Differential
Equations
& Applications
Volume 6, Number 1 (2014), 1–19
doi:10.7153/dea-06-01

POSITIVE SOLUTIONS FOR SINGULAR NONLOCAL BOUNDARY VALUE PROBLEMS INVOLVING NONLINEAR INTEGRAL CONDITIONS

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(*Communicated by Qingkai Kong*)

Abstract. In this paper, using the fixed point theory on a cone and Leray-Schauder fixed point theorem, we present some existence results for singular nonlocal boundary value problems involving nonlinear integral conditions. Our nonlinearity may be singular in its dependent variable and it is allowed to change sign.

1. Introduction

In this paper, we consider the existence of positive solutions of nonlinear nonlocal boundary value problem(BVP) of the form

$$-y'' = q(t)f(t,y(t)), t \in (0,1) \tag{1.1}$$

with integral boundary conditions

$$y'(0) = 0, y(1) = \alpha[y] = \int_0^1 (y(s))^\beta dA(s) \tag{1.2}$$

involving a Stieltjes integral, which generalizes the boundary conditions in [9-10]. J.R.L. Webb, G. Infante, G.S.Goodrich discussed the existence of at least one positive solutions and multiplicity of positive solutions for BVP(1.1)-(1.2) under the nonlinear boundary conditions or the case $\beta = 1$ and $f(t,y)$ is positive and continuous on $(0,1) \times [0,+\infty)$, that is, f has no singularity at $y = 0$(see [6, 8, 13-14]). But the study of singular boundary value problems (singular in the dependent variable) is very important and there are many results on the existence of positive solutions(see [1-4,11-12,15]). Inspired by the above works, we consider the case that f is singular at $y = 0$ and may be sign changing. In order to get the existence of positive solutions for BVP(1.1)-(1.2), we establish some new conditions. Using the fixed point theorems on a cone and the Leray-Schauder fixed point theorem, some new existence results are obtained for the BVP(1.1)-(1.2).

Our paper is organized as follows. In Section 2, we present some lemmas and preliminaries. Section 3 discusses the existence of multiple positive solutions for BVP(1.1) -(1.2) when f is positive. In Section 4, we discuss the multiplicity of positive solutions for the semi-positone BVP(1.1)-(1.2). In sction 5, we present the existence of positive solutions of BVP(1.1)-(1.2) when f is changing sign and singular at $y = 0$.

Mathematics subject classification (2010): 34B10, 34B15, 34B18.
Keywords and phrases: integral boundary conditions, singularity, cone, positive solutions.
This research is supported by Young Award of Shandong Province(ZR2013AQ008).

Paper DEA-06-01

2. Preliminaries

Let $C[0,1]=\{y:[0,1]\to\mathbb{R}|y(t)$ is continuous on $[0,1]\}$ with norm

$$\|y\|=\max_{t\in[0,1]}|y(t)|.$$

It is easy to see that $C[0,1]$ is a Banach space. Define

$$P=\Big\{y\in C[0,1]|y \text{ is concave and nonincreasing on } [0,1] \quad \text{with } y(t)\geqslant 0 \text{ for all } t\in[0,1]\Big\}.$$

It is easy to prove P is a cone of $C[0,1]$.

LEMMA 2.1. (see [7]) *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\overline{\Omega}\cap P\to P$ be continuous and compact. Suppose $\lambda Ax\neq x$, $\forall x\in\partial\Omega\cap P$, $\lambda\in(0,1]$. Then*

$$i(A,\Omega\cap P,P)=1.$$

LEMMA 2.2. (see [7]) *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\overline{\Omega}\cap P\to P$ be continuous and compact. Suppose $Ax\nleqslant x$, $\forall x\in\partial\Omega\cap P$. Then*

$$i(A,\Omega\cap P,P)=0.$$

LEMMA 2.3. (see [5]) *Let E be a Banach space, $R>0$, $B_R=\{x\in E:\|x\|\leqslant R\}$, $F:B_R\to E$ be a continuous compact operator. If $x\neq\lambda F(x)$ for any $x\in E$ with $\|x\|=R$ and $0<\lambda<1$, then F has a fixed point in B_R.*

LEMMA 2.4. *Let $y\in P$. Then*

$$y(t)\geqslant(1-t)\|y\| \text{ for } t\in[0,1]. \tag{2.1}$$

Proof. For $t\in(0,1)$, since $y(t)\geqslant 0$ is nonincreasing on $(0,1)$, we have $y(0)=\|y\|$. From the concavity of y, we have

$$y(t)=y((1-t)0+t\cdot 1)\geqslant(1-t)y(0)+ty(1)\geqslant(1-t)\|y\|.$$

Then (2.1) is true. The proof is complete. □

Now we present following conditions for convenience:

