27, 28, 29, 30, 31, 33], Lupulescu [2, 22, 23, 24],Park and Jeong [45]. We consider the random set-valued functional differential equations with second type Hukuhara derivative. The paper will be organized as follows. As preliminaries we recall some basic results set-valued mapping, set-valued stochastic process. In section 3, we concerns with the initial value problem for random set-valued functional differential equations with second type Hukuhara derivative. By using the techniques of successive approximations, the existence and uniqueness of solutions are established. Two kinds of boundedness of the solution are also established. In addition, the problem at least one solution under some conditions is proven and two examples illustrate the results.

## 2. Preliminaries

Let $\mathscr{K}_c(\mathbb{R}^d)$ denoted the collection of nonempty, compact and convex subsets of $\mathbb{R}^d$ . The following operations can be naturally defined on it:

$$X+Y=\{x+y:x\in X,y\in Y\};\ \lambda X=\{\lambda x:x\in X\},\ \lambda\in\mathbb{R}_+.$$

The Hausdorff metric is defined as

$$D[X,Y]=\max\{\sup_{y\in Y}\inf_{x\in X}d(y,x);\ \sup_{x\in X}\inf_{y\in Y}d(x,y)\}$$

where $X,Y$ are bounded subsets of $\mathbb{R}^d$. It is clear that the Hausdorff metric satisfies the relations of the ordinary metric.

It is known that $(\mathscr{K}_c(\mathbb{R}^d),D)$ is a complete metric space. Moreover, $\mathscr{K}_c(\mathbb{R}^d)$ equipped with the above-mentioned natural algebraic operations of addition and non-negative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space. On the other hand, the Hausdorff metric $D$ is compatible with the operations defined on it as described by the following properties: for any $X,Y,Z,W\in\mathscr{K}_c(\mathbb{R}^d)$

$$D[X+Z,Y+Z]=D[X,Y],$$

$$D[kX,kY]=|k|D[X,Y],$$

$$D[X+Z,Y+W]\leqslant D[X,Y]+D[Z,W].$$

Let $X,Y \in \mathscr{K}_c(\mathbb{R}^d)$. If there exists a subset $Z \in \mathscr{K}_c(\mathbb{R}^d)$ such that $X = Y + Z$, then we call $Z$ the Hukuhara difference of $X$ and $Y$. The set $Z$ we denote by $X \ominus Y$. Note that $X \ominus Y \neq X + (-1)Y$.

The problem of the existence of Hukuhara difference $X \ominus Y$ is often inconvenient although it is known ([20]) that for $X,Y \in \mathscr{K}_c(\mathbb{R}^d)$ the Hukuhara difference $X \ominus Y$ exists iff the following if satisfied: if $x$ belongs to the boundary of $X$ then there exists a point $z$ such that $x \in Y + z \subset A$.

Next, we recall the definition of a derivative which will be used in the paper:

DEFINITION 1. ([29]) A mapping $F : [a,b] \to \mathscr{K}_c(\mathbb{R}^d)$ is second type Hukuhara differentiable at $t_0 \in [a,b]$ if there exists $F'(t_0) \in \mathscr{K}_c(\mathbb{R}^d)$ such that the limits

$$\lim_{h\to 0^+}(-h^{-1})(F(t_0-h)\ominus F(t_0)), \text{ and } \lim_{h\to 0^+}(-h^{-1})(F(t_0)\ominus F(t_0+h)),$$

exist and are equal to $F'(t_0)$. The set $F'(t_0)$ is said to be the second type Hukuhara derivative of set-mapping $F$ at the point $t_0$.

In this definition it is implicit that for all $h > 0$ (sufficiently small) the Hukuhara differences $F(t_0-h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0+h)$ have to exist.

DEFINITION 2. The function $F : [a,b] \to \mathscr{K}_c(\mathbb{R}^d)$ is called second type Hukuhara differentiable on $[a,b]$ if $F$ is second type Hukuhara differentiable at every point $t_0 \in [a,b]$.

REMARK 1. ([29]) Let $F : [a,b] \to \mathscr{K}_c(\mathbb{R}^d)$ be a second type Hukuhara differentiable on $[a,b]$. Then

i) $F$ is continuous on $[a,b]$;

ii) the function $\operatorname{diam}(F) : [a,b] \to [0,\infty)$ is nonincreasing on $[a,b]$.

The mapping $F : [a,b] \to \mathscr{K}_c(\mathbb{R}^d)$ is said to be integrable if the set $S(F)$ of integrable selectors of $F$ is nonempty. Then

$$\int_a^b F(t)dt = \left\{ \int_a^b f(t)dt \mid f \in S(F) \right\}.$$

It is also known that for the integrable set-valued mapping $F,G : [a,b] \to \mathscr{K}_c(\mathbb{R}^d)$ we have $D[F(\cdot),G(\cdot)]$ is integrable and the following property is valid

$$D\Big[\int_a^b F(t)dt, \int_a^b G(t)dt\Big] \leqslant \int_a^b D[F(t),G(t)]dt.$$

If a set-valued function $F$ is second type Hukuhara differentiable on $[a,b]$ and $F'$ is integrable, then for $t \in [a,b]$ ,

$$F(a) = F(t) + (-1)\int_a^t F'(s)ds.$$

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space. A set-valued mapping $F : \Omega \to \mathscr{K}_c(\mathbb{R}^d)$ is called a set-valued random variable if

$$\{\omega \in \Omega : F(\omega) \cap \mathscr{O} \neq \emptyset\} \in \mathscr{A}, \text{ for every closed set } \mathscr{O} \subset \mathbb{R}^d.$$

DEFINITION 3. ([26]) A set-valued mapping $F : [0,\infty) \times \Omega \to \mathscr{K}_c(\mathbb{R}^d)$ is said to be set-valued stochastic process if $F(\cdot,\omega)$ is a set-valued function with any fixed $\omega \in \Omega$ and $F(t,\cdot)$ is a set-valued random variable for any fixed $t \in [0,\infty)$.

