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# EXISTENCE AND MULTIPLICITY OF SOLUTIONS TO A $p(x)$–LAPLACIAN EQUATION WITH NONLINEAR BOUNDARY CONDITION ON UNBOUNDED DOMAIN

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*Abstract.* We study the existence and multiplicity of positive solutions for the nonlinear boundary value problems involving the $p(x)$-Laplacian of the form

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u)+b(x)|u|^{p(x)-2}u=f(x,u) & \text{in } \Omega\subset\mathbb{R}^N,\\ a(x)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}=g(x,u) & \text{on } \Gamma=\partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is an unbounded domain with non-compact, smooth boundary $\Gamma=\partial\Omega$, $p\in C^{0,1}(\Omega)$ and $1<p^-\leqslant p(x)\leqslant p^+<N$, $a,b$ are suitable weights. By using the variational methods, we prove that there exist multiple solutions provided $f$ and $g$ are given appropriate assumptions.

## 1. Introduction

In this paper, we study the existence and multiplicity of solutions for the nonlinear elliptic boundary value problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u)+b(x)|u|^{p(x)-2}u=f(x,u) & \text{in } \Omega\subset\mathbb{R}^N,\\ a(x)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}=g(x,u) & \text{on } \Gamma=\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is an unbounded domain with non-compact, smooth boundary $\Gamma=\partial\Omega$ (for example $\Omega$ is a cylinder domain); $\frac{\partial}{\partial\nu}$ is the outer unite normal derivative; $p(x)\in C^{0,1}(\Omega)$, $1<p^-\leqslant p(x)\leqslant p^+<N$, $0<a_0<a(x)\in L^\infty(\Omega)$, and $b(x)$ is a positive and continuous function defined in $\mathbb{R}^N$, such that

$$\frac{c}{(1+p(x))^{p(x)}}\leqslant b(x)\leqslant \frac{C}{(1+p(x))^{p(x)}},$$

where $c,C$ are two positive constants; $f(x,u)$ and $g(x,u)$, are two Carathéodory functions defined on $\Omega\times\mathbb{R}$ and $\Gamma\times\mathbb{R}$ respectively, and satisfy

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**($\mathbf{f_0}$)**, $f(\cdot,0)=0,\ |f(x,t)|\leqslant f_0(x)+f_1(x)|t|^{q(x)-1}$, and

$$1<q^-\leqslant q(x)<<\frac{Np(x)}{N-p(x)},^{1}$$

where $f_i$ are nonnegative, measurable functions which satisfy the hypothesis: There exists a function $\alpha_1(x)$ defined in $\mathbb{R}^N$ such that $-N<<\alpha_1<<q(x)\frac{N-p(x)}{Np(x)}-N$, and for $w_1=\frac{1}{(1+|x|)^{\alpha_1}}$, we have

$$0\leqslant f_i(x)\leqslant C_f w_1 \quad \text{a.e. } x\in\Omega, i=1,2, \text{ and } f_1\in L^{\frac{q(x)}{q(x)-1}}(\Omega;w_1^{\frac{1}{1-q(x)}}),$$

where $C_f$ is a positive constant;

**($\mathbf{g_0}$)**, $g(\cdot,0)=0,\ |g(x,t)|\leqslant g_0(x)+g_1(x)|t|^{r(x)-1}$, and

$$1<r^-\leqslant r(x)<<\frac{(N-1)p(x)}{N-p(x)},$$

where $g_i$ are nonnegative, measurable functions which satisfy the hypothesis: There exists a function $\alpha_2(x)$ defined in $\mathbb{R}^N$ such that $-N<\alpha_2<<r(x)\frac{N-p(x)}{p(x)}-N+1$, and for $w_2=\frac{1}{(1+|x|)^{\alpha_2}}$, we have

$$0\leqslant g_i(x)\leqslant C_g w_2 \quad \text{a.e. } x\in\Gamma, i=1,2, \text{ and } g_0\in L^{\frac{r(x)}{r(x)-1}}(\Omega;w_2^{\frac{1}{1-r(x)}}),$$

where $C_g$ is a positive constant.

Throughout this paper, we also assume that

**($\mathbf{f_1}$)**, $\lim_{t\to 0}\frac{f(x,t)}{|t|^{p^+-1}}=0$, uniformly in $x$;

**($\mathbf{f_2}$)**, $\exists M_1>1$ and $\theta_1>p^+$, such that, $0<\theta_1F(x,t)\leqslant f(x,t)t$ for all $|t|\geqslant M_1$ a.e. $x\in\Omega$;

**($\mathbf{f_3}$)**, $f(x,-t)=-f(x,t)$ for all $x\in\Omega, t\in\mathbb{R}^N$;

**($\mathbf{g_1}$)**, $\lim_{t\to 0}\frac{g(x,t)}{|t|^{p^+-1}}=0$ uniformly in $x$;

**($\mathbf{g_2}$)**, $\exists M_2>1$ and $\theta_2>p^+$, such that $0<\theta_2G(x,t)\leqslant g(x,t)t$, for all $|t|\geqslant M_2$ a.e. $x\in\Gamma$;

**($\mathbf{g_3}$)**, $g(x,-t)=-g(x,t)$.

The $p(x)$-Laplace operator in (1.1) is a special case of the divergence form operator $-\operatorname{div}(a(x,\nabla u))$ which appears in many fields such as nonlinear electrorheological fluid (see [21]) and elastic mechanics (see [28]), the nonlinear boundary condition describes a flux through the boundary which depends on the solution itself in a nonlinear manner. For the physical motivation of such boundary conditions, we refer to [17].

In recent years, many authors have studied the nonlinear boundary problems involving the $p(x)$-Laplacian when $\Omega$ is bounded or unbounded domain, see e.g. [11,

[1] We say $f(x)<<g(x)$ on $\Omega$ to indicate the fact that $\inf_{x\in\Omega}(g(x)-f(x))>0$.

15, 24]. When $p(x) \equiv p$ ($p$ is a constant), there have been numerous studies on the $p$-Laplace equation, we refer the readers to see [2, 3, 4, 6, 7, 12, 22, 23, 25, 26] and the references therein. More precisely, In [2, 12], the authors studied the "convex concave" case; and in [3], P. Amster, M. C. Mariani and O. Mendez used the degree theory and the upper and lower solutions method to studied the nonlinear bounded problem. A. C. Cavalheiro's paper [5] and K. Pflüger's papers [18, 19, 20] studied the existence of solutions to the $p$-Laplace equation with nonlinear boundary conditions by using variational methods. We notice that the method used in [5, 18, 20]is base on K. Pflüfer's works of the compact embedding and compact trace of the weighted Sobolev space defined on unbounded domain (see [19]). For the $p(x)$-laplace equation, the corresponding problems are new and interesting. J. H. Zhao and P. H. Zhao in [25, 26] studied the $p(x)$-Laplace equation with the case $f(x,u) = |u|^{\frac{Np(x)}{N-p(x)}-2}u$ which is a critical case. X. L. Fan in [10] studied the existence solutions of the Dirichlet problems $p(x)$-Laplacian equation. Q. H. Zhang [27] studied the radial solutions of the $p(x)$-Laplacian problem in $\mathbb{R}^N$. For other problems for the $p(x)$-Laplacian, we refer the readers to [9, 10, 27].

