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ASYMPTOTICS FOR NONLOCAL EVOLUTION PROBLEMS BY SCALING ARGUMENTS

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Abstract. In this paper we consider a nonlocal evolution problem and obtain by a scaling method the first term in the asymptotic behavior of the solutions. The method employed treats in different way the smooth and the rough part of the solution.

1. Introduction

In this paper we study a nonlocal equation of the form:

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}} J(x-y)(u(y,t)-u(x,t))\,dy, & x\in\mathbb{R},\ t>0, \\ u(x,0) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.1}$$

We consider $J:\mathbb{R}\to\mathbb{R}$ a nonnegative, smooth, even function rapidly decaying at infinity, with $\int_{\mathbb{R}} J(s)ds = 1$ and the initial data $u_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$.

Equations like (1.1) and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. For the interested reader we refer to [2], [3], [8], [9] and the references therein.

In this paper we will obtain the first term in the asymptotic behavior of the solution of system (1.1) by using a scaling method. The main result of this paper is the following one:

THEOREM 1.1. *Let $u_0 \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. For any $p\in[1,\infty]$ the solution $u(x,t)$ of equation* (1.1) *satisfies:*

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})}\|u(t)-MG_{At}\|_{L^p(\mathbb{R})} = 0 \tag{1.2}$$

where

$$G_t(x) = \frac{1}{\sqrt{4\pi t}}\exp(-\frac{x^2}{4t})$$

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is the heat kernel and

$$M=\int_{\mathbb{R}} u_0(x)dx, \quad A=\frac{1}{2}\int_{\mathbb{R}} J(z)z^2dz.$$

Similar results have been obtained in [4] and [12] by using different methods, under various assumptions on the regularity of the initial data u_0 and on J. The goal of this paper is to prove that the asymptotic behavior of the nonlocal evolution problems of type (1.1) can be analyzed by scaling arguments even if the equation does not support a self-similar solution due to the lack of homogeneity of the kernel J.

The main difficulty in applying scaling arguments in nonlocal problems is the lack of smoothness of the solution. As observed in [4], the solution at any positive time is as smooth as the initial data is. More precisely the solution of equation (1.1) can be written as

$$u(x,t)=e^{-t}u_0(x)+v(x,t), \tag{1.3}$$

where v is the smooth part of the solution while $e^{-t}u_0$ remains as smooth as the initial data is. By a simple computation, it follows that $v(x,t)$ verifies the equation:

$$\begin{cases} v_t(x,t)=e^{-t}(J*u_0)(x)+(J*v-v)(x,t), & x\in\mathbb{R},\ t>0,\\ v(x,0)=0, & x\in\mathbb{R}. \end{cases} \tag{1.4}$$

The key point in using the scaling method to the nonlocal model considered here is to apply this method to the regular part of the solution v. To obtain the decay in Theorem 1.1 we will prove a similar asymptotic behavior for v:

$$\lim_{t\to\infty} t^{1/2(1-1/p)}\|v(t)-MG_{At}\|_{L^p(\mathbb{R})}=0. \tag{1.5}$$

To fix the ideas, for $v(x,t)$ solution of problem (1.4) we define a family of functions $\{v_\lambda\}_{\lambda>0}$ as follows:

$$v_\lambda(x,t)=\lambda v(\lambda x,\lambda^2 t), \quad x\in\mathbb{R}, t\geqslant 0.$$

In order to obtain the asymptotic behavior of v in (1.5) we will prove that, at the time $t=1$ the rescaled family $v_\lambda(1)$ strongly converges as $\lambda\to\infty$ in the $L^p(\mathbb{R})$-norm to the solution of the heat equation, $\overline{v}_t=A\overline{v}_{xx}$ with $M\delta_0$ initial data, i.e. converges to MG_A. To do that we prove that for any $0<t_1<t_2<\infty$ the sequence $\{v_\lambda\}_{\lambda>0}$ is relatively compact on $C([t_1,t_2],L^1(\mathbb{R}))$ and that the limit point is the solution of the heat equation.

When we rescale function v in fact we can write a similar scaling for u with the difference that for the new family $\{u_\lambda\}_{\lambda>0}$ we will not be able to prove the compactness (by the lack of regularity with respect to the initial data). Our method not only rescale the solution but also the initial data. The limit of the rescaled solutions u_λ when the initial data remains unchanged, i.e. the hyperbolic-parabolic relaxation limit, has been considered in [1, Ch. 1, p. 23].

In the context of classical diffusion problems, linear or nonlinear, the scaling method has been successfully applied. We cite here just a few references [6], [7], [14].

This paper shows that the nonlocal evolution problems involving operators as in (1.1), where the smoothing effect is not present, could be treated by means of scaling methods. The extension of the method to nonlinear models as the ones analyzed in [11, 5] remains open. However, the main difficulty in the context of the nonlinear problems will be to separate the smooth and rough parts of the solutions, an argument that is immediate in the case of linear problems. We recall that there are cases when nonlinearity can help. We recall here the results in [15] where a simplified model for radiating gases has been analyzed. The asymptotic profile is obtained there by using some Oleinik type estimates which are not available for the model we have considered here.