For a positive number $\sigma$, we denote by $C_\sigma$ the space $C([-\sigma,0],\mathscr{K}_c(\mathbb{R}^d))$. Also we denote by

$$D_\sigma[X,Y] = \sup_{t\in[-\sigma,0]} D[X(t),Y(t)],$$

the metric on the space $C_\sigma$. Let $X \in C([-\sigma,b],\mathscr{K}_c(\mathbb{R}^d))$. Then for each $t \in [0,b]$ we denote by $X_t \in C_\sigma$ defined by $X_t = X(t+s)$, $s \in [-\sigma,0]$.

Assume that $F : [0,b] \times \Omega \times C_\sigma \to \mathscr{K}_c(\mathbb{R}^d)$ satisfies the following hypotheses:

(F1) $F_\cdot(t,\varphi) : \Omega \to \mathscr{K}_c(\mathbb{R}^d)$ is a set-valued random variable for $t \in [0,b]$, $\varphi \in C_\sigma$.

(F2) with $\mathbb{P}.1$ the mapping $F_\omega(\cdot,\cdot) : [0,b] \times C_\sigma \to \mathscr{K}_c(\mathbb{R}^d)$ is a continuous set-valued mapping at every at $(t_0,\varphi_0) \in [0,b] \times C_\sigma$, i.e., there exists $\Omega_* \subset \Omega$ with $\mathbb{P}(\Omega_*) = 1$ such that for every $\omega \in \Omega_*$ the following is true : for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $t \in [0,b]$ and $\varphi, \varphi_0 \in C_\sigma$ it holds

$$\max\{|t-t_0|, D_\sigma[\varphi,\varphi_0]\} < \delta \Longrightarrow D[F_\omega(t,\varphi), F_\omega(t_0,\varphi_0)] < \varepsilon.$$

For convenience, from now on, the fact that there that exists $\Omega_* \subset \Omega$ such that $\mathbb{P}(\Omega_*) = 1$ and for every $\omega \in \Omega_*$ it holds $X(\omega) = Y(\omega)$, where $X,Y$ random elements, will be written as $X(\omega) \stackrel{\mathbb{P}.1}{=} Y(\omega)$. Similarly, for the inequalities. Also if there exists $\Omega_* \subset \Omega$ such that $\mathbb{P}(\Omega_*) = 1$ and for every fixed $\omega \in \Omega_*$ it holds $X(t,\omega) = Y(t,\omega)$ for every $t \in [-\sigma,b]$, where $X,Y$ are stochastic process, then we will write $X(t,\omega) \stackrel{[-\sigma,b]\,\mathbb{P}.1}{=} Y(t,\omega)$ in short, or $X(t,\omega) = Y(t,\omega)$ for every $t \in [-\sigma,b]$ with $\mathbb{P}.1$. Similarly, for the inequality.

## 3. Main results

In this paper, we will consider the random set-valued functional differential equation as follows:

$$\begin{cases} X'(t,\omega) \stackrel{[0,b],\,\mathbb{P}.1}{=} F_\omega(t,X_t) \\ X(t,\omega) \stackrel{[-\sigma,0],\,\mathbb{P}.1}{=} \varphi(t,\omega) \end{cases} \tag{3.1}$$

where $F : [0,b] \times \Omega \times C_\sigma \to \mathscr{K}_c(\mathbb{R}^d)$ is a set-valued stochastic process and the symbol $'$ denotes the derivative from Definition 1.

DEFINITION 4. By the solution of (3.1) we mean a continuous set-valued stochastic process $X : [-\sigma, b] \times \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ that satisfies (3.1). A solution $X$ is unique if it holds

$$D[X(t,\omega), Y(t,\omega)] \overset{[-\sigma,b]\,\mathbb{P}.1}{=} 0$$

for any set-valued stochastic process $Y : [-\sigma, b] \times \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ which is a solution of (3.1).

LEMMA 1. A set-valued mapping $X : [-\sigma, b] \times \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ is a solution to the problem (3.1) if and only if $X$ is a continuous set-valued stochastic process and $X$ satisfies the following set-valued stochastic integral equation

$$\begin{cases} X(t,\omega) \overset{[-\sigma,0],\,\mathbb{P}.1}{=} \varphi(t,\omega), \\ \varphi(0,\omega) \overset{[0,b],\,\mathbb{P}.1}{=} X(t,\omega) + (-1)\int_0^t F_\omega(s, X_s)ds. \end{cases} \tag{3.2}$$

THEOREM 1. *Assume that $F : [0,b] \times \Omega \times C_\sigma \to \mathcal{K}_c(\mathbb{R}^d)$ satisfies (F1) - (F2) and with $\mathbb{P}.1$ for every $t \in [0,b]$ and every $\varphi, \psi \in C_\sigma$ it holds*

$$D[F_\omega(t,\varphi), F_\omega(t,\psi)] \leqslant L(t,\omega) D_\sigma[\varphi,\psi], \tag{3.3}$$

*where $L : [0,b] \times \Omega \to (0,\infty)$ such that $L(\cdot,\omega)$ is continuous with $\mathbb{P}.1$. Suppose that there exist non-negative constants $\gamma$ and $Q$ such that*

$$D[F_\omega(t,\varphi), \{0\}] \overset{[0,b]\times C_\sigma,\,\mathbb{P}.1}{\leqslant} Q, \tag{3.4}$$