This paper is divided into three sections. In section 2, we recall some basic facts about the weighted variable exponent Lebesgue and Sobolev spaces. In section 3, we give the main results, which contains four Theorems and a Corollary corresponding to $f,g$ both are "sublinear" (Theorem 1); "superlinear" (Theorems 2 and 3) and $f,g$ satisfy "convex concave" case (Theorem 4 and Corollary 1).

## 2. Weighted variable exponent Lebesgue space and weighted variable exponent Sobolev space

Let $\Omega \subset \mathbb{R}^N$ be a domain with non-empty boundary $\partial\Omega$, denote

$$L_+^\infty(\Omega) = \{p \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1\}.$$

For $p \in L_+^\infty(\Omega)$, denote

$$p^- = p^-(\Omega) = \text{essinf}_{x\in\Omega} p(x), \quad p^+ = p^+(\Omega) = \text{esssup}_{x\in\Omega} p(x).$$

Let $w, v_0, v_1$ are positive measurable real valued and a.e. finite functions defined in $\mathbb{R}^N$. For $p \in L_+^\infty(\Omega)$, define

$$L^{p(x)}(\Omega;w) = \left\{u | u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega w(x)|u(x)|^{p(x)}\mathrm{d}x < \infty\right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega;w)} = |u|_{p(x),\Omega,w} = \inf\left\{\lambda > 0 : \int_\Omega w(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\}.$$

When $w(x) \equiv 1$, we denote $L^{p(x)}(\Omega)$ instead of $L^{p(x)}(\Omega;w)$ and denote $|u|_{p(x),\Omega}$ instead of $|u|_{p(x),\Omega,w}$.

Define

$$W^{1,p(x)}(\Omega;v_0,v_1)=\left\{u\in L^{p(x)}(\Omega;v_0):|\nabla u(x)|\in L^{p(x)}(\Omega;v_1)\right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega;v_0,v_1)}=\|u\|_{1,p(x),\Omega,v_0,v_1}=|u|_{p(x),\Omega,v_0}+|\nabla u|_{p(x),\Omega,v_1}.$$

When $v_0\equiv v_1\equiv 1$, we denote $W^{1,p(x)}(\Omega)$ instead of $W^{1,p(x)}(\Omega;1,1)$ and $\|u\|_{1,p(x),\Omega}$ for the norm on it.

As usual, we denote $C,C_i,i=1,,2,\cdots,$ by the generic positive constants throughout his paper. We also assume throughout this paper that $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$ functions restricted on $\Omega$, and $E$ be the weighted Sobolev space as the completion of $C_\delta^\infty(\Omega)$ under the norm

$$\|u\|=\inf\left\{\lambda>0:\int_\Omega|\nabla\frac{u}{\lambda}|^{p(x)}+(1+|x|)^{-p(x)}|\frac{u}{\lambda}|^{p(x)}\mathrm{d}x\leqslant 1\right\},$$

and from the assumptions in Section 1, it is easy to verify that the norm

$$\|u\|_E=\inf\left\{\lambda>0:\int_\Omega a(x)|\nabla\frac{u}{\lambda}|^{p(x)}+b(x)|\frac{u}{\lambda}|^{p(x)}\mathrm{d}x\leqslant 1\right\}$$

is an equivalent norm of $\|\cdot\|$.

On the basic properties of the space $L^{p(x)}(\Omega;w)$ and $E$, we refer to [1, 8, 14, 15] for more details. Here we only display some facts that will be used later.

PROPOSITION 1. (See [8, 14].) The spaces $L^{p(x)}(\Omega;w)$ and $E$ are separable and reflexive Banach spaces.

PROPOSITION 2. (See [8, 14, 15].) Set $\phi(u)=\int_\Omega w(x)|u(x)|^{p(x)}\mathrm{d}x$, for $u,u_k\in L^{p(x)}(\Omega;w)$, we have

(1) For $u\neq 0,|u|_{p(x),\Omega,w}=\lambda\Leftrightarrow\phi(\frac{u}{\lambda})=1$;
(2) $|u|_{p(x),\Omega,w}<1(=1;>1)\Leftrightarrow\phi(u)<1(=1;>1)$;
(3) If $|u|_{p(x),\Omega,w}>1$, then $|u|_{p(x),\Omega,w}^{p^-}\leqslant\phi(u)\leqslant|u|_{p(x),\Omega,w}^{p^+}$;
(4) If $|u|_{p(x),\Omega,w}<1$, then $|u|_{p(x),\Omega,w}^{p^+}\leqslant\phi(u)\leqslant|u|_{p(x),\Omega,w}^{p^-}$;
(5) $\lim_{k\to\infty}|u_k|_{p(x),\Omega,w}=0\Leftrightarrow\lim_{k\to\infty}\phi(u_k)=0$;
(6) $|u_k|_{p(x),\Omega,w}\to\infty\Leftrightarrow\phi(u_k)\to\infty$.

Similar to Proposition 2, we have

PROPOSITION 3. Set $I(u)=\int_\Omega\left(a(x)|\nabla u(x)|^{p(x)}+b(x)|u(x)|^{p(x)}\right)\mathrm{d}x$, for $u,u_k\in E$, we have

(1) For $u\neq 0,\|u\|_E=\lambda\Leftrightarrow I(\frac{u}{\lambda})=1$;
(2) $\|u\|_E<1(=1;>1)\Leftrightarrow I(u)<1(=1;>1)$;

(3) If $\|u\|_E > 1$, then $\|u\|_E^{p^-} \leqslant I(u) \leqslant \|u\|_E^{p^+}$;
(4) If $\|u\|_E < 1$, then $\|u\|_E^{p^+} \leqslant I(u) \leqslant \|u\|_E^{p^-}$;
(5) $\lim_{k\to\infty} \|u_k\|_E = 0 \Leftrightarrow \lim_{k\to\infty} I(u_k) = 0$;
(6) $\|u_k\|_E \to \infty \Leftrightarrow I(u_k) \to \infty$.

PROPOSITION 4. (See [15].) If $1 < p(x) \leqslant q(x) < +\infty, 0 < w(x) \leqslant v(x)$ a.e. $x \in \Omega$, and $|\Omega| < \infty$, then

$$|u(x)|_{p(x),\Omega,w} \leqslant C|u(x)|_{q(x),\Omega,v},$$

where $C$ is independent of $u$.

Let $L^{q(x)}(\Omega;w_1)$ and $L^{r(x)}(\Gamma;w_2)$ be the weighted variable exponent Sobolev spaces with weight functions $w_i = (1+|x|)^{\alpha_i(x)}, i = 1,2, \alpha_i \in C(\mathbb{R}^N)$, then we have following embedding and trace embedding theorem.