We have considered here the case when J is a smooth function rapidly decaying at infinity. In fact more general kernels can be considered. Essentially, as observed in [12] we need the following assumptions on J:

$$\hat{J}(\xi) = 1 - A\xi^2 + o(\xi^2) \quad \text{as}\, \xi \to 0 \tag{1.6}$$

and for some $m > 2$

$$|\hat{J}(\xi)| \leqslant \frac{C}{|\xi|^m} \quad \text{as}\, \xi \to \infty. \tag{1.7}$$

Obviously when J is an even function and has decay faster than $1/|x|^2$ at infinity, i.e. $J \in L^1(1+|x|^2)$ for example, the first hypothesis (1.6) is satisfied with

$$A = \frac{1}{2}\int_{\mathbb{R}} J(z) z^2 dz.$$

Condition (1.7) holds for example when J has at least three derivatives in $L^1(\mathbb{R})$. These restrictions are assumed in order to prove decay properties for the solution v of system (1.4) and its derivative in the L^p-norms for all $2 \leqslant p \leqslant \infty$. If we only need to have estimates in the $L^2(\mathbb{R})$ norm then only $m > 3/2$ is needed (see carefully the proof of Lemma 1.16 in [1]). This happens if J is of class $W^{2,1}(\mathbb{R}) \cap L^1(1+|x|^2)$.

We recall here that in order to obtain $L^1 - L^2$ estimates for u, a solution of (1.1), and thus for v solution of (1.4), only $J \in L^1(1+|x|^2)$ is sufficient as proved by energy methods in [13], [16]. The obtention of all the estimates involved in the proof by using energy methods (see [10, Ch. 1, p. 25] for the case of the heat equation) remains to be analyzed.

2. Proof of main results

We first recall some preliminary results that will help us during the proof.

We point out that as long as the initial data u_0 is nonnegative, v a solution of system (1.4) is a supersolution for system (1.1) with initial data identically zero. Then v is nonnegative since the comparison principle holds (see [1, Ch. 2, p. 37]). We will consider here, without loss of generality, the case of nonnegative initial data u_0, so nonnegative solutions.

The following lemma shows that (1.5) is equivalent with the strong convergence of the sequence $\{v_\lambda(t_0)\}$ toward the heat kernel at the time t_0 multiplied by the mass of the initial data MG_{At_0}.

LEMMA 2.1. *Let $p\in[1,\infty]$. The following statements are equivalent:*

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})}\|v(t)-MG_{At}\|_{L^p(\mathbb{R})}=0 \tag{2.1}$$

and

$$v_\lambda(1)\to MG_A \tag{2.2}$$

in the $L^p(\mathbb{R})$-norm as $\lambda\to\infty$, i.e.

$$\lim_{\lambda\to\infty}\|v_\lambda(1)-MG_A\|_{L^p(\mathbb{R})}=0.$$

Proof. Observe that at time $t_0=1$ the rescaled solution v_λ satisfies

$$\begin{aligned}\|v_\lambda(x,1)-MG_A(x)\|_{L^p(\mathbb{R})}&=\|\lambda v(\lambda x,\lambda^2)-M\lambda G_{\lambda^2A}(\lambda x)\|_{L^p(\mathbb{R})}\\&=\lambda^{1-1/p}\|v(x,\lambda^2)-MG_{\lambda^2A}(x)\|_{L^p(\mathbb{R})}\\&=t^{1/2(1-1/p)}\|v(x,t)-MG_{At}(x)\|_{L^p(\mathbb{R})},\end{aligned}$$

where $t=\lambda^2$. Then (2.1) holds if and only if (2.2) holds.

In the following we prove

$$\lim_{\lambda\to\infty}\|v_\lambda(1)-MG_A\|_{L^1(\mathbb{R})}=0. \tag{2.3}$$

The proof is divided into four steps. We mainly follow the ideas of [14]. In Step I we obtain estimates on v_λ and its derivative. In Step II, using the Aubin-Lions compactness principle (see for example [17]) we prove that v_λ strongly converges to a function $\overline{v}$ in $C([t_1,t_2],L^1_{loc}(\mathbb{R}))$. We then improve the convergence to $C([t_1,t_2],L^1(\mathbb{R}))$. In Step III we finish the proof of (2.3) by showing that any limit point $\overline{v}$ satisfies the heat equation $\overline{v}_t=A\overline{v}_{xx}$ with $M\delta_0$ as initial data. Since the limit point is unique then the whole family $\{v_\lambda\}_{\lambda>0}$ converges to that limit. We then use (1.3) to prove the result stated in Theorem 1.1.