(C_1) A is of bounded variation with a positive measure, $0<\int_0^1 dA(s)<1$, $0<\beta\leqslant 1$,

(C_2)

$$\begin{cases}\text{for each constant } r>0 \text{ there exists a function } \psi_r\\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ such that}\\ f(t,y)\geqslant\psi_r(t) \text{ on } (0,1)\times(0,r]\end{cases}$$

(C_3)

$$q \in C(0,1),\ q > 0 \text{ on } (0,1) \text{ and } \int_0^1 (1-t)q(t)\,dt < \infty,$$

(C_4)

$$f : [0,1] \times (0,\infty) \to (0,\infty) \text{ is continuous.}$$

REMARK 2.5. From (C_4), if $y \in C([0,1],\mathbb{R})$ is a solution to BVP(1.1)-(1.2), it is easy to see that $y \in C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$. Then the solution space to BVP(1.1)-(1.2) must be at least $C([0,1],\mathbb{R}) \cap C^2((0,1),\mathbb{R})$.

3. Multiplicity of positive solutions for singular boundary value problems with positive nonlinearities

In this section, we consider the existence of multiple positive solutions for BVP(1.1) - (1.2). To show that BVP(1.1)-(1.2) has a solution, since f may be singular at $y = 0$, for $y \in P$, define

$$(T_c y)(t) = \alpha[y] + \int_0^1 k(t,s)q(s)f(s,\max\{c,y(s)\})ds,\ \ t \in [0,1],\ \ c > 0 \tag{3.1}$$

where

$$k(t,s) = \begin{cases} 1-t,\ 0 \leqslant s \leqslant t \leqslant 1; \\ 1-s,\ 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

REMARK 3.1. The idea of the definition of T_c comes from [1-2, 10].

LEMMA 3.2. *Suppose* (C_1)-(C_4) *hold. Then* $T_c : P \to P$ *is continuous and compact for all* $c > 0$.

Proof. It is easy to prove that T_c is well defined and $(T_c y)(t) \geqslant 0$ for all $t \in P$. For $y \in P$, we have

$$\begin{cases} (T_c y)''(t) \leqslant 0 \text{ on } (0,1) \\ (T_c y)'(0) = 0,\ (T_c y)(1) = \alpha[y], \end{cases} \tag{3.2}$$

so

$$T_c y(t) \text{ is concave and nonincreasing on } [0,1]. \tag{3.3}$$

Consequently, $T_c : P \to P$. A standard argument shows that $T_c : P \to P$ is continuous and compact(see [5-8,13]). □

THEOREM 3.3. *Suppose* (C_1)-(C_4) *hold and the following conditions are satisfied:*

$$\begin{cases} 0 \leqslant f(t,y) \leqslant g(y) + h(y) \ \text{on}\ [0,1] \times (0,\infty) \ \text{with} \\ g > 0 \ \text{continuous and nonincreasing on}\ (0,\infty), \\ h \geqslant 0 \ \text{continuous on}\ [0,\infty), \ \text{and}\ \frac{h}{g} \\ \text{nondecreasing on}\ (0,\infty) \end{cases} \tag{3.4}$$

and

$$\sup_{r\in(0,+\infty)} \frac{1}{\left\{1+\frac{h(r)}{g(r)}\right\}} \int_{c_0 r^\beta}^{r} \frac{dy}{g(y)} > b_0 \tag{3.5}$$

hold; here

$$c_0 = \int_0^1 dA(s),\ b_0 = \int_0^1 (1-t)\,q(t)\,dt; \tag{3.6}$$

there exists an $a \in \left(0, \frac{1}{2}\right)$ *such that*

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty \tag{3.7}$$

uniformly on $[a, 1-a]$*. Then BVP(1.1)-(1.2) has at least two positive solutions.*

Proof. Choose $\varepsilon > 0$ and $r > 0$ with $\varepsilon < \min\{1, c_0 r^\beta\}$ and

$$\frac{1}{\left\{1+\frac{h(r)}{g(r)}\right\}} \int_{c_0 r^\beta}^{r} \frac{dy}{g(y)} > b_0. \tag{3.8}$$

From (3.7), there exists an $R > r$ such that

$$f(t,y) \geqslant N^* y, \forall y \geqslant R, \tag{3.9}$$

where

$$N^* > \frac{2}{a\int_a^{1-a}(1-s)q(s)ds}.$$

Let

$$\Omega_1 = \{y \in C[0,1] \,|\, \|y\| < r\},$$

$$\Omega_2 = \{y \in C[0,1] \,|\, \|y\| < \frac{R}{a}\}.$$

For $y \in P$, define

$$(T_\varepsilon y)(t) = \alpha[y] + \int_0^1 k(t,s)q(s)f(s, \max\{\varepsilon, y(s)\})ds,\ \ t \in [0,1],$$

where $k(t,s)$ is defined in (3.1). Lemma 3.1 shows that $T_\varepsilon : P \to P$ is continuous and compact.