*and the sequence $\{X\}_{n=0}^\infty$, $X^n : [-\sigma,\gamma] \times \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ given by*

$$X^0(t,\omega) = \begin{cases} \varphi(t,\omega) & \text{for } [-\sigma,0], \\ \varphi(0,\omega) & \text{for } [0,\gamma], \end{cases} \tag{3.5}$$

*and for $n = 1, 2, \ldots$*

$$X^n(t,\omega) = \begin{cases} \varphi(t,\omega) & \text{for } [-\sigma,0], \\ \varphi(0,\omega) \ominus (-1)\int_0^t F_\omega(s, X_s^{n-1})ds & \text{for } [0,\gamma], \end{cases} \tag{3.6}$$

*is well defined (i.e. the foregoing H-differences do exist). Then there exists a constant $\vartheta > 0$ such that the random set-valued functional differential equation (3.1) has a unique local solution $X(t,\omega)$ on the interval $[0,\vartheta]$.*

*Proof.* To prove the theorem we shall use the method of successive approximations. Let us define $\theta \in C_\sigma$ by $\theta \equiv \{0\}$ and set $\vartheta = \min\{b, \gamma, \frac{1}{2Q}\}$. Then for $t \in [0,\vartheta]$ we have

$$D[F_\omega(t, X_t^0), \{0\}] \leqslant D[F_\omega(t, X_t^0), F_\omega(t,\theta)] + D[F_\omega(t,\theta), \{0\}]$$

$$\leqslant L(t,\omega)D_\sigma[X_t^0(\cdot,\omega),\theta]+D[F_\omega(t,\theta),\{0\}]$$
$$\leqslant L(t,\omega)\sup_{t\in[0,\vartheta]} D[X^0(t,\omega),\theta]+D[F_\omega(t,\theta),\{0\}]$$
$$\leqslant LM+K<\infty,$$

where

$$M\overset{P.1}{=}\sup_{t\in[0,\vartheta]} D[X^0(t,\omega),\theta],\ K\overset{[0,\vartheta],\mathbb{P}.1}{=}\sup_{t\in[0,\vartheta]} D[F(t,\theta),\{0\}]\quad\text{and}\quad L\overset{\mathbb{P}.1}{=}\sup_{t\in[0,\vartheta]} L(t,\omega).$$

For $t\in[0,\vartheta]$, from (3.4) and (3.5), it is easy to see that

$$\begin{aligned}D[X^1(t,\omega),X^0(t,\omega)]&=D\Big[\varphi(0,\omega)\ominus(-1)\int_0^t F_\omega(s,X_s^0)ds,\varphi(0,\omega)\Big]\\&\leqslant D\Big[\int_0^t F_\omega(s,X_s^0)ds,\{0\}\Big]\\&\leqslant\int_0^t D[F_\omega(s,X_s^0),\{0\}]ds\\&\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} Qt\\&\leqslant Q\vartheta.\end{aligned}$$

where $Q=LM+K$.

Using the conditions (3.3)-(3.6), we have

$$\begin{aligned}D[X^2(t,\omega),X^1(t,\omega)]&=D\Big[\ominus(-1)\int_0^t F_\omega(s,X_s^1)ds,\ominus(-1)\int_0^t F_\omega(s,X_s^0)ds\Big]\\&=D\Big[\int_0^t F_\omega(s,X_s^1)ds,\int_0^t F_\omega(s,X_s^0)ds\Big]\\&\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant}\int_0^t D[F_\omega(s,X_s^1),F_\omega(s,X_s^0)]ds\\&\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant}\int_0^t L(s,\omega)D_\sigma[X_s^1(\cdot,\omega),X_s^0(\cdot,\omega)]ds\\&\overset{[0,\vartheta],\mathbb{P}.1}{=}L(\omega)\int_0^t\sup_{r\in[-\sigma,0]} D[X^1(s+r,\omega),X^0(s+r,\omega)]ds\\&\overset{[0,\vartheta],\mathbb{P}.1}{=}L(\omega)\int_0^t\sup_{\iota\in[s-\sigma,s]} D[X^1(\iota,\omega),X^0(\iota,\omega)]ds\\&\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant}\frac{Q}{L(\omega)}\cdot\frac{[L(\omega)t]^2}{2!},\end{aligned}$$

where $L(\omega)=\sup_{t\in[0,\vartheta]} L(t,\omega)$.

Further for every $n>2$ and $t\in[0,\vartheta]$ we obtain

$$D[X^n(t,\omega),X^{n-1}(t,\omega)]=D\Big[\ominus(-1)\int_0^t F_\omega(s,X_s^{n-1})ds,\ominus(-1)\int_0^t F_\omega(s,X_s^{n-2})ds\Big]$$

$$\leqslant \int_0^t D[F_\omega(s,X_s^{n-1}),F_\omega(s,X_s^{n-2})]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^t D_\sigma[X_s^{n-1}(\cdot,\omega),X_s^{n-2}(\cdot,\omega)]ds$$
$$= L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X_s^{n-1}(r,\omega),X_s^{n-2}(r,\omega)]ds$$
$$= L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X^{n-1}(r+s,\omega),X^{n-2}(r+s,\omega)]ds$$
$$= L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,s]} D[X^{n-1}(\iota,\omega),X^{n-2}(\iota,\omega)]ds.$$

If we assume that

$$D[X^{n-1}(t,\omega),X^{n-2}(t,\omega)] \overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \frac{Q}{L(\omega)}\cdot\frac{[L(\omega)t]^{n-1}}{(n-1)!}.$$

Thus, by mathematical induction, for $n\in\mathbb{N}$ and $t\in[0,\vartheta]$

$$D[X^n(t,\omega),X^{n-1}(t,\omega)] \overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \int_0^t L(\omega)\frac{Q}{L(\omega)}\cdot\frac{[L(\omega)s]^{n-1}}{(n-1)!}ds \leqslant \frac{Q}{L(\omega)}\cdot\frac{[L(\omega)\vartheta]^n}{n!}. \tag{3.7}$$

It is easy to see that for $n\in\mathbb{N}$ the functions $X^n(\cdot,\omega):[-\sigma,\vartheta]\to\mathscr{K}_c(\mathbb{R}^d)$ are continuous with $\mathbb{P}.1$.