PROPOSITION 5. (See [15].) Let $\Omega$ be a (bounded or unbounded) domain in $\mathbb{R}^N$ with smooth boundary, $p,q,r \in L_+^\infty(\Omega)$ and $p \in C^{0,1}(\overline{\Omega})$, $p^+ < N$, then
(1) If $1 < q(x) < \infty$, $\frac{N}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ and

$$-N < \alpha_1(x) < N(p(x)-1), \frac{\alpha_1(x)}{q(x)} + \frac{N}{q(x)} - \frac{N}{p(x)} + 1 \geqslant 0,$$

then the embedding $E \hookrightarrow L^{q(x)}(\Omega;w_1)$ is continuous; if the two inequalities above are replaced by

$$\operatorname{essinf}_{x\in\Omega}\left(\frac{N}{q(x)} - \frac{N}{p(x)} + 1\right) > 0 \text{ and } \operatorname{esssup}_{x\in\Omega}\left(\frac{\alpha_1(x)}{q(x)} + \frac{N}{q(x)} - \frac{N}{p(x)} + 1\right) < 0,$$

then the corresponding embedding is compact.
(2) If $1 < r(x) < \infty$, $\frac{N-1}{r(x)} - \frac{N}{p(x)} + 1 \geqslant 0$ and

$$\operatorname*{ess\,sup}_{x\in\Omega}\left(\frac{\alpha_2(x)}{r(x)} + \frac{N-1}{r(x)} - \frac{N}{p(x)} + 1\right) < 0,$$

then the corresponding trace embedding is compact.

PROPOSITION 6. (see [9]) Denote

$$I(u) = \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x, \qquad \text{for all } u \in E.$$

Then $I \in C^1(E,\mathbb{R})$ and the derivative operator $I'$ of $I$ satisfies

$$< I'(u), v >= \int_\Omega (a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv)\mathrm{d}x,$$

and we have
(1), $I' : E \to E^*$ is a continuous, bounded and strictly monotone operator;
(2), $I'$ is a mapping of type $(S_+)$, i.e., if $u_n \rightharpoonup u$ in $E$ and $\limsup_{n\to\infty} < I'(u_n) - I'(u), u_n - u > \leqslant 0$, then $u_n \to u$ in $E$;
(3), $I' : E \to E^*$ is a homeomorphism.

## 3. Main results and proofs

DEFINITION 1. We say that $u \in E$ is a weak solution of Problem (1.1) if

$$\int_\Omega (a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v + b(x)|u|^{p(x)-2}uv)\mathrm{d}x = \int_\Omega f(x,u)v\mathrm{d}x + \int_\Gamma g(x,u)v\mathrm{d}\Gamma,$$

holds for $\forall v \in E$. The corresponding energy functional of problem (1.1) is

$$J(u) = \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x - \int_\Gamma G(x,u)\mathrm{d}\Gamma,$$

where $F$ and $G$ denote the primitive functions of $f$ and $g$ with respect to the second variable, i.e., $F(x,t) = \int_0^t f(x,s)\mathrm{d}s, G(x,t) = \int_0^t g(x,s)\mathrm{d}s$.

We notice that the operator $J$ is a $C^1$ functional and the critical points of $J$ are weak solutions of the problem (1.1). We denote by $N_f, N_F, N_g, N_G$ the corresponding Nemytskii operators. Under the assumptions $(\mathbf{f_0})$, $(\mathbf{g_0})$, we have

LEMMA 1. *The operators*

$$N_f : L^{q(x)}(\Omega; w_1) \to L^{\frac{q(x)}{q(x)-1}}(\Omega; w_1^{\frac{1}{1-q(x)}}), \qquad N_F : L^{q(x)}(\Omega; w_1) \to L^1(\Omega);$$
$$N_g : L^{r(x)}(\Gamma; w_2) \to L^{\frac{r(x)}{r(x)-1}}(\Gamma; w_2^{\frac{1}{1-r(x)}}), \qquad N_G : L^{r(x)}(\Gamma; w_2) \to L^1(\Gamma)$$

*are bounded and continuous.*

*Proof.* We only prove the statements of $N_f$ and $N_F$, the arguments for $N_g$ and $N_G$ can be obtained in a similar way.

Let $q'(x) = \frac{q(x)}{q(x)-1}$ and $u \in L^{q(x)}(\Omega; w_1)$, then by assumption $(\mathbf{f_0})$

$$\begin{aligned}
&\int_\Omega |N_f(u)|^{q'(x)} w_1^{\frac{1}{1-q(x)}}\mathrm{d}x \\
&\qquad \leqslant 2^{q'(x)-1}\left(\int_\Omega f_0^{q'(x)} w_1^{\frac{1}{1-q(x)}}\mathrm{d}x + \int_\Omega f_1^{q'(x)}|u|^{q(x)} q_1^{\frac{1}{1-q(x)}}\mathrm{d}x\right) \\
&\qquad \leqslant 2^{q'(x)-1}\left(C + C_f\int_\Omega w_2|u|^{q(x)}\mathrm{d}x\right) \\
&\qquad \leqslant 2^{(q')^+-1}\left(C + C_f\int_\Omega w_1|u|^{q(x)-1}\mathrm{d}x\right) \\
&\qquad \leqslant 2^{\frac{q^-}{q^--1}-1}\left(C + C_f\int_\Omega w_1|u|^{q(x)-1}\mathrm{d}x\right) \\
&\qquad \leqslant 2^{\frac{1}{q^--1}}\left(C + C_f\int_\Omega w_1|u|^{q(x)-1}\mathrm{d}x\right).
\end{aligned}$$

This shows that $N_f$ is bounded. In a similar way, we obtain

$$\begin{aligned}\int_\Omega |N_F(u)|\mathrm{d}x &\leqslant \int_\Omega f_0|u|\mathrm{d}x + \int_\Omega f_1|u|^{q(x)}\mathrm{d}x \\ &\leqslant |f_0|_{q'(x),\Omega,w_1^{\frac{1}{1-q(x)}}} |u|_{q(x),\Omega,w_1} + C_d\int_\Omega w_1|u|^{1(x)}\mathrm{d}x,\end{aligned}$$

which implies that $N_F$ is bounded. The continuity of these operators follows from the well-know properties of Nemytskii operators.