Before starting the proof of the main result let us recall that the smooth part v can be written as

$$v(x,t)=K_t*u_0$$

where

$$K_t(x)=\int_{\mathbb{R}}(e^{t(\hat{J}(\xi)-1)}-e^{-t})e^{ix\xi}d\xi \tag{2.4}$$

or in terms of the Fourier transform

$$\hat{K}_t(\xi)=e^{t(\hat{J}(\xi)-1)}-e^{-t}. \tag{2.5}$$

Moreover, the rescaled solutions $\{v_\lambda\}_{\lambda>0}$ satisfy the following system

$$\begin{cases}(v_\lambda)_t=\lambda^2e^{-\lambda^2t}(J_\lambda*u_{0\lambda})+\lambda^2(J_\lambda*v_\lambda-v_\lambda), & x\in\mathbb{R},t>0,\\ v_\lambda(x,0)=0, & x\in\mathbb{R}.\end{cases} \tag{2.6}$$

Observe that the $L^1(\mathbb{R})$-norm of the nonnegative solution v_λ is uniformly bounded by the mass of the initial data:

$$\int_{\mathbb{R}} v_\lambda(x,t)dx = (1-e^{-\lambda^2 t})\int_{\mathbb{R}} u_0(x)dx. \tag{2.7}$$

Step I. Estimates for v_λ. We estimate the $L^p(\mathbb{R})$-norm, $p \geqslant 2$ of v_λ and $(v_\lambda)_x$. Similar estimates could be obtained for $p \in [1,2)$ under stronger assumptions on function J (see [12]). We point out that if only the case $p=2$ is needed then we only need to assume hypotesis (1.7) with $m > 1/2$ in Lemma 2.2 and $m > 3/2$ in Lemma 2.3 below.

LEMMA 2.2. *For any $p \in [2,\infty]$ there exists a positive constant $C(p,J)$ such that:*

$$\|v_\lambda(t)\|_{L^p(\mathbb{R})} \leqslant C(p,J)t^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_{L^1(\mathbb{R})}$$

for any $t>0$ and any $\lambda>0$.

REMARK 2.1. We emphasize that Lemma 2.2 can be proved under weaker assumptions on J as in [13], [16]. Essentially $J \in L^1(1+|x|^2)$ is enough to obtain bounds for u solution of (1.1), so for v and v_λ. We point out that we do not know if the energy methods (see [10, Ch. 1, p. 25]) that work in the classical heat equation to establish a bound of the type $\|u_x(t)\|_{L^2} \lesssim t^{-1/4}\|\varphi\|_{L^1}$ can be adapted to the nonlocal setting to obtain similar estimates for v and then for v_λ.

Proof. Using the definition of v_λ we have

$$\|v_\lambda(t)\|_{L^p(\mathbb{R})} = \lambda^{1-\frac{1}{p}}\|v(\lambda^2 t)\|_{L^p(\mathbb{R})}.$$

It is then sufficient to prove the same estimate for v. Using the results in [12], under the hypotheses (1.6) and (1.7) the kernel K_t defined by (2.4) satisfies

$$\|K_t\|_{L^p(\mathbb{R})} \leqslant C(p,J)t^{-\frac{1}{2}(1-\frac{1}{p})}.$$

Therefore

$$\|v(t)\|_{L^p(\mathbb{R})} \leqslant C(p,J)t^{-\frac{1}{2}(1-\frac{1}{p})}\|u_0\|_{L^1(\mathbb{R})}$$

and the proof of the Lemma is finished.

LEMMA 2.3. *For each $p \in [2,\infty]$ there exists a positive constant C such that:*

$$\|(v_\lambda(t))_x\|_{L^p(\mathbb{R})} \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_{L^1(\mathbb{R})}$$

for any $t>0$ and any $\lambda>0$.

Proof. Using the same arguments as in the previous lemma it is sufficient to prove that
$$\|(K_t)_x\|_{L^p(\mathbb{R})} \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}.$$
Previous results in [12] guarantee the desired estimates for K_t and the proof is finished.

Step II. Compactness in $C([t_1,t_2]),L^1_{loc}(\mathbb{R})$. Let us first recall the Aubin-Lions compactness criterion (see [17] for related results).

THEOREM 2.1. *Let X, B and Y be Banach spaces satisfying $X \subset B \subset Y$ with compact embedding $X \subset B$. Assume, for $1 \leqslant p \leqslant \infty$ and $T > 0$, that*
1) F is bouned in $L^p(0,T;X)$
2) $\{(f_t) : f \in F\}$ is bounded in $L^p(0,T;Y)$.
Then F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p=\infty$).

The following lemma gives the compactness of $\{v_\lambda\}_{\lambda>0}$ in $C([t_1,t_2]),L^1_{loc}(\mathbb{R})$.

LEMMA 2.4. *For any $0<t_1<t_2<\infty$ and for each $R>0$ the set*
$$\{v_\lambda\}_{\lambda>0} \subseteq C([t_1,t_2];L^1(-R,R))$$
is relatively compact.

Proof. We first prove the compactness in $C([t_1,t_2];L^2(-R,R))$ since we need estimates for v_λ in the $L^2(\mathbb{R})$-norm and these are given by Lemma 2.2 and Lemma 2.3. Using estimates on the L^1-norm of v_λ will require more assumptions on $\hat{J}$ in these lemmas.