Now we show that

$$y \neq \lambda T_\varepsilon y, \forall y \in \partial\Omega_1 \cap P, \lambda \in [0,1]. \tag{3.10}$$

Suppose that there is a $y_0 \in \partial\Omega_1 \cap P$ and $\lambda_0 \in [0,1]$ with $y_0 = \lambda_0 T_\varepsilon y_0$. Then, y_0 satisfies

$$\begin{cases} y_0'' + \lambda_0 q(t) f(t, \max\{\varepsilon, y_0(t)\}) = 0,\ \ 0 < t < 1, \\ y_0'(0) = 0, y_0(1) = \alpha[y_0]. \end{cases} \tag{3.11}$$

Then $y_0''(t) \leqslant 0$ on $(0,1)$ and $y_0'(0) = 0$, $y_0(1) = \alpha[y_0] \leqslant r^\beta \int_0^1 dA(s) = c_0 r^\beta < r$. For $t \in (0,1)$ we have

$$\begin{aligned} -y_0''(t) &\leqslant g(\max\{\varepsilon, y_0(t)\})\left\{1 + \frac{h(\max\{\varepsilon, y_0(t)\})}{g(\max\{\varepsilon, y_0(t)\})}\right\} q(t) \\ &\leqslant g(\max\{\varepsilon, y_0(t)\})\left\{1 + \tfrac{h(r)}{g(r)}\right\} q(t),\ t \in (0,1). \end{aligned} \tag{3.12}$$

Integrate from 0 to t to obtain

$$-y_0'(t) \leqslant g(\max\{\varepsilon, y_0(t)\})\left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^t q(s)\,ds \leqslant g(y_0(t))\left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^t q(s)\,ds$$

and then integrate from 0 to 1 to obtain

$$\int_{\alpha[y_0]}^{y_0(0)} \frac{dy}{g(y)} \leqslant \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^1 (1-s)\,q(s)\,ds,$$

which together with $\alpha[y_0] \leqslant c_0 r^\beta$ means that

$$\int_{c_0 r^\beta}^{r} \frac{dy}{g(y)} \leqslant \int_{\alpha[y_0]}^{r} \frac{dy}{g(y)} \leqslant \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^1 (1-s)\,q(s)\,ds.$$

This contradict (3.8), which yields (3.10) is true. Lemma 2.1 implies that

$$i(T_\varepsilon, \Omega_1 \cap P, P) = 1. \tag{3.13}$$

Next we show

$$T_\varepsilon y \not\leqslant y \ \text{ for } \ y \in P \cap \partial\Omega_2. \tag{3.14}$$

Suppose that there exists a $y_0 \in P \cap \partial\Omega_2$ with $T_\varepsilon y_0 \leqslant y_0$. Then, $\|y_0\| = \frac{R}{a}$. Also since $y_0(t)$ is concave on $[0,1]$ (since $y_0 \in P$) we have from Lemma 2.4 that $y_0(t) \geqslant (1-t)\|y_0\| \geqslant (1-t)\frac{R}{a}$ for $t \in [0,1]$. Also for $s \in [a, 1-a]$ we have

$$y_0(t) \geqslant a\frac{R}{a} = R, \forall\ t \in [a, 1-a],$$

which together with (3.9) yields that

$$f(t, y_0(t) \geqslant N^* y_0(t) \geqslant N^* R, \forall\ \ t \in [a, 1-a]. \tag{3.15}$$

Then we have using (3.15),

$$\begin{aligned} y_0(0) \geqslant T_\varepsilon y_0(0) &= \alpha[y_0] + \int_0^1 (1-s)\,q(s)\,f(s, \max\{\varepsilon, y_0(s)\})\,ds \\ &\geqslant \int_a^{1-a} (1-s)\,q(s)\,f(s, \max\{\varepsilon, y_0(s)\})\,ds \\ &= \int_a^{1-a} (1-s)\,q(s)\,f(s, y_0(s))\,ds \end{aligned}$$

$$\geqslant N^* \frac{R}{a} a \int_a^{1-a} (1-s) q(s) ds$$
$$> \frac{R}{a} = \|y_0\|,$$

which is a contradiction. Hence (3.14) is true. Lemma 2.2 guarantees that

$$i(T_\varepsilon, \Omega_2 \cap P, P) = 0,$$

and so

$$i(T_\varepsilon, (\Omega_2 - \overline{\Omega}_1) \cap P, P) = -1. \tag{3.16}$$