THEOREM 1. *If* ($\mathbf{f_0}$), ($\mathbf{g_0}$) *hold and* $q^+, r^+ < p^-$, *then the Problem* (1.1) *has a weak solution.*

*Proof.* From ($\mathbf{f_0}$), ($\mathbf{g_0}$), the above Lemma 1 and Proposition 5, we have for $\|u\|_E \geqslant 1$,

$$\begin{aligned}J(u) =& \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x - \int_\Gamma G(x,u)\mathrm{d}\Gamma \\ \geqslant& \frac{1}{p^+}\int_\Omega (a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x - \int_\Omega (f_0|u| + \frac{1}{q(x)}f_1(x)|u|^{q(x)})\mathrm{d}x \\ &- \int_\Gamma (g_0(x)|u| + \frac{1}{r(x)}g_1(x)|u|^{r(x)})\mathrm{d}\Gamma \\ \geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - C_1|u|_{q(x),\Omega,w_1} - \frac{C_f}{q^-}\max\{|u|^{q^-}_{q(x),\Omega,w_1}, |u|^{q^+}_{q(x),\Omega,w_1}\} \\ &- C_2|u|_{r(x),\Gamma,w_2} - \frac{C_g}{r^-}\max\{|u|^{r^-}_{r(x),\Gamma,w_2}, |u|^{r^+}_{r(x),\Gamma,w_2}\} \\ \geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - C_3(\max\{|u|_{q(x),\Omega,w_1}, |u|^{q^-}_{q(x),\Omega,w_1}, |u|^{q^+}_{q(x),\Omega,w_1}\} \\ &+ \max\{|u|_{r(x),\Gamma,w_2}, |u|^{r^-}_{r(x),\Gamma,w_2}, |u|^{r^+}_{r(x),\Gamma,w_2}\}) \\ \geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - C(\|u\|_E + \|u\|_E^{q^-} + \|u\|_E^{q^+} + \|u\|_E^{r^-} + \|u\|_E^{r^+}) \\ \geqslant& \frac{1}{p^+}\|u\|_E^{p^+} - C'(\|u\|_E^{q^+} + \|u\|_E^{r^+}) \\ \to& \infty \qquad \text{as } \|u\|_E \to \infty,\end{aligned}$$

where

$$C_1 = |f_0|_{q'(x),\Omega,w_1^{\frac{1}{1-q(x)}}}, \ \ C_2 = |g_0|_{r'(x),\Gamma,w_2^{\frac{1}{1-r(x)}}}, (r'(x) = \frac{r(x)}{1-r(x)}).$$

Since $q^+, r^+ < p^-$, so the operator $J$ is coercive, and from Proposition 6 and Lemma 1, it is easy to verify that the operator $J$ is weakly lower semicontinuous. Thus $J$ has a minimum point $u$ in $E$, i.e., $u$ is a weak solution of (1.1).

LEMMA 2. *If* $(\mathbf{f_0})$, $(\mathbf{f_2})$, $(\mathbf{g_0})$, $(\mathbf{g_2})$ *hold, then the operator* $J$ *satisfies the (PS) condition.*

*Proof.* Let $\{u_n \in E\}$ is a (PS) sequence, i.e.,

$$\|J(u_n)\| \leqslant M, \quad J'(u_n) \to 0 \text{ as } n \to \infty,$$

we want to prove that $\{u_n\}$ has a convergence subsequence $\{u_k\}$. Define

$$K_F(u) = \int_\Omega F(x,u)\mathrm{d}x, \qquad K_G(u) = \int_\Gamma G(x,u)\mathrm{d}\Gamma.$$

Hence, the directional derivative of the operator $J$ in direction $v \in E$ is

$$< J'(u), v >=< I'(u), v > - < K_F'(u), v > - < K_G'(u), v >,$$

where $< K_F'(u), v >= \int_\Omega f(x,u)v\mathrm{d}x$, $< K_G'(u), v >= \int_\Gamma g(x,u)v\mathrm{d}\Gamma$ and $< I'(u), v >$ is the same as in Proposition 6.

Clearly, under the assumptions $(\mathbf{f_0})$ and $(\mathbf{g_0})$, by using Proposition 6, we have $I' : E \to E^*$ is continuous; From Proposition 5 and Lemma 1, we know $K_F(u)$ and $K_G(u)$ are both weakly continuous and their derivative operators are compact.

On the other hand, for $k$ large enough, we have $| < J'(u_k), u_k > | \leqslant \|u_k\|_E$, and under the assumptions $(\mathbf{f_2})$, $(\mathbf{g_2})$, we have (for convenience, we suppose that $\|u_k\|_E \geqslant 1$)

$$\begin{aligned}
M + \|u_k\|_E \geqslant & J(u_k) - \frac{1}{\theta} < J'(u_k), u_k > \\
\geqslant & (\frac{1}{p^+} - \frac{1}{\theta})\|u_k\|_E^{p^-} - \int_\Omega (F(x,u_k) - \frac{1}{\theta} f(x,u_k)u_k)\mathrm{d}x \\
& - \int_\Gamma (G(x,u_k) - \theta g(x,u_k)u_k)\mathrm{d}x \\
\geqslant & (\frac{1}{p^+} - \frac{1}{\theta})\|u_k\|_E^{p^-},
\end{aligned}$$

where $\theta = \min\{\theta_1, \theta_2\}$, this shows that $\{u_k\}$ is bounded in $E$.

To show that $\{u_k\}$ is a cauchy sequence, we use the following inequalities for $\xi, \eta \in \mathbb{R}^N$ (see [9, 20]).

$$|\xi - \eta|^p \leqslant C(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta), \quad \text{for } p \geqslant 2; \tag{3.1}$$

$$|\xi - \eta|^2 \leqslant C(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta)(|\xi| + |\eta|)^{2-p}, \quad \text{for } 1 < p < 2. \tag{3.2}$$

Assume $\|u_n - u_k\|_E \leqslant 1$, then we can obtain that in the case $p \geqslant 2$,

$$\begin{aligned}
\|u_n - u_k\|_E^{p^+} \leqslant & \int_\Omega (a(x)|\nabla u_n - \nabla u_k|^{p(x)} + b(x)|u_n - u_k|^{p(x)})\mathrm{d}x \\
\leqslant & C(< I'(u_n, u_n - u_k) > - < I'(u_k), u_n - u_k >)
\end{aligned}$$

$$\begin{aligned}&\leqslant C(<J'(u_n),u_n-u_k>-<J'(u_k),u_n-u_k>+<K_F'(u_n)\\&\quad+K_G'(u_n),u_n-u_k>-<K_F'(u_k)+K_G'(u_k),u_n-u_k>)\\&\leqslant C(\|J'(u_n)\|_{E^*})+\|J'(u_k)\|_{E^*}+\|K_F'(u_n)-K_F'(u_k)\|_{E^*}\\&\quad+\|K_G'(u_n)-K_G'(u_k)\|_{E^*})\|u_n-u_k\|_E.\end{aligned}$$

Since $J'(u_k)\to 0$ and $K_F', K_G'$ are compact, there exists a subsequence of $\{u_k\}$ which convergence in $E$.

For the case $1<p^-\leqslant p(x)<2$, we use (3.2) and the Hölder's inequality to obtain that

$$\begin{aligned}\|u_n-u_k\|_E^2\leqslant & C|<I'(u_n,u_n-u_k)>-<I'(u_k),u_n-u_k>|\\&\times\max\{\|u_n\|_E^{p^-}+\|u_k\|_E^{p^-},\|u_n\|_E^{p^+}+\|u_k\|_E^{p^+}\}.\end{aligned}$$

Since $\|u_k\|_E$ is bounded, the same arguments as the above yield a convergent subsequence.