We apply the above compactness principle with $p=\infty$ and the following spaces $X=H^1(-R,R)$, $B=L^2(-R,R)$ and $Y=H^{-1}(-R,R)$. We prove that for some $M=M(t_1,R)$ the following estimates hold uniformly with respect to the parameter λ:
$$\|v_\lambda\|_{L^\infty([t_1,t_2];H^1(-R,R))} \leqslant M \tag{2.8}$$
and
$$\|(v_\lambda)_t\|_{L^\infty([t_1,t_2];H^{-1}(-R,R))} \leqslant M. \tag{2.9}$$
Using Lemma 2.2 and Lemma 2.3 we immediately obtain estimate (2.8).

We now prove the second estimate (2.9). For a function $\Phi \in C_c^\infty(-R,R)$ we set $\overline{\Phi}$ its extension as zero outside $(-R,R)$. Using that v_λ satisfies equation (2.6) we get:
$$\begin{aligned}
\langle (v_\lambda)_t, \Phi\rangle_{H^{-1},H^1_0(-R,R)} &= \int_{-R}^{R} (v_\lambda)_t \Phi(x)dx \\
&= \int_{\mathbb{R}} (v_\lambda)_t \overline{\Phi}(x)dx \\
&= \int_{\mathbb{R}} [\lambda^2 e^{-\lambda^2 t}(J_\lambda * u_{0\lambda})(x) + \lambda^2 (J_\lambda * v_\lambda - v_\lambda)(x,t)]\overline{\Phi}(x)dx \\
&= \lambda^2 e^{-\lambda^2 t} \int_{\mathbb{R}} (J_\lambda * u_{0\lambda})(x)\overline{\Phi}(x)dx
\end{aligned}$$

$$
\begin{aligned}
&+\lambda^2\int_{\mathbb{R}}(J_\lambda * v_\lambda - v_\lambda)(x,t)\overline{\Phi}(x)dx\\
&= B_1 + B_2.
\end{aligned}
$$

Using Hölder and Young's inequalities, we obtain the following bounds for B_1:

$$
\begin{aligned}
|B_1| &\leqslant \lambda^2 e^{-\lambda^2 t}\|J_\lambda * u_{0\lambda}\|_{L^2(\mathbb{R})}\|\overline{\Phi}\|_{L^2(\mathbb{R})}\\
&\leqslant \lambda^2 e^{-\lambda^2 t}\|J_\lambda\|_{L^2(\mathbb{R})}\|u_{0\lambda}\|_{L^1(\mathbb{R})}\|\overline{\Phi}\|_{L^2(\mathbb{R})}\\
&\leqslant \lambda^{3-\frac{1}{2}} e^{-\lambda^2 t_1}\|J\|_{L^2(\mathbb{R})}\|u_0\|_{L^1(\mathbb{R})}\|\overline{\Phi}\|_{L^2(\mathbb{R})}\\
&\leqslant M\|\overline{\Phi}\|_{L^2(\mathbb{R})} = M\|\Phi\|_{L^2(-R,R)}.
\end{aligned}
$$

To obtain bounds for B_2, we use Cauchy's inequality, the identity

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(\phi(y)-\phi(x))\psi(x)dxdy \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(\phi(y)-\phi(x))(\psi(y)-\psi(x))dxdy
\end{aligned}
\tag{2.10}
$$

and the following Lemma.

LEMMA 2.5. *There exists a positive constant* $C(J)=\int_{\mathbb{R}}J(z)z^2dz$ *such that*

$$
\lambda^2\int_{\mathbb{R}}\int_{\mathbb{R}} J_\lambda(x-y)(u(y)-u(x))^2dydx \leqslant C(J)\int_{\mathbb{R}}|u_x(x)|^2dx \tag{2.11}
$$

holds for all $u\in H^1(\mathbb{R})$ *and* $\lambda>0$.

It follows that

$$
B_2 = -\frac{\lambda^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} J_\lambda(x-y)(v_\lambda(y,t)-v_\lambda(x,t))(\overline{\Phi}(y)-\overline{\Phi}(x))dydx.
$$

Applying Lemma 2.5 we get

$$
\begin{aligned}
|B_2| &\leqslant \frac{1}{2}\left(\lambda^2\int_{\mathbb{R}}\int_{\mathbb{R}} J_\lambda(x-y)(v_\lambda(y,t)-v_\lambda(x,t))^2dydx\right)^{\frac{1}{2}}\\
&\qquad\times\left(\lambda^2\int_{\mathbb{R}}\int_{\mathbb{R}} J_\lambda(x-y)(\overline{\Phi}(y)-\overline{\Phi}(x))^2dydx\right)^{\frac{1}{2}}\\
&\leqslant C\|(v_\lambda)_x(t)\|_{L^2(\mathbb{R})}\|\overline{\Phi}_x\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Applying Lemma 2.3 we have

$$
|B_2| \leqslant C(J)\|(v_\lambda)_x(t)\|_{L^2(\mathbb{R})}\|\Phi_x\|_{L^2((-R,R))} \leqslant C(J,t_1)\|u_0\|_{L^1(\mathbb{R})}\|\Phi\|_{L^2((-R,R))}.
$$

The above estimates on B_1 and B_2 show that estimate (2.9) also holds. By Theorem 2.1 we obtain that $\{v_\lambda\}_{\lambda>0}$ is relatively compact in $C([t_1,t_2];L^2(-R,R))$, then in $C([t_1,t_2];L^1(-R,R))$ and the proof of Lemma 2.4 is now complete.