Now (3.13) and (3.16) imply that there exists a $y_1 \in \Omega_1 \cap P$ and a $y_2 \in (\Omega_2 - \overline{\Omega}_1) \cap P$ such that

$$T_\varepsilon y_1 = y_1,\ T_\varepsilon y_2 = y_2.$$

Define

$$H = \Big\{ x \in C([0,1],\mathbb{R}) \cap C^1([0,1),\mathbb{R}) \cap C((0,1),(0,+\infty)) \cap C^2((0,1),\mathbb{R}) \\ \mid x \text{ satisfies } x''(t) + q(t) f(t, \max\{\varepsilon, x(t)\}) = 0,\ 0 < t < 1, \\ x'(0) = 0,\ x(1) = \alpha[x] = \int_0^1 x^\beta(s) dA(s),\ \forall \varepsilon > 0 \Big\}.$$

Since y_1, $y_2 \in H$, we know that $H \neq \emptyset$. Let $\overline{c} = \inf_{x \in H} \min_{t \in [0,1]} x(t)$. Now we show that

$$\overline{c} > 0.$$

In fact, if $x \in H$, there are two cases to consider:

(1) $\|x\| > 1$. Lemma 2.4 implies that

$$x(t) \geqslant (1-t)\|x\| \geqslant (1-t),\ t \in [0,1]$$

and so

$$x(t) = \alpha[x] + \int_0^1 k(t,s) q(s) f(s, \max\{\varepsilon, x(s)\}) ds \geqslant \int_0^1 (1-s)^\beta dA(s),\ \ t \in [0,1].$$

(2) $0 < \|x\| \leqslant 1$. Then (C_2) means that

$$x(t) = \alpha[x] + \int_0^1 k(t,s) q(s) f(s, \max\{\varepsilon, x(s)\}) ds$$
$$\geqslant \int_0^1 k(t,s) q(s) \psi_1(s) ds = \gamma_0(t),\ \ t \in [0,1]$$

and so

$$x(t) = \alpha[x] + \int_0^1 k(t,s) q(s) f(s, \max\{\varepsilon, x(s)\}) ds \geqslant \int_0^1 \gamma_0^\beta(s) dA(s),\ \ t \in [0,1].$$

Then

$$\overline{c} \geqslant \min\{\int_0^1 (1-s)^\beta dA(s), \int_0^1 \gamma_0^\beta(s) dA(s)\} > 0.$$

Let $0 < \varepsilon < \overline{c}$ and $\varepsilon < r$ (r is defined in (3.8)). (3.13) and (3.16) guarantee that there exists a $y_1 \in \Omega_1 \cap P$ and a $y_2 \in (\Omega_2 - \overline{\Omega}_1) \cap P$ such that

$$T_\varepsilon y_1 = y_1,\ T_\varepsilon y_2 = y_2,$$

i.e., y_1 and y_2 satisfy

$$\begin{cases} y_1'' + q(t) f(t, \max\{\varepsilon, y_1(t)\}) = 0, & 0 < t < 1, \\ y_1'(0) = 0, y_1(1) = \alpha[y_1], \end{cases} \tag{3.17}$$

$$\begin{cases} y_2'' + q(t) f(t, \max\{\varepsilon, y_2(t)\}) = 0, & 0 < t < 1, \\ y_2'(0) = 0, y_2(1) = \alpha[y_2], \end{cases} \tag{3.18}$$

and $\min_{t\in[0,1]} y_1(t) \geqslant \overline{c} > \varepsilon$, $\min_{t\in[0,1]} y_2(t) \geqslant \overline{c} > \varepsilon$. And then

$$\begin{cases} y_1'' + q(t) f(t, y_1(t)) = 0, & 0 < t < 1, \\ y_1'(0) = 0, y_1(1) = \alpha[y_1] \end{cases} \tag{3.19}$$

and

$$\begin{cases} y_2'' + q(t) f(t, y_2(t)) = 0, & 0 < t < 1, \\ y_2'(0) = 0, y_2(1) = \alpha[y_2]. \end{cases} \tag{3.20}$$