THEOREM 2. *If* ($\mathbf{f_0}$), ($\mathbf{f_1}$), ($\mathbf{f_2}$), ($\mathbf{g_0}$), ($\mathbf{g_1}$), ($\mathbf{g_2}$) *hold and* $q^-, r^->p^+$, *then the Problem* (1.1) *has a nontrivial weak solution.*

*Proof.* It is easy to show that problem (1.1) satisfies all the geometric assumptions of the Mountain Pass Theorem (see [23], Theorem 2.10), and the solution is the Mountain Pass solution.

By Lemma 2, the operator $J$ satisfies $(PS)$ condition on $E$. Since

$$p^+<q^-<q(x)<<\frac{Np(x)}{N-p(x)}\quad\text{and}\quad p^+<r^-<r(x)<<\frac{(N-1)p(x)}{N-p(x)},$$

from Proposition 5, the embeddings $E\hookrightarrow L^{p^+}(\Omega;w_1)$ and $E\hookrightarrow L^{p^+}(\Gamma;w_2)$ are compact.

From the assumptions ($\mathbf{f_0}$), ($\mathbf{f_1}$), ($\mathbf{g_0}$), ($\mathbf{g_1}$), we observe that for any given $\varepsilon>0$, there exists a $C_\varepsilon>0$ such that $|F(x,u)|\leqslant\varepsilon f_0(x)|u|^{p^+}+C_\varepsilon f_1(x)|u|^{q(x)}$ and $|G(x,u)|\leqslant\varepsilon g_0(x)|u|^{p^+}+C_\varepsilon g_1(x)|u|^{r(x)}$, consequently, the inequality

$$\begin{aligned}J(u)\geqslant&\frac{1}{p^+}\int_\Omega(a(x)|\nabla u|^{p(x)}+b(x)|u|^{p(x)})\mathrm{d}x-\int_\Omega(\varepsilon f_0|u|^{p^+}+C_\varepsilon f_1|u|^{q(x)})\mathrm{d}x\\&-\int_\Gamma(\varepsilon g_0|u|^{p^+}+C_\varepsilon g_1|u|^{q(x)})\mathrm{d}\Gamma\\\geqslant&\frac{1}{p^+}\|u\|_E^{p^+}-\varepsilon C_1\|u\|_E^{p^+}-C_\varepsilon C_2(\|u\|_E^{q^-}+\|u\|_E^{r^-})\end{aligned}$$

holds for $\|u\|_E\leqslant 1$, and the right hand side is strictly bigger than $0$. Hence, when $\varepsilon$ and $\|u\|_E=\rho$ sufficiently small, we have $J(u)>0$. In order to use the Mountain Pass Theorem, it remain to show that there exists $u_0\in E, \|u\|_E>\rho$ satisfies $J(u_0)\leqslant 0$.

From ($\mathbf{f_2}$), ($\mathbf{g_2}$), it follows that

$$F(x,u)\geqslant C_3|u|^{\theta_1},\qquad\forall x\in\Omega, |u|\geqslant M_1;$$

$$G(x,u) \geqslant C_4|u|^{\theta_2}, \qquad \forall x \in \Gamma, |u| \geqslant M_2.$$

for $\omega \in E \setminus \{0\}$ and $t > 1$, we have

$$\begin{aligned}
J(t\omega) = &\int_\Omega \frac{1}{p(x)}(a(x)|\nabla t\omega|^{p(x)} + b(x)|t\omega|^{p(x)})\mathrm{d}x \\
&- \int_\Omega F(x,t\omega)\mathrm{d}x - \int_\Gamma G(x,t\omega)\mathrm{d}\Gamma \\
\leqslant & t^{p^+} \int_\Omega (a(x)|\nabla\omega|^{p(x)} + b(x)|\omega|^{p(x)})\mathrm{d}x - Ct^{\theta_1}\int_\Omega w_1|\omega|^{\theta_1}\mathrm{d}x \\
&- Ct^{\theta_2}\int_\Gamma w_2|\omega|^{\theta_2}\mathrm{d}\Gamma \\
\to & -\infty \quad \text{as } t \to +\infty, \quad \text{since } \theta_2 > p^+.
\end{aligned}$$

Notice that there still holds $J(0) = 0$, hence $J$ satisfies the geometric conditions of the Mountain Pass Theorem, and the operator $J$ admits at least one nontrivial critical point.

LEMMA 3. *(see [9]) Let $E$ be a reflexive and separable Banach space, then there exist $\{e_j\} \subset E$ and $\{e_j^* \subset E^*\}$ such that*

$$E = \overline{span\{e_j | j = 1,2,\cdots\}}, \quad E^* = \overline{span\{e_j^* | j = 1,2,\cdots\}},$$

*and*

$$< e_i^*, e_j >= \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $E_j = \mathrm{span}\{e_j\}, Y_k = \oplus_{j=1}^k E_j, Z_k = \oplus_{j=k}^\infty E_j$.

LEMMA 4. (see [9], Lemma 4.9) *If $q, r \in L_+^\infty(\Omega)$ and*

$$1 < q^- \leqslant q(x) << \frac{Np(x)}{N - p(x)}, \quad 1 < r^- \leqslant r(x) << \frac{(N-1)p(x)}{N - p(x)},$$

*denote*

$$\alpha_k = \sup\{|u|_{q(x),\Omega,w_1}; \|u\|_E = 1, u \in Z_k\}; \quad \beta_k = \sup\{|u|_{r(x),\Omega,w_2}; \|u\|_E = 1, u \in Z_k\},$$

*then* $\lim_{k\to\infty}\alpha_k = 0, \lim_{k\to\infty}\beta_k = 0$.

THEOREM 3. *If ($\mathbf{f_0}$), ($\mathbf{f_2}$), ($\mathbf{f_3}$), ($\mathbf{g_0}$), ($\mathbf{g_2}$), ($\mathbf{g_3}$) hold and $q^-, r^- > p^+$, then $J$ has a sequence of critical point $\{u_n\}$ such that $J(u_n) \to +\infty$ and the problem* (1.1) *has infinite many pairs of solutions.*

*Proof.* Under the assumptions ($\mathbf{f_0}$), ($\mathbf{f_2}$), ($\mathbf{f_3}$), ($\mathbf{g_0}$), ($\mathbf{g_2}$), ($\mathbf{g_3}$), it is easy to show that $J$ is an even functional and satisfies $(PS)$ condition, we will prove that if $k$ is a large enough, then there exist $\rho_k > \gamma_k > 0$ such that

$$(A_1): b_k = \inf\{J(u) | u \in Z_k, \|u\|_E = \gamma_k\} \to \infty, \quad (k \to \infty)$$

$$(A_2): a_k = \max\{J(u) | u \in Y_k, \|u\|_E = \rho_k\} \leqslant 0.$$

The assertion of the Theorem can be obtained by the Fountain Theorem (see [23], Theorem 3.6), we assume that $\|u\|_E \geqslant 1$.
$(A_1)$, For any $u \in Z_k$, denote $\|u\|_\Omega = |u|_{q(x),\Omega,w_1}$, $\|u\|_\Gamma = |u|_{r(x),\Gamma,w_2}$ for simplicity, then