Proof. [Proof of Lemma 2.5] To prove inequality (2.11) we use Cauchy's inequality and Fubini's theorem. Let us denote by I_λ the right hand side in (2.11). It follows that

$$
\begin{aligned}
I_\lambda &= \lambda \int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)\Big(u\Big(\frac{y}{\lambda}\Big)-u\Big(\frac{x}{\lambda}\Big)\Big)^2 dydx \\
&= \frac{1}{\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(y-x)^2\Big[\int_0^1 u_x(\frac{x}{\lambda}+\theta\frac{y-x}{\lambda})d\theta\Big]^2 dydx \\
&\leqslant \frac{1}{\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}} J(x-y)(y-x)^2\int_0^1\Big[u_x(\frac{x}{\lambda}+\theta\frac{y-x}{\lambda})\Big]^2 d\theta dydx \\
&= \frac{1}{\lambda}\int_{\mathbb{R}} J(z)z^2\int_0^1\int_{\mathbb{R}}\Big[u_x(\frac{z+y}{\lambda}-\theta\frac{z}{\lambda})\Big]^2 dyd\theta dz \\
&= \int_{\mathbb{R}} J(z)z^2dz\int_{\mathbb{R}}|u_x(x)|^2dx
\end{aligned}
$$

and the proof of Lemma 2.5 is finished.

Step II. Compactness in $C([t_1,t_2],L^1(\mathbb{R}))$. The previous step gives us that for any $R>0$ the family $\{v_\lambda\}_{\lambda>0}$ is relatively compact in $C([t_1,t_2],L^1(-R,R))$. Using a standard diagonal argument the compactness in $C([t_1,t_2],L^1(\mathbb{R}))$ is reduced to the fact that

$$
\sup_{t\in[t_1,t_2]}\|v_\lambda(t)\|_{L^1(|x|>R)}\to 0,\quad \text{as } R\to\infty, \tag{2.12}
$$

uniformly on $\lambda\geqslant 1$. This follows from the following Lemma.

LEMMA 2.6. *There exists a constant $C=C(J,\|u_0\|_{L^1(\mathbb{R})})$ such that*

$$
\int_{|x|>2R} v_\lambda(t,x)dx\leqslant\int_{|x|>R}(J*u_0)(x)dx+C\Big(\frac{t}{R^2}+\frac{t^{1/2}}{R}\Big) \tag{2.13}
$$

holds for any $t>0$, $R>0$, uniformly for $\lambda\geqslant 1$.

Proof. Let $\Psi\in C_c^\infty(\mathbb{R})$ be a nonnegative function satisfying $0\leqslant\Psi\leqslant 1$ and

$$
\Psi(x)=\begin{cases}0, & |x|<1,\\ 1, & |x|>2.\end{cases}
$$

Put $\Psi_R(x)=\Psi(\frac{x}{R})$ for every $R>0$. Multiplying equation (2.6) by $\Psi_R(x)$ and integrating in space and time we obtain:

$$
\begin{aligned}
&\int_{\mathbb{R}} v_\lambda(x,t)\Psi_R(x)dx \\
&\quad=\int_0^t\int_{\mathbb{R}}\Big(\lambda^2e^{-\lambda^2 s}(J_\lambda*u_{0\lambda})(x)+\lambda^2(J_\lambda*v_\lambda-v_\lambda)(x,s)\Big)\Psi_R(x)dxds \\
&\quad=B_1+B_2.
\end{aligned}
$$

Using identity (2.10) we obtain:

$$\begin{aligned} B_2 &= -\frac{\lambda^2}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J_\lambda(x-y)[v_\lambda(y,t)-v_\lambda(x,t)][\Psi_R(y)-\Psi_R(x)]dxds \\ &= \lambda^2\int_0^t\int_{\mathbb{R}}(J_\lambda*\Psi_R-\Psi_R)(x)v_\lambda(x,s)dxds. \end{aligned}$$

We now need the following result.

LEMMA 2.7. *There exists a positive constant* $C(J)$ *such that*

$$\|\lambda^2(J_\lambda*\psi-\psi)\|_{L^\infty(\mathbb{R})} \leqslant C(J)\|\psi_{xx}\|_{L^\infty(\mathbb{R})}$$

holds for all $\lambda>0$ *and* $\psi\in C_c^2(\mathbb{R})$.