Now, for any $n\in\mathbb{N}$ and $t\in[0,\vartheta]$ we shall show that the sequence $\{X^n(t,\omega)\}$ is a Cauchy sequence uniformly in $t$ with $\mathbb{P}.1$ and then $\{X^n(\cdot,\omega)\}$ is uniformly convergent with $\mathbb{P}.1$. For $n>m>0$, from (3.7) we obtain

$$D[X^n(t,\omega),X^m(t,\omega)] \leqslant \sum_{k=m}^{n} D[X^{k+1}(t,\omega),X^k(t,\omega)] \overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} Q\sum_{k=m}^{n} L^k(\omega)\frac{\vartheta^{k+1}}{(k+1)!} \tag{3.8}$$

The almost sure convergence of the series $\sum_{n=1}^{\infty} L^{n-1}(\omega)\frac{\vartheta^n}{n!}$ with $\mathbb{P}.1$, then for each $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ large enough such that $n,m\geqslant n_0$

$$\sup_{t\in[0,\vartheta]} D[X^n(t,\omega),X^m(t,\omega)]\leqslant\varepsilon. \tag{3.9}$$

Since $(\mathscr{K}_c(\mathbb{R}^d),D)$ is a complete metric space, it follows that there exists $\Omega_*\subset\Omega$ such that $\mathbb{P}(\Omega_*)=1$ and for every $\omega\in\Omega_*$ the sequence $\{X^n(\cdot,\omega)\}$ is uniformly convergent.

We shall show that $X(t,\omega)$ is a solution of (3.1). For any $\varepsilon>0$, there is $n_0$ large enough such that for every $n\geqslant n_0$, $n\in\mathbb{N}$ we derive

$$D\Big[\int_0^t F_\omega(s,X_s^n)ds,\int_0^t F_\omega(s,X_s)ds\Big]$$

$$\leqslant \int_0^t D[F_\omega(s,X_s^n),F_\omega(s,X_s)]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^t D_\sigma[X_s^n(\cdot,\omega),X_s(\cdot,\omega)]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{=} L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X_s^n(r,\omega),X_s(r,\omega)]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{=} L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X^n(r+s,\omega),X(r+s,\omega)]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{=} L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,s]} D[X^n(\iota,\omega),X(\iota,\omega)]ds \overset{\mathbb{P}.1}{\longrightarrow} 0.$$

Hence, by virtue of Lebesgue Dominated Convergence theorem,

$$D\Big[\int_0^t F_\omega(s,X_s^n)ds, \int_0^t F_\omega(s,X_s)ds\Big] \to 0,$$

as $n \longrightarrow \infty$ for any $t \in [0,\vartheta]$, with $\mathbb{P}.1$.

Consequently, we have

$$D\Big[\varphi(0,\omega),X(t,\omega)+(-1)\int_0^t F_\omega(s,X_s)ds\Big]$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[X^n(t,\omega),X(t,\omega)] + D\Big[\int_0^t F_\omega(s,X_s^{n-1})ds, \int_0^t F_\omega(s,X_s)ds\Big]$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \sup_{t\in[0,b]} D[X^n(t,\omega),X(t,\omega)] + D\Big[\int_0^t F_\omega(s,X_s^{n-1})ds, \int_0^t F_\omega(s,X_s)ds\Big]$$

Thus, in view of the two previous convergence results and the fact that the second term of the right-hand side is equal to zero, we have

$$D\Big[\varphi(0,\omega),X(t,\omega)+(-1)\int_0^t F_\omega(s,X_s)ds\Big] \overset{[0,\vartheta],\mathbb{P}.1}{=} 0.$$

Hence, $X(t,\omega)$ is the solution of (3.1). By Lemma 1, we have that $X(t,\omega)$ is a solution of (3.1).

Finally, we prove the uniqueness of the solution (3.2). Let us assume that $X,Y:[-\sigma,\vartheta]\times\Omega \to \mathscr{K}_c(\mathbb{R}^d)$ are two continuous set-valued stochastic process which are solutions of (3.1).

Then we have

$$D\Big[X(t,\omega),Y(t,\omega)\Big] \leqslant \int_0^t D[F_\omega(s,X_s),F_\omega(s,Y_s)]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^t D_\sigma[X_s(\cdot,\omega),Y_s(\cdot,\omega)]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{=} L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X(r+s,\omega),Y(r+s,\omega)]ds$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{=} L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,s]} D[X(\iota,\omega),Y(\iota,\omega)]ds \tag{3.10}$$

If we take

$$\xi(s,\omega) = \sup_{\iota\in[s-\sigma,s]} D[X(\iota,\omega),Y(\iota,\omega)]$$

for any $s\in[0,t]$ , then from (3.10), we have

$$\xi(t,\omega) \overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^t \xi(s,\omega)ds$$

Applying Lemma 2 in ([26]), we obtain $\xi(t,\omega)=0$ on $t\in[0,\vartheta]$ with $\mathbb{P}.1$. Hence,

$$X(t,\omega) \overset{[0,\vartheta],\mathbb{P}.1}{=} Y(t,\omega).$$

This completes the proof.