$$\begin{aligned}
J(u) =& \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x - \int_\Gamma G(x,u)\mathrm{d}\Gamma \\
\geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - \int_\Omega (f_0 u + \frac{1}{q(x)} f_1 |u|^{q(x)})\mathrm{d}x - \int_\Gamma (g_0 u + \frac{1}{r(x)} g_1 |u|^{r(x)})\mathrm{d}\Gamma \\
\geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - \int_\Omega (f_0 u + \frac{C_f}{q^-} |u|^{q(x)})\mathrm{d}x - \int_\Gamma (g_0 u + \frac{C_g}{r^-} w_2 |u|^{r(x)})\mathrm{d}\Gamma \\
\geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - C_1\|u\|_\Omega - \frac{C_f}{q^-}\max\{\|u\|_\Omega^{q^+}; \|u\|_\Omega^{q^-}\} \\
& - C_2\|u\|_\Gamma - \frac{C_g}{r^-}\max\{\|u\|_\Gamma^{r^+}; \|u\|_\Gamma^{r^-}\} \\
\geqslant& \begin{cases}
\frac{1}{p^+}\|u\|_E^{p^-} - C_1 - \frac{C_f}{q^-} - C_2 - \frac{C_g}{r^-}, & \text{if } \|u\|_\Omega \leqslant 1, \|u\|_\Gamma \leqslant 1 \\
\frac{1}{p^+}\|u\|_E^{p^-} - C_1 - \frac{C_f}{q^-} - (C_2 - \frac{C_g}{r^-})\|u\|_\Gamma^{r^+} & \text{if } \|u\|_{\Omega \leqslant 1}, \|u\|_\Gamma \geqslant 1 \\
\frac{1}{p^+}\|u\|_E^{p^-} - (C_1 + \frac{C_f}{q^-})\|u\|_\Omega^{q^+} - C_2 - \frac{C_g}{r^-} & \text{if } \|u\|_\Omega \geqslant 1, \|u\|_\Gamma \leqslant 1 \\
\frac{1}{p^+}\|u\|_E^{p^-} - (C_1 + \frac{C_f}{q^-})\|u\|_\Omega^{q^+} - (C_2 - \frac{C_g}{r^-})\|u\|_\Gamma^{r^+} & \text{if } \|u\|_\Omega \geqslant 1, \|u\|_\Gamma \geqslant 1
\end{cases} \\
\geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - (C_1 + \frac{C_f}{q^-} + C_2 + \frac{C_g}{r^-}) \cdot \max\{\|u\|_\Omega^{q^+}, \|u\|_\Gamma^{r^+}\}.
\end{aligned}$$

If $\max\{\|u\|_\Omega^{q^+}, \|u\|_\Gamma^{r^+}\} = \|u\|_\Omega^{q^+}$, we have

$$\begin{aligned}
J(u) \geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - C\|u\|_\Omega^{q^+} \\
\geqslant& \frac{1}{p^+}\|u\|_E^{p^-} - C\alpha_k^{q^+} \frac{1}{p^+}\|u\|_E^{q^+},
\end{aligned}$$

choose $\gamma_k = (C\alpha_k^{q^+} q^+)^{\frac{1}{p^- - q^+}}$, then

$$\begin{aligned}
J(u) \leqslant& \frac{1}{p^+}(C\alpha_k^{q^+} q^+)^{\frac{1}{p^- - q^+}} - C\alpha_k^{q^+}(C\alpha_k^{q^+} q^+)^{\frac{1}{p^- - q^+}} \\
=& \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\gamma_k \to \infty \quad (k \to \infty),
\end{aligned}$$

because of $p^+ < q^-$ and $\alpha_k \to 0$ as $k \to \infty$.

If $\max\{\|u\|_\Omega^{q^+}, \|u\|_\Gamma^{r^+}\} = \|u\|_\Gamma^{r^+}$, we have

$$J(u) \geqslant \frac{1}{p^+}\|u\|_E^{p^-} - C\|u\|_\Gamma^{r^+}$$

$$\geqslant \frac{1}{p^+}\|u\|_E^{p^-} - C\beta_k^{r^+}\frac{1}{p^+}\|u\|_E^{r^+},$$

choose $\gamma_k = (C\beta_k^{r^+} r^+)^{\frac{1}{p^- - r^+}}$, then

$$J(u) \leqslant \Big(\frac{1}{p^+} - \frac{1}{r^+}\Big)\gamma_k^{p^-} \to \infty \quad (k \to \infty),$$

because of $p^+ < r^-$ and $\beta_k \to 0$ as $k \to \infty$.

$(A_2)$, From $(\mathbf{f_2})$, $(\mathbf{g_2})$, we have

$$\begin{aligned} F(x,t) &\leqslant C_3|t|^{\theta_1} - C_4, \quad \forall x \in \Omega, |t| \geqslant M_1;\\ G(x,t) &\leqslant C_5|t|^{\theta_2} - C_6, \quad \forall x \in \Gamma, |t| \geqslant M_2. \end{aligned}$$

By $\theta_1 > p^+, \theta_2 > p^+$

$$\begin{aligned} J(u) &= \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x - \int_\Gamma G(x,u)\mathrm{d}\Gamma\\ &\leqslant \frac{1}{p^-}\|u\|_E^{p^+} - \int_\Omega (C_3|u|^{\theta_1} - C_4)\mathrm{d}x - \int_\Gamma (C_5|u|^{\theta_2} - C_6)\mathrm{d}\Gamma\\ &\leqslant \frac{1}{p^-}\|u\|_E^{p^+} - C_3\int_\Omega |u|^{\theta_1}\mathrm{d}x - C_5\int_\Gamma |u|^{\theta_2}\mathrm{d}\Gamma + C. \end{aligned}$$

Notice that $\dim Y_k = k$ implies that all norms are equivalent in $Y_k$, hence we have

$$\begin{aligned} J(u) \leqslant &\frac{1}{p^-}\|u\|_E^{p^+} - C_3'\|u\|_E^{\theta_1} - C_5'\|u\|_E^{\theta_2} + C\\ \to &-\infty \qquad \text{as } \|u\|_E \to \infty \text{ for } u \in Y_k, \end{aligned}$$

because of $p^+ < \theta_1, \theta_2$. So we can choose $\rho_k$ big enough, then the proof of the Theorem 3 is completed.