Let us now recall that (2.7) shows that the mass of $v_\lambda(t)$ is bounded by the mass of u_0. Using this fact and Lemma 2.7 we get

$$\begin{aligned} |B_2| &\leqslant \int_0^t \lambda^2\|J_\lambda*\Psi_R-\Psi_R\|_{L^\infty(\mathbb{R})}\|v_\lambda(s)\|_{L^1(\mathbb{R})}ds \\ &\leqslant C(J)t\|u_0\|_{L^1(\mathbb{R})}\|(\Psi_R)_{xx}\|_{L^\infty(\mathbb{R})} \leqslant \frac{C(J)t}{R^2}\|u_0\|_{L^1(\mathbb{R})}. \end{aligned}$$

Now we analyze B_1. Observe that

$$\begin{aligned} B_1 &= \lambda^2(1-e^{-\lambda^2 t})\int_{\mathbb{R}}\int_{\mathbb{R}} J(\lambda(x-y))u_0(\lambda y)\Psi_R(x)dydx \\ &= (1-e^{-\lambda^2 t})\int_{\mathbb{R}}(J*u_0)(x)\Psi(\frac{x}{\lambda R})dx \\ &\leqslant (1-e^{-\lambda^2 t})\int_{|x|>R\lambda}(J*u_0)(x)dx \leqslant \int_{|x|>R}(J*u_0)(x)dx. \end{aligned}$$

The proof of Lemma 2.6 is now complete.

Proof. [Proof of Lemma 2.7] Using Taylor's formula we have for any x and y that

$$|\psi(x)-\psi(y)-(y-x)\psi_x(x)| \leqslant \frac{(y-x)^2}{2}\|\psi_{xx}\|_{L^\infty(\mathbb{R})}.$$

Taking into account the symmetry of J, we obtain:

$$\begin{aligned} &\lambda^2\Big|\int_{\mathbb{R}} J_\lambda(x-y)[\psi(y)-\psi(x)]dy\Big| \\ &\quad\leqslant \lambda^2\Big|\int_{\mathbb{R}} J_\lambda(x-y)(y-x)\psi'(x)dx\Big| + \frac{\lambda^2}{2}\|\psi_{xx}\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} J_\lambda(x-y)(y-x)^2dy \\ &\quad= \frac{\|\psi_{xx}\|_{L^\infty(\mathbb{R})}}{2}\int_{\mathbb{R}} J(z)z^2dz \end{aligned}$$

and the desired result follows.

Step III. Identification of the limit. By Step II, for any $0<t_1<t_2<\infty$, the family $\{v_\lambda\}_{\lambda>0}$ is relatively compact in $C([t_1,t_2],L^1(\mathbb{R}))$. Thus, there exists a subsequence $\{v_\lambda\}_{\lambda>0}$ (not relabeled) and a function $\overline{v}\in C((0,\infty),L^1(\mathbb{R}))$ such that

$$v_\lambda \to \overline{v} \quad \text{in } C([t_1,t_2],L^1(\mathbb{R})) \quad \text{as } \lambda\to\infty. \tag{2.14}$$

Moreover, for any $t>0$, since $\|v_\lambda(t)\|_{L^1(\mathbb{R})}\leqslant \|u_0\|_{L^1(\mathbb{R})}$ we also have this property for function $\overline{v}$: $\|\overline{v}(t)\|_{L^1(\mathbb{R})}\leqslant\|u_0\|_{L^1(\mathbb{R})}$.

We multiply equation (2.6) with a function $\varphi\in C_c^\infty([0,T]\times\mathbb{R})$. Integrating equation (2.6) over $[0,T]\times\mathbb{R}$ we get:

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}(v_\lambda)_t(x,t)\varphi(x,t)dxdt &= \int_0^T\int_{\mathbb{R}}\lambda^2 e^{-\lambda^2 t}(J_\lambda * u_{0\lambda})(x)\varphi(x,t)dxdt\\ &\quad +\int_0^T\int_{\mathbb{R}}\lambda^2(J_\lambda * v_\lambda - v_\lambda)\varphi(x,t)dxdt.\end{aligned}$$

Integrating by parts with respect to variables t and x and using that $v_\lambda(x,0)=0$, φ has compact support and identity (2.10), we have:

$$\begin{aligned}-\int_0^T\int_{\mathbb{R}}v_\lambda(x,t)\varphi_t(x,t)dxdt &= \int_0^T\lambda^2 e^{-\lambda^2 t}\int_{\mathbb{R}}(J_\lambda * u_{0\lambda})(x)\varphi(x,t)dxdt\\ &\quad +\int_0^T\int_{\mathbb{R}}\lambda^2(J_\lambda * \varphi - \varphi)v_\lambda(x,t)dxdt.\end{aligned}$$