THEOREM 2. *Let $F:[0,b]\times\Omega\times C_\sigma\to\mathscr{K}_c(\mathbb{R}^d)$ satisfies the conditions of Theorem 1, $\varphi\in C_\sigma$ and let $X:[-\sigma,\vartheta]\times\Omega\to\mathscr{K}_c(\mathbb{R}^d)$ be the solution of (3.1). Then we have*

$$\sup_{t\in[0,\vartheta]} D[X(t,\omega),\{0\}]$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \Big(D[\varphi(0,\omega),\{0\}]+K\vartheta+bL(\omega)D[\varphi(0,\omega),\{0\}]\Big)\exp(tL(\omega)),$$

*where $\theta\in C_\sigma$ is such that $\theta\equiv\{0\}$ and $K=\sup_{t\in[0,\vartheta]} D[F_\omega(t,\theta),\{0\}]$.*

*Proof.* Since $X(t,\omega)$ is the solution of (3.1), by Lemma 1, for $t\in[0,\vartheta]$ we have

$$\begin{aligned}
D[X(t,\omega),\{0\}] &= D\Big[\varphi(0,\omega)\ominus(-1)\int_0^t F_\omega(s,X_s)ds,\{0\}\Big]\\
&\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\{0\}]+\int_0^t D[F_\omega(s,X_s),\{0\}]ds\\
&\leqslant D[\varphi(0,\omega),\{0\}]+\int_0^t D[F_\omega(s,X_s),F_\omega(s,\theta)]ds\\
&\qquad +\int_0^t D[F_\omega(s,\theta),\{0\}]ds\\
&\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\{0\}]+Kt+L(\omega)\int_0^t D_\sigma[X_s(\cdot,\omega),\theta]ds\\
&\overset{[0,\vartheta],\mathbb{P}.1}{=} D[\varphi(0,\omega),\{0\}]+Kt+L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X_s(r,\omega),\theta]ds\\
&\overset{[0,\vartheta],\mathbb{P}.1}{=} D[\varphi(0,\omega),\{0\}]+Kt+L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,s]} D[X(\iota,\omega),\theta]ds
\end{aligned}$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\{0\}] + Kt + L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,0]} D[X(\iota,\omega),\theta]ds$$
$$+L(\omega)\int_0^t \sup_{\iota\in[0,s]} D[X(\iota,\omega),\theta]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\{0\}] + K\vartheta + \vartheta L(\omega)D[\varphi(0,\omega),\theta]$$
$$+L(\omega)\int_0^t \sup_{\iota\in[0,s]} D[X(\iota,\omega),\theta]ds.$$

Thus we infer that for $t\in[0,\vartheta]$ it holds

$$\sup_{\iota\in[0,t]} D[X(\iota,\omega),\{0\}] \overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\{0\}] + K\vartheta + \vartheta L(\omega)D[\varphi(0,\omega),\theta]$$
$$+L(\omega)\int_0^t \sup_{\iota\in[0,s]} D[X(\iota,\omega),\theta]ds.$$

Applying Lemma 2 in ([26]), we obtain

$$\sup_{\iota\in[0,t]} D[X(\iota,\omega),\{0\}]$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \Big(D[\varphi(0,\omega),\{0\}] + K\vartheta + \vartheta L(\omega)D[\varphi(0,\omega),\theta]\Big)\exp(tL(\omega)).$$

This completes the proof.

THEOREM 3. *Assume that $F:[0,b]\times\Omega\times C_\sigma\to\mathscr{K}_c(\mathbb{R}^d)$ satisfies the conditions of Theorem 1, $\varphi,\psi\in C_\sigma$ and let $X,Y:[-\sigma,\vartheta]\times\Omega\to\mathscr{K}_c(\mathbb{R}^d)$ be the solutions of (3.1) with $X(t,\omega)=\varphi(t,\omega)$ and $Y(t,\omega)=\psi(t,\omega)$ for $t\in[-\sigma,0]$. Then we have*

$$\sup_{t\in[0,\vartheta]} D[X(t,\omega),Y(t,\omega)]$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \Big(D[\varphi(0,\omega),\psi(0,\omega)] + \vartheta L(\omega)D[\varphi(0,\omega),\psi(0,\omega)]\Big)\exp(tL(\omega)).$$

*Proof.* Since $X(t,\omega)$ and $Y(t,\omega)$ are the solutions of (3.1), we obtain

$$D[X(t,\omega),Y(t,\omega)]$$
$$= D\Big[\varphi(0,\omega)\ominus(-1)\int_0^t F_\omega(s,X_s)ds, \psi(0,\omega)\ominus(-1)\int_0^t F_\omega(s,Y_s)ds\Big]$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\psi(0,\omega)] + \int_0^t D\Big[F_\omega(s,X_s),F_\omega(s,Y_s)\Big]ds$$
$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\psi(0,\omega)] + L(\omega)\int_0^t D_\sigma[X_s(\cdot,\omega),Y_s(\cdot,\omega)]ds$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\psi(0,\omega)] + L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X_s(r,\omega),Y_s(r,\omega)]ds$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{=} D[\varphi(0,\omega),\psi(0,\omega)] + L(\omega)\int_0^t \sup_{r\in[-\sigma,0]} D[X(r+s,\omega),Y(r+s,\omega)]ds$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{=} D[\varphi(0,\omega),\psi(0,\omega)] + L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,s]} D[X(\iota,\omega),Y(\iota,\omega)]ds$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{=} D[\varphi(0,\omega),\psi(0,\omega)] + L(\omega)\int_0^t \sup_{\iota\in[s-\sigma,0]} D[X(\iota,\omega),Y(\iota,\omega)]ds + L(\omega)\int_0^t \sup_{\iota\in[0,s]} D[X(\iota,\omega),Y(\iota,\omega)]ds$$

$$\overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} D[\varphi(0,\omega),\psi(0,\omega)] + bL(\omega)D[\varphi(0,\omega),\psi(0,\omega)] + L(\omega)\int_0^t \sup_{\iota\in[0,s]} D[X(\iota,\omega),Y(\iota,\omega)]ds.$$