To study "concave and convex" problem, we focus on the form

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in}\Omega \subset \mathbb{R}^N,\\ a(x)|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \mu g(x,u) & \text{on}\Gamma = \partial\Omega, \end{cases} \tag{3.3}$$

where $\mu, \lambda \in \mathbb{R}$, and $\Omega \subset \mathbb{R}^n$ is still an unbounded domain with non-compact, smooth boundary $\Gamma$; $p(x)$, $a(x)$, $b(x)$ as the Section 1; $f(x,u)$ and $g(x,u)$ are Carathéodory functions on $\Omega \times \mathbb{R}^n$ and $\Gamma \times \mathbb{R}^n$. For the Problem (3.3), we have the following theorem:

THEOREM 4. *If* $(\mathbf{f_0})$, $(\mathbf{f_2})$, $(\mathbf{f_3})$, $(\mathbf{g_0})$, $(\mathbf{g_3})$ *hold and* $q^- > p^+, r^+ < p^-$, *then*
(a) *for every* $\lambda > 0, \mu \in \mathbb{R}$ *the problem* (3.3) *has a sequence of solutions* $\{u_k\}$ *such*

*that* $J_{\lambda,\mu}(u_k) \to \infty, k \to \infty$*;*
(b) *for every* $\mu > 0, \lambda \in \mathbb{R}$ *and in (* $f_0$ *), (* $g_0$ *), we have*

$$|f(x,t)| = f_1(x)|t|^{q(x)-1}, \ \ |g(x,t)| = g_1(x)|t|^{r(x)-1},$$

*the problem* (3.3) *has a sequence of solutions* $\{u_k\}$ *such that*

$$J_{\lambda,\mu}(u_k) < 0 \quad and \quad J_{\lambda,\mu}(u_k) \to 0, k \to \infty.$$

*Proof.* The proof of part (a) follows by the Fountain Theorem, and part (b) follows by the Dual Fountain Theorem (see [23], Theorem 3.18).

(a). It is sufficient to prove that $J_{\lambda,\mu}(u)$ satisfies the $(PS)$ condition, other proofs are similar to the proof of Theorem 2. Assume

$$\{u_n\} \subset E, J_{\lambda,\mu}(u_n) < M, J^{'}(u_n) \to 0 \text{ as } n \to \infty.$$

Let $\|u\|_E \geqslant 1$ for convenience, by the conditions $(\mathbf{f_0})$, $(\mathbf{f_2})$, $(\mathbf{g_0})$, Lemma 3.1 and Proposition 5, we have for $n$ big enough

$$\begin{aligned}
M + \|u\|_E &\geqslant J(u_n) - \frac{1}{\theta_1} < J'(u_n), u_n > \\
&\geqslant (\frac{1}{p^+} - \frac{1}{\theta_1}) \int_\Omega \frac{1}{p(x)} (a(x)|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}) \mathrm{d}x \\
&\quad - \lambda \int_\Omega (F(x,u_n) - \frac{1}{\theta_1} f(x,u_n)u_n) \mathrm{d}x \\
&\quad - \mu \int_\Gamma (G(x,u_n) - \frac{1}{\theta_1} g(x,u_n)u_n) \mathrm{d}\Gamma \\
&\geqslant (\frac{1}{p^+} - \frac{1}{\theta_1}) \|u_n\|_E^{p^+} - \mu \int_\Gamma (G(x,u_n) - \frac{1}{\theta_1} g(x,u_n)u_n) \mathrm{d}\Gamma \\
&\geqslant (\frac{1}{p^+} - \frac{1}{\theta_1}) \|u_n\|_E^{p^+} - (1 + \frac{1}{\theta_1}) |\mu| \int_\Gamma (g_0 u_n + \frac{g_1}{r(x)} |u_n|^{r(x)}) \mathrm{d}\Gamma \\
&\geqslant (\frac{1}{p^+} - \frac{1}{\theta_1}) \|u_n\|_E^{p^+} \\
&\quad - (1 + \frac{1}{\theta_1}) |\mu| \left( C_1 \|u_n\|_\Omega + \frac{C_g}{r^-} \max\{\|u_n\|_\Gamma^{r^+}, \|u_n\|_\Gamma^{r^-}\} \right) \\
&\geqslant (\frac{1}{p^+} - \frac{1}{\theta_1}) \|u_n\|_E^{p^+} - C|\mu| (\|u_n\|_E + \|u_n\|_E^{r^-} + \|u_n\|_E^{r^+}),
\end{aligned}$$

where $C_1 = C_1(g_0)$. Since $\theta_1 > p^+, r^+ < p^-$, it is easy to verify that $\{u_n\}$ is bounded in $E$.

(b) By the Dual Fountain Theorem, we only need to prove that: there exits $\rho_k > \gamma_k > 0$ such that

$(B_1)$ $\quad a_k := \inf\{J_{\lambda,\mu}(u) | u \in Z_k, \|u\|_E = \rho_k\} \geqslant 0;$

$(B_2)$ $b_k := \max\{J_{\lambda,\mu}(u) | u \in Y_k, \|u\|_E = \gamma_k\} \leqslant 0$;

$(B_3)$ $d_k := \inf\{J_{\lambda,\mu}(u) | u \in Z_k, \|u\|_E \leqslant \rho_k\} \to 0, k \to \infty$;

$(B_4)$ $J_{\lambda,\mu}(u)$ satisfies the $(PS)_C^*$ (see [23], Definition 3.17) condition$\forall C \in [d_{k_0}, 0[$.

Now, we show that the above conditions hold.

$(B_1)$ Let $u \in Z_k$, for convenience, we may assume that $\|u\|_E < 1$, then

$$\begin{aligned} J_{\lambda,\mu}(u) = & \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x \\ & - \lambda \int_\Omega \frac{f_1}{q(x)}|u|^{q(x)}\mathrm{d}x - \mu \int_\Gamma \frac{g_1}{r(x)}|u|^{r(x)}\mathrm{d}\Gamma \\ \geqslant & \frac{1}{p^+}\|u\|_E^{p^+} - \frac{C_f}{q^-}\max\{\|u\|_\Omega^{q^-}, \|u\|_\Omega^{q^+}\} - \frac{C_g}{r^-}\max\{\|u\|_\Gamma^{r^-}, \|u\|_\Gamma^{r^+}\} \\ \geqslant & \frac{1}{p^+}\|u\|_E^{p^+} - \frac{C_f}{q^-}\max\{\alpha_k^{q^-}\|u\|_E^{q^-}, \alpha_k^{q^+}\|u\|_E^{q^+}\} \\ & - \frac{C_g}{r^-}\max\{\beta_k^{r^-}\|u\|_E^{r^-}, \beta_k^{r^+}\|u\|_E^{r^+}\} \\ \geqslant & \frac{1}{p^+}\|u\|_E^{p^+} - \left(\frac{C_f}{q^-}\max\{\alpha_k^{q^-}, \alpha_k^{q^+}\} + \frac{C_g}{r^-}\max\{\beta_k^{r^-}, \beta_k^{r^+}\}\right)\|u\|_E \\ = & \frac{1}{p^+}\|u\|_E^{p^+} - C\|u\|_E. \end{aligned}$$

Choose $\rho_k = (p^+C)^{\frac{1}{p^+-1}}$, then we have

$$\begin{aligned} J_{\lambda,\mu}(u) \geqslant & \frac{1}{p^+}(\rho_k)^{p^+} - C\rho_k \\ \geqslant & \frac{1}{p^+}(p^+C)^{\frac{1}{p^+-1}} - C(p^+C)^{\frac{1}{p^+-1}} = 0, \end{aligned}$$

which implies that $(B_1)$ holds.