We will prove later that as $\lambda\to\infty$ the following convergences hold:

$$\int_0^T\int_{\mathbb{R}}v_\lambda(x,t)\varphi_t(x,t)dxdt \to \int_0^T\int_{\mathbb{R}}\overline{v}(x,t)\varphi_t(x,t)dxdt, \tag{2.15}$$

$$\int_0^T\int_{\mathbb{R}}\lambda^2(J_\lambda*\varphi-\varphi)(x,t)v_\lambda(x,t)dxdt \to \int_0^T\int_{\mathbb{R}}\overline{v}(x,t)A\varphi_{xx}(x,t)dxdt \tag{2.16}$$

and

$$\int_0^T\int_{\mathbb{R}}\lambda^2 e^{-\lambda^2 t}(J_\lambda * u_{0\lambda})(x)\varphi(x,t)dxdt \to M\varphi(0,0), \tag{2.17}$$

where

$$A=\frac{1}{2}\int_{\mathbb{R}}J(z)z^2dz \quad\text{and}\quad M=\int_{\mathbb{R}}u_0(x)dx.$$

The above results show that $\overline{v}\in C((0,\infty),L^1(\mathbb{R}))$ satisfies

$$-\int_0^T\int_{\mathbb{R}}\overline{v}(x,t)\varphi_t(x,t)dxdt = M\varphi(0,0)+A\int_0^T\int_{\mathbb{R}}\overline{v}(x,t)\varphi_{xx}(x,t)dxdt.$$

Hence $\overline{v}$ is a solution of the heat equation

$$\begin{cases} \overline{v}_t(x,t) = A\overline{v}_{xx}(x,t)\,dy,\ x\in\mathbb{R},\ t>0, \\ \overline{v}(0) = M\delta_0. \end{cases} \tag{2.18}$$

Since this equation has a unique solution $\overline{v}(t) = MG_{At}$, G_t being the heat kernel, the whole family $\{v_\lambda\}_{\lambda>0}$ converges to $\overline{v}$ not only to a subsequence. Hence

$$\lim_{\lambda\to\infty} \|v_\lambda(1) - MG_A\|_{L^1(\mathbb{R})} = 0$$

and by Lemma 2.1, v the solution of system (1.4) satisfies

$$\lim_{t\to\infty} \|v(t) - MG_{At}\|_{L^1(\mathbb{R})} = 0.$$

This immediately implies that u, the solution of system (1.1) satisfies

$$\lim_{t\to\infty} \|u(t) - MG_{At}\|_{L^1(\mathbb{R})} = 0.$$

The case $p \geqslant 1$ easily follows since by Step I,

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leqslant C(p, \|\varphi\|_{L^1(\mathbb{R})}, \|\varphi\|_{L^\infty(\mathbb{R})})t^{-\frac{1}{2}}$$

and then

$$\begin{aligned} \|u(t) - MG_{At}\|_{L^p(\mathbb{R})} &\leqslant \|u(t) - MG_{At}\|_{L^1(\mathbb{R})}^{1/p} (\|u(t)\|_{L^\infty(\mathbb{R})} + M\|G_{At}\|_{L^\infty(\mathbb{R})})^{1-\frac{1}{p}} \\ &= o(t^{-\frac{1}{2}(1-\frac{1}{p})}). \end{aligned}$$

To finish the proof of Theorem 1.1 it remains to prove (2.15), (2.16) and (2.17). Before starting the proof we observe that

$$\lim_{\lambda\to\infty} \int_0^T \|v_\lambda(t) - \overline{v}(t)\|_{L^1(\mathbb{R})} dt = 0. \tag{2.19}$$

Indeed, for any $\varepsilon > 0$ we have

$$\begin{aligned} \int_0^T \|v_\lambda(t) - \overline{v}(t)\|_{L^1(\mathbb{R})} dt &= \int_0^\varepsilon \|v_\lambda(t) - \overline{v}(t)\|_{L^1(\mathbb{R})} dt + \int_\varepsilon^T \|v_\lambda(t) - \overline{v}(t)\|_{L^1(\mathbb{R})} dt \\ &\leqslant 2\varepsilon \|u_0\|_{L^1(\mathbb{R})} + \int_\varepsilon^T \|v_\lambda(t) - \overline{v}(t)\|_{L^1(\mathbb{R})} dt. \end{aligned}$$

Since v_λ is relatively compact in $C([\varepsilon,T], L^1(\mathbb{R}))$ we obtain that (2.19) holds.

Let us now prove (2.15). We have

$$\left| \int_0^T \int_{\mathbb{R}} (v_\lambda(x,t) - \overline{v}(x,t))\varphi_t(x,t)\,dxdt \right| \leqslant \int_0^T \|v_\lambda(t) - \overline{v}(t)\|_{L^1(\mathbb{R})} \|\varphi_t(t)\|_{L^\infty(\mathbb{R})} dt$$

$$\leqslant C(\varphi)\int_0^T \|v_\lambda(t)-\overline{v}(t)\|_{L^1(\mathbb{R})}dt,$$

and (2.19) shows that (2.15) holds.