Applying Lemma 2 in ([26]), we obtain

$$\sup_{\iota\in[0,t]} D[X(\iota,\omega),Y(\iota,\omega)] \overset{[0,\vartheta],\mathbb{P}.1}{\leqslant} \Big(D[\varphi(0,\omega),\psi(0,\omega)] + \vartheta L(\omega)D[\varphi(0,\omega),\psi(0,\omega)]\Big)\exp(tL(\omega)),$$

for $t\in[0,\vartheta]$ . This completes the proof.

THEOREM 4. *Suppose that the function* $F:[0,b]\times\Omega\to\mathscr{K}_c(\mathbb{R}^d)$ *satisfies the conditions of Theorem 1. Assume that there exists a real-valued stochastic process* $f:[0,b]\times\Omega\to[0,\infty)$ *satisfying* $\int_0^b f(t,\omega)dt \overset{\mathbb{P}.1}{\leqslant} K$ *with* $K>0$*, and such that with* $\mathbb{P}.1$

$$D[F_\omega(t,\varphi),\{0\}] \overset{\mathbb{P}.1}{\leqslant} f(t,\omega)$$

*for every* $t\in[0,b]$ *and every* $\varphi\in C_\sigma$. *Suppose that there exists a non-negative constant* $r<b$ *such that for* $t\in[0,r]$ *the sequence* $\{X^n\}_{n=0}^\infty$, $X^n:[-\sigma,r]\times\Omega\to\mathscr{K}_c(\mathbb{R}^d)$ *given by*

$$X^0(t,\omega) \overset{\mathbb{P}.1}{=} \begin{cases} \varphi(t,\omega) & \text{if } t\in[-\sigma,0],\\ \varphi(0,\omega) & \text{if } t\in[0,r],\end{cases}$$

*and for* $n=1,2,\ldots$

$$X^n(t,\omega) \overset{\mathbb{P}.1}{=} \begin{cases} \varphi(t,\omega) & \text{if } t\in[-\sigma,0],\\ \varphi(0,\omega) & \text{if } t\in[0,r]_1^n,\\ \varphi(0,\omega)\ominus(-1)\int_0^t F_\omega(s,X_s^n)ds & \text{if } t\in[0,r]_2^n\cup\ldots\cup[0,r]_n^n,\end{cases}$$

*is well defined (i.e. the foregoing H-differences do exist). Then there exists at least one solution $X(t,\omega)$ to the random set-valued differential equation (3.1) on the interval $[0,r]$.*

*Proof.* Let us now define for $n \in \mathbb{N}$ the interval

$$[0,r]_k^n = \Big[\frac{k-1}{n}r; \frac{k}{n}r\Big],\ k = \overline{1,n}.$$

We have

$$\bigcup_{k=1}^{n} [0,r]_k^n = [0,r] \text{ for } n \in \mathbb{N}..$$

Let us observe that

$$\begin{aligned} D[X^n(t,\omega),\{0\}] &\overset{\mathbb{P}.1}{\leqslant} \sup_{t\in[-\sigma,0]} D[\varphi(t,\omega),\{0\}] + \int_0^{t-r/n} D[F_\omega(s,X_s^n),\{0\}]ds \\ &\overset{\mathbb{P}.1}{\leqslant} \sup_{t\in[-\sigma,0]} D[\varphi(t,\omega),\{0\}] + \int_0^{t-r/n} f(s,\omega)ds \\ &\overset{\mathbb{P}.1}{\leqslant} \sup_{t\in[-\sigma,0]} D[\varphi(t,\omega),\{0\}] + \int_0^{r} f(s,\omega)ds < \infty. \end{aligned}$$

where $\theta \in C_\sigma$ is such that $\theta \equiv \{0\}$ . Therefore the sequence $\{X^n(\cdot,\omega)\}$ is uniformly bounded with $\mathbb{P}.1$.

For $t_1,t_2 \in [0,r]$, $\omega \in \Omega$ and $n \geqslant 2$ we have

$$\begin{aligned} D[X^n(t_1,\omega),X^n(t_2,\omega)] &\overset{\mathbb{P}.1}{\leqslant} \int_{\min\{t_1,t_2\}-r/n}^{\max\{t_1,t_2\}-r/n} D[F_\omega(s,X_s^n),\{0\}]ds \\ &\overset{\mathbb{P}.1}{\leqslant} \int_{\min\{t_1,t_2\}-r/n}^{\max\{t_1,t_2\}-r/n} f(s,\omega)ds. \end{aligned}$$

Hence, if $|t_1-t_2| \to 0$ then $D[X^n(t_1,\omega),X^n(t_2,\omega)] \to 0$ with $\mathbb{P}.1$. This implies the sequence $\{X^n(\cdot,\omega)\}$ is equi-continuous with $\mathbb{P}.1$.