$(B_2)$ For $u \in Y_k$ and $\mu > 0, \lambda \in \mathbb{R}$,

$$\begin{aligned} J_{\lambda,\mu}(u) = & \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\mathrm{d}x \\ & - \lambda \int_\Omega \frac{f_1}{q(x)}|u|^{q(x)}\mathrm{d}x - \mu \int_\Gamma \frac{g_1}{r(x)}|u|^{r(x)}\mathrm{d}\Gamma \\ \leqslant & \frac{1}{p^-}\|u\|_E^{p^-} + |\lambda|\frac{C_f}{q^-}\int_\Omega w_1|u|^{q(x)}\mathrm{d}x - \mu\frac{C_g}{r^-}\int_\Gamma w_2|u|^{r(x)}\mathrm{d}\Gamma. \end{aligned}$$

Notice that $\dim Y_k < \infty$, $r^+ < p^-$ and $q^- > p^+$, we find that $(B_2)$ holds if we choose $\gamma_k > 0$.

$(B_3)$ From the proof of $(B_1)$ and $Y_k \cap Z_k \neq \varnothing$, we know for $u \in Z_k, \|u\|_E \leqslant \rho_k$ small enough,

$$J_{\lambda,\mu}(u) \geqslant -\left(\frac{C_f}{q^-}\max\{\alpha_k^{q^-},\alpha_k^{q^+}\} + \frac{C_g}{r^-}\max\{\beta_k^{r^-},\beta_k^{r^+}\}\right)\|u\|_E,$$

since $\max\{\alpha_k^{q^-},\alpha_k^{q^+}\} \to 0$ and $\max\{\beta_k^{r^-},\beta_k^{r^+}\} \to 0$. That is $(B_3)$ holds. Moreover, from the above proofs, we can choose that $\rho_k > \gamma_k > 0$.

$(B_4)$ Now we prove the $(PS)_C^*$ condition holds. For the sequence $\{u_{n_j}\} \subset E$ such that

$$n_j \to \infty, \quad u_{n_j} \in Y_{n_j}, \quad J_{\lambda,\mu}(u_{n_j}) \to C, \quad J_{\lambda,\mu}\,|'_{Y_{n_j}}(u_{n_j}) \to 0.$$

Assume $\|u_{n_j}\|_E \leqslant 1$, for $n$ big enough, we have

$$\begin{aligned}
C+1+\|u_{n_j}\|_E^p &\geqslant J_{\lambda,\mu}(u_{n_j}) - \frac{1}{q^-} < J'_{\lambda,\mu}(u_{n_j}), u_{n_j} > \\
&\geqslant (\frac{1}{p^+}-\frac{1}{q^-})\|u_{n_j}\|_E^{p^-} - \lambda\int_\Omega(\frac{1}{q(x)}f_1 - \frac{1}{q^-})|u|^{q(x)}\mathrm{d}x \\
&\qquad - \mu\int_\Gamma(\frac{1}{r(x)}-\frac{1}{q^-})g_1|u|^{r(x)}\mathrm{d}\Gamma \\
&\geqslant (\frac{1}{p^+}-\frac{1}{q^-})\|u_{n_j}\|_E^{p^-} - \frac{2|\lambda|}{q^-}\int_\Omega f_1|u|^{q(x)}\mathrm{d}x \\
&\geqslant (\frac{1}{p^+}-\frac{1}{q^-})\|u_{n_j}\|_E^{p^-} - \frac{2|\lambda|}{q^-}C_f\int_\Omega w_1|u|^{q(x)}\mathrm{d}x \\
&\geqslant (\frac{1}{p^+}-\frac{1}{q^-})\|u_{n_j}\|_E^{p^-} - \frac{2|\lambda|}{q^-}C_f\max\{\alpha_0^{q^-,\alpha_0^{q^+}}\}\|u_{n_j}\|_E^{q^+},
\end{aligned}$$

where $\alpha_0$ defined in Lemma 4, and we get that $\{u_{n_j}\}$ is bounded in $E$. Going if necessary to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in $E$, as $E = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{nj} \in Y_{n_j}$ such that $v_{n_j} \to u$, hence

$$\begin{aligned}
&\lim_{n_j\to\infty} < J'_{\lambda,\mu}(u_{n_j}), u_{n_j}-u > \\
&= \lim_{n_j\to\infty} < J'_{\lambda,\mu}(u_{n_j}), u_{n_j}-v_{n_j} > + \lim_{n_j\to\infty} < J'_{\lambda,\mu}(u_{n_j}), v_{n_j}-u > \\
&= \lim_{n_j\to\infty} < J'_{\lambda,\mu}\,|'_{Y_{n_j}}(u_{n_j}), u_{n_j}-u >= 0.
\end{aligned}$$

From Proposition 6, we conclude that $u_{n_j} \to u$. Furthermore, we have $J'_{\lambda,\mu}(u_{n_j}) \to J'_{\lambda,\mu}(u)$.

Taking arbitrary $v_k \in Y_k$, notice that when $n_j \geqslant k$ we have

$$\begin{aligned}
< J'_{\lambda,\mu}(u), v_k > &= < J'_{\lambda,\mu}(u) - J'_{\lambda,\mu}(u_{n_j}), v_k > + < J'_{\lambda,\mu}(u_{n_j}), v_k > \\
&= < J'_{\lambda,\mu}(u) - J'_{\lambda,\mu}(u_{n_j}), v_k > + < J'_{\lambda,\mu}\,|'_{Y_{n_j}}(u_{n_j}), v_k >
\end{aligned}$$

$$\to 0 \quad \text{as } n_j \to \infty.$$

Hence $J'_{\lambda,\mu}(u) = 0$, this shows that $J_{\lambda,\mu}$ satisfies the $(PS)_C^*$ condition for every $C \in [d_{k_0}, 0[$. This completes the proof of Theorem 3.

From the the proof of Theorem 4, we can obtain the following corollary of another concave and convex case problem.

COROLLARY 1. *If* ($\mathbf{f_0}$), ($\mathbf{f_3}$), ($\mathbf{g_0}$), ($\mathbf{g_2}$), ($\mathbf{g_3}$) *hold and* $q^+ < p^-, r^- > p^+$, *then* $(a')$ *for every* $\lambda > 0, \mu \in \mathbb{R}$ *the problem* (3.3) *has a sequence of solutions* $\{u_k\}$ *such that* $J_{\lambda,\mu}(u_k) \to \infty, k \to \infty$;
$(b')$ *for every* $\mu > 0, \lambda \in \mathbb{R}$ *and in* ($\mathbf{f_0}$), ($\mathbf{g_0}$), *we have*

$$|f(x,t)| = f_1(x)|t|^{q(x)-1}, \;\; |g(x,t)| = g_1(x)|t|^{r(x)-1},$$

*the problem* (3.3) *has a sequence of solutions* $\{u_k\}$ *such that*

$$J_{\lambda,\mu}(u_k) < 0 \quad \textit{and} \quad J_{\lambda,\mu}(u_k) \to 0, k \to \infty.$$

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