In the case of (2.16) we have

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}}\lambda^2(J_\lambda*\varphi-\varphi)(x,t)v_\lambda(x,t)dxdt-\int_0^T\int_{\mathbb{R}}\overline{v}(x,t)A\varphi_{xx}(x,t)dxdt\right|\\
&\leqslant\left|\int_0^T\int_{\mathbb{R}}\lambda^2(J_\lambda*\varphi-\varphi)(x,t)(v_\lambda(x,t)-\overline{v}(x,t))dxdt\right|\\
&\quad+\left|\int_0^T\int_{\mathbb{R}}\overline{v}(x,t)\Big(\lambda^2(J_\lambda*\varphi-\varphi)(x,t)-A\varphi_{xx}(x,t)\Big)dxdt\right|\\
&=A_\lambda+B_\lambda.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
A_\lambda&\leqslant\int_0^T\int_{\mathbb{R}}\left|\lambda^2(J_\lambda*\varphi-\varphi)(x,t)\right||v_\lambda(x,t)-v(x,t)|dxdt\\
&\leqslant\int_0^T\|\lambda^2(J_\lambda*\varphi-\varphi)(t)\|_{L^\infty(\mathbb{R})}\|v_\lambda(t)-v(t)\|_{L^1(\mathbb{R})}dt.
\end{aligned}$$

Using Lemma 2.7 and (2.19) we obtain that $A_\lambda\to 0$ as $\lambda\to\infty$.

For the second term, B_λ, we obtain:

$$\begin{aligned}
B_\lambda&\leqslant\int_0^T\int_{\mathbb{R}}|\overline{v}(x,t)|\left|\lambda^2(J_\lambda*\varphi-\varphi)(x,t)-A\varphi_{xx}(x,t)\right|dxdt\\
&=\int_0^T\int_{\mathbb{R}}|\overline{v}(x,t)|\left|\lambda^3\int_{\mathbb{R}}J(\lambda(x-y))(\varphi(y,t)-\varphi(x,t))dy-A\varphi_{xx}(x,t)\right|dxdt.
\end{aligned}$$

Since $\overline{v}$ belongs to $L^1((0,T)\times\mathbb{R})$ it is sufficient to prove that the second term in the last integral goes to zero. Let us observe that

$$\begin{aligned}
\lambda^3\int_{\mathbb{R}}J(\lambda(x-y))(\varphi(y,t)-\varphi(x,t))dy&=\lambda^2\int_{\mathbb{R}}J(z)\big(\varphi(x-\frac{z}{\lambda})-\varphi(x)\big)dz\\
&=\lambda^2\int_{\mathbb{R}}J(z)\Big[-\frac{z}{\lambda}\varphi_x(x)+\frac{1}{\lambda^2}\int_0^1(1-s)\varphi_{xx}(x-\frac{sz}{\lambda})z^2ds\Big]dz\\
&=-\frac{\varphi_x(x)}{\lambda}\int_{\mathbb{R}}J(z)zdz+\int_{\mathbb{R}}J(z)z^2\int_0^1(1-s)\varphi_{xx}(x-\frac{sz}{\lambda})dsdz\\
&=\int_{\mathbb{R}}J(z)z^2\int_0^1(1-s)\varphi_{xx}(x-\frac{sz}{\lambda})dsdz\to A\varphi_{xx}(x)\quad\text{as}\,\lambda\to\infty.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem we obtain that B_λ goes to zero as $\lambda\to\infty$.

Before entering in the proof of (2.17) let us remark that

$$\int_0^\infty e^{-t}\int_{\mathbb{R}}(J*u_0)(x)dxdt=\int_{\mathbb{R}}u_0(x)dx=M. \tag{2.20}$$

Since φ has compact support we have that

$$\left|\varphi\left(\frac{x}{\lambda},\frac{t}{\lambda^2}\right)-\varphi(0,0)\right| \leqslant \left(\frac{|x|}{\lambda}+\frac{t}{\lambda^2}\right)\|\nabla\varphi\|_{L^\infty(\mathbb{R}^2)} \leqslant \frac{C(\varphi)}{\lambda}. \tag{2.21}$$

Using (2.20) and (2.21) we have that when $\lambda\to\infty$

$$\begin{aligned}
&\Big|\int_0^\infty\int_{\mathbb{R}}\lambda^2 e^{-\lambda^2 t}(J_\lambda * u_{0\lambda})(x)\varphi(x,t)dxdt - M\varphi(0,0)\Big| \\
&\qquad = \left|\int_0^\infty\int_{\mathbb{R}}\lambda^2 e^{-\lambda^2 t}\lambda(J*u_0)(\lambda x)\varphi(x,t)dxdt - M\varphi(0,0)\right| \\
&\qquad \leqslant \int_0^\infty e^{-t}\int_{\mathbb{R}}(J*u_0)(x)\left|\varphi(\frac{x}{\lambda},\frac{t}{\lambda^2})-\varphi(0,0)\right|dxdt \\
&\qquad \leqslant \frac{C(\varphi)}{\lambda}\int_0^\infty e^{-t}\int_{\mathbb{R}}(J*u_0)(x)dxdt = \frac{C(\varphi)}{\lambda}M\to 0.
\end{aligned}$$

The proof of (2.15), (2.16), (2.17) is now complete and the proof of the main result finishes.

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