By Arzela- Ascoli theorem, then there exists a subsequence $\{X^{n_m}\} \subset X^n$ which is uniformly convergent to some $X : [-\sigma,r]\times\Omega \to \mathscr{K}_c(\mathbb{R}^d)$ with $\mathbb{P}.1$, i.e, there exists $\Omega_* \subset \Omega$ with $\mathbb{P}(\Omega_*) = 1$ such that for every $\omega \in \Omega$ it holds

$$\sup_{t\in[0,r]} D[X^{n_m}(t,\omega),X(t,\omega)] \to 0, \text{ as } m \to \infty.$$

We shall show that $X : [-\sigma,r]\times\Omega \to \mathscr{K}_c(\mathbb{R}^d)$ is a solution to (3.1). Let $\{n_m\} \subset \mathbb{N}$ be the sequence defined in the preceding steps. For $(t,\omega) \in [\frac{r}{n_m};r]\times\Omega$ we have

$$X^{n_m}(t,\omega) \overset{\mathbb{P}.1}{=} \varphi(0,\omega) \ominus (-1)\int_0^{t-r/n_m} F_\omega(s,X_s^{n_m})ds.$$

Let us notice the following: for every $(t,\omega)\in[0,r]\times\Omega$, there exists $m_0\in\mathbb{N}$ such that for every $m\geqslant m_0$, we can write

$$\begin{aligned}
D[\varphi(0,\omega),&X(t,\omega)+(-1)\int_0^t F_\omega(s,X_s)ds]\\
&\leqslant D[\varphi(0,\omega),X^{n_m}(t,\omega)+(-1)\int_0^{t-r/n_m} F_\omega(s,X_s^{n_m})ds]\\
&\quad + D\Big[X^{n_m}(t,\omega)+(-1)\int_0^{t-r/n_m} F_\omega(s,X_s^{n_m})ds, X(t,\omega)\\
&\quad\quad +(-1)\int_0^t F_\omega(s,X_s)ds\Big]\\
&\leqslant D[X^{n_m}(t,\omega),X(t,\omega)]+D\Big[\int_0^{t-r/n_m} F_\omega(s,X_s^{n_m})ds,\int_0^t F_\omega(s,X_s)ds\Big].
\end{aligned}$$

The first term of the right-hand side of the inequality uniformly converges to zero with $\mathbb{P}.1$. It remains to show that the second summand converges to zero. Let us observe that

$$\begin{aligned}
D&\Big[\int_0^{t-r/n_m} F_\omega(s,X_s^{n_m})ds,\int_0^t F_\omega(s,X_s)ds\Big]\\
&\leqslant D\Big[\int_0^{t-r/n_m} F_\omega(s,X_s^{n_m})ds,\int_0^{t-r/n_m} F_\omega(s,X_s)ds\Big]\\
&\quad + D\Big[\int_0^{t-r/n_m} F_\omega(s,X_s)ds,\int_0^t F_\omega(s,X_s)ds\Big]\\
&\overset{[0,r],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^{t-r/n_m} D_\sigma[F_\omega(s,X_s^{n_m}),F_\omega(s,X_s)]ds + D\Big[\int_{t-r/n_m}^t F_\omega(s,X_s)ds,\{0\}\Big]\\
&\overset{[0,r],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^{t-r/n_m} D_\sigma[X_s^{n_m}(\cdot,\omega),X_s(\cdot,\omega)]ds + \int_{t-r/n_m}^t D[F_\omega(s,X_s),\{0\}]ds\\
&\overset{[0,r],\mathbb{P}.1}{\leqslant} L(\omega)\int_0^{t-r/n_m} \sup_{\overline{r}\in[-\sigma,0]} D[X^{n_m}(\overline{r}+s,\omega),X(\overline{r}+s,\omega)]ds + \int_{t-r/n_m}^t f(t,\omega)ds\\
&\overset{[0,r],\mathbb{P}.1}{=} L(\omega)\int_0^{t-r/n_m} \sup_{\iota\in[s-\sigma,s]} D[X^{n_m}(\iota,\omega),X(\iota,\omega)]ds + \int_{t-r/n_m}^t f(t,\omega)ds.
\end{aligned}$$

As

$$D[X^{n_m}(t,\omega),X(t,\omega)] \overset{[0,r],\mathbb{P}.1}{\leqslant} 2f(t,\omega)$$

we have Lebesgue Dominated Convergence Theorem that

$$\int_0^t \sup_{\iota\in[s-\sigma,s]} D[X^{n_m}(\iota,\omega),X(\iota,\omega)]ds\to 0,\ \text{as } m\to\infty, \text{for every } t\in[0,r] \text{ with } \mathbb{P}.1,$$

and

$$\int_{t-r/n_m}^t f(t,\omega)ds\to 0 \text{ as } m\to\infty,\ \text{with } \mathbb{P}.1,$$

for every $t \in [0,r]$, $\omega \in \Omega$. Hence, we obtain

$$\varphi(0,\omega) \overset{[0,r],\mathbb{P}.1}{=} X(t,\omega) + (-1)\int_0^t F_\omega(s, X_s^n)ds.$$

This completes the proof.

## 4. Illustrative examples

In this section we consider two examples which illustrate the main results of the paper.

EXAMPLE 1. Let us consider a class of random set-valued functional differential equations with distributed delay. For $m \in \mathbb{N}$ and times $0 < \sigma_1 < \ldots < \sigma_m < \sigma$. Consider the problem initial condition as follows:

$$\begin{cases} X'(t,\omega) \overset{\mathbb{P}.1}{=} \int_{-\sigma}^0 G_0(s,\omega,X(t+s,\omega)) + \sum\limits_{i=1}^m G_i(s,\omega,X(t-\sigma_i,\omega)) & \text{for } t \in [0,b], \\ X(t,\omega) \overset{\mathbb{P}.1}{=} \varphi(t,\omega) & \text{for } t \in [-\sigma,0] \end{cases} \tag{4.1}$$

where $(\Omega, \mathscr{A}, \mathbb{P})$ is a complete probability space, and

$$C_\sigma = C([-\sigma,0] \times \Omega; \mathscr{K}_c(\mathbb{R}^d)),$$
$$X : [-\sigma,b] \times \Omega \to \mathscr{K}_c(\mathbb{R}^d),$$
$$G_i : [0,b] \times \Omega \times C_\sigma \to \mathscr{K}_c(\mathbb{R}^d),\ i = \overline{1,m}$$

are some set-valued random mappings.

Assume that $G_{i,\omega} : [0,b] \times \Omega \to \mathscr{K}_c(\mathbb{R}^d)$, $i = \overline{1,m}$ satisfy the following hypotheses:

(G1) $G_{i,\cdot} : \Omega \to \mathscr{K}_c(\mathbb{R}^d)$, $i = \overline{1,m}$ are set-valued random variables for $t \in [0,b]$, $\varphi \in C_\sigma$ and $\varphi$ is a set-valued stochastic process,

(G2) with $\mathbb{P}.1$ the mapping $G_{i,\omega} : [0,b] \times \Omega \to \mathscr{K}_c(\mathbb{R}^d)$, $i = \overline{1,m}$ are continuous set-valued mapping for every $(t_0, \varphi_0) \in [0,b] \times C_\sigma$ and $\omega \in \Omega$,

(G3) there exist stochastic processes $L_i : [0,b] \times \Omega \to (0,\infty)$ such that $L_i(\cdot,\omega)$ with $\mathbb{P}.1$ and for $i = \overline{1,m}$, $t \in [0,b]$, $\varphi, \psi \in C_\sigma$,

$$D[G_{i,\omega}(t,\varphi), G_{i,\omega}(t,\psi)] \overset{\mathbb{P}.1}{\leqslant} L_i(t,\omega) D_\sigma[\varphi,\psi]$$

(G4) for $\varphi \in C_\sigma$,

$$D[G_{i,\omega}(t,\varphi), \{0\}] \overset{\mathbb{P}.1}{\leqslant} Q_i,$$

where $Q_i > 0$, $i = \overline{1,m}$.

PROPOSITION 1. Assume that $G_{i,\omega} : [0,b] \times \Omega \to \mathscr{K}_c(\mathbb{R}^d)$, $i = \overline{1,m}$ satisfy assumptions (G1)-(G4), then the problem (4.1) has a unique solution.

*Proof.* It is easy to prove assumptions (F1)-(F2) are satisfied from (G1)-(G2). By (G3) we have the right-hand side of (4.1) satisfies (3.3), and (G4) satisfies (3.4). This completes the proof.

EXAMPLE 2. Suppose that $X : [-1,1] \times \Omega \to \mathscr{K}_c(\mathbb{R})$ and $\Omega = (0, \pi/2)$. Let us consider the following random interval-valued functional differential equation:

$$\begin{cases} X'(t,\omega) \overset{[0,1],\mathbb{P}.1}{=} -X(t-1,\omega), \\ X(t,\omega) \overset{[-1,0],\mathbb{P}.1}{=} [-1, 1-2t]\cos\omega. \end{cases} \tag{4.2}$$

Let us denote $X(t,\omega) \overset{\mathbb{P}.1}{=} [\underline{X}(t,\omega), \overline{X}(t,\omega)]$ for each $t$. Then it holds

$$X'(t,\omega) \overset{\mathbb{P}.1}{=} [\overline{X}'(t,\omega), \underline{X}'(t,\omega)].$$

Therefore we arrive to the system random interval-valued functional differential equation:

$$\begin{cases} \underline{X}'(t,\omega) \overset{[0,1],\mathbb{P}.1}{=} -\underline{X}(t-1,\omega), \\ \overline{X}'(t,\omega) \overset{[0,1],\mathbb{P}.1}{=} -\overline{X}(t-1,\omega), \\ \underline{X}(t,\omega) \overset{[-1,0],\mathbb{P}.1}{=} -\cos\omega, \\ \overline{X}(t,\omega) \overset{[-1,0],\mathbb{P}.1}{=} (1-2t)\cos\omega. \end{cases} \tag{4.3}$$

By solving (4.3), then the solution of the system random interval-valued functional differential equation (4.2) is derived as follows:

$$X(t,\omega) = \begin{cases} [-1, 1-2t]\cos\omega, & \text{for } t \in [-1,0], \\ [-1-t, 1-3t+t^2]\cos\omega, & \text{for } t \in [0,1]. \end{cases} \tag{4.4}$$

The boundaries of $X$ together with the solution (4.2) are illustrated in Figure 1 on the next page.

*Acknowledgements.* The author would like to express his gratitude to the Editor-in-Chief, the Associate Editor and the anonymous referees for their helpful comments.

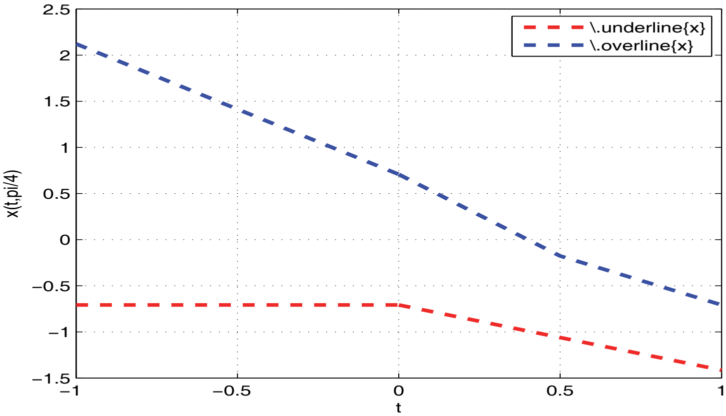


Figure 1: Graph of the solution to (4.4).

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(Received April 16, 2013)

(Revised September 3, 2013)

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