(3.19) and (3.20) guarantee that y_1 and y_2 are two positive solutions. □

EXAMPLE 3.4. Consider

$$y''(t) + \mu(y^{-\delta_1}(t) + y^{\delta_2}(t)) = 0,\ 0 < t < 1, \tag{3.21}$$

$$y'(0) = 0,\ y(1) = \int_0^1 y^{\frac{1}{2}}(s) dA(s), dA(s) = \frac{1}{2} ds, \tag{3.22}$$

where $\delta_1 > 0$, $\delta_2 > 1$. Let

$$q(t) = \mu,\ f(t,y) = y^{-\delta_1} + y^{\delta_2},\ g(y) = y^{-\delta_1},\ h(y) = y^{\delta_2},$$
$$c_0 = \int_0^1 dA(s) = \frac{1}{2},\ b_0 = \frac{1}{4}\mu.$$

It is easy to see that (C_1)-(C_4) and (3.4) hold. Since

$$\frac{1}{1 + \frac{h(1)}{g(1)}} \int_{c_0 1^{\frac{1}{2}}}^1 \frac{1}{g(y)} dy = \frac{1 - (\frac{1}{2})^{\delta_1 + 1}}{2(1 + \delta_1)},$$

letting $\mu_0 < 2\frac{1-(\frac{1}{2})^{\delta_1+1}}{2(1+\delta_1)}$, we have

$$\sup_{r\in(0,+\infty)} \frac{1}{1 + \frac{h(r)}{g(r)}} \int_{c_0 r^{\frac{1}{2}}}^r \frac{1}{g(y)} dy > \int_0^1 (1-t) q(t) dt = b_0$$

for all $\mu \leqslant \mu_0$, which means that (3.5) is true. Moreover, since

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$$

uniformly on $[0,1]$, (3.7) is true.

Then, Theorem 3.1 implies that (3.21)-(3.22) has at least two positive solutions.

4. Multiplicity of positive solutions for singular semi-positone boundary value problems

In this section, we consider the case

$$f(t,y) = F(t,y) - \gamma(t), \;\; t \in (0,1),$$

where the conditions (C_1), (C_3), (C_4) for $F(t,y)$ instead of $f(t,y)$ hold and $\gamma \in C((0,1),(0,+\infty))$

$$w(t) = \int_0^1 k(t,s)\gamma(s)ds < +\infty, \;\; t \in [0,1]$$

throughout this section, where $k(t,s)$ is the same as one in (3.1).

THEOREM 4.1. *Suppose the following conditions are satisfied:*

$$\begin{cases} 0 \leqslant F(t,y) \leqslant g(y)+h(y) \;\; on \;\; [0,1]\times(0,\infty) \;\; with \\ g>0 \;\; continuous\ and\ nonincreasing\ on \;\; (0,\infty), \\ h \geqslant 0 \;\; continuous\ on \;\; [0,\infty), \;\; and \;\; \frac{h}{g} \\ nondecreasing\ on \;\; (0,\infty), \end{cases} \tag{4.1}$$

$$\begin{cases} there\ exists\ a\ function \;\; \psi_{4c_1} \\ continuous\ on \;\; [0,1] \;\; and\ positive\ on \;\; (0,1) \;\; such\ that \\ F(t,y) \geqslant \psi_{4c_1}(t) \;\; on \;\; (0,1)\times(0,4c_1], \end{cases} \tag{4.2}$$

with

$$\int_0^1 (1-s)q(s)\psi_{4c_1}(s)ds > 2c_1$$

and

$$\sup_{r\in(2c_1,+\infty)} \frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 r^\beta}^{r} \frac{dy}{g(\frac{1}{2}y)} > b_0 \tag{4.3}$$

hold; here

$$c_0 = \int_0^1 dA(s), \; c_1 = \int_0^1 \gamma(s)ds, \quad b_0 = \int_0^1 (1-t)\,q(t)\,dt; \tag{4.4}$$

there exists an $a \in \left(0, \frac{1}{2}\right)$ *such that*

$$\lim_{y\to+\infty} \frac{F(t,y)}{y} = +\infty \tag{4.5}$$

uniformly on $[a,1-a]$. *Then BVP(1.1)-(1.2) has at least two positive solutions.*

Proof. From (4.3), choose $r > 2c_1$, $\varepsilon > 0$ with $\varepsilon < \min\{\frac{1}{2}c_0 r^\beta, r\}$ with

$$\frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 r^\beta}^{r} \frac{dy}{g(\frac{1}{2}y)} > b_0. \tag{4.6}$$

From (4.5), there exists an $R > r$ such that

$$F(t,y) \geqslant N^* y, \forall y \geqslant R, \tag{4.7}$$

where

$$N^* > \frac{2}{a\int_a^{1-a}(1-s)q(s)ds}.$$

Let

$$\Omega_1 = \{y \in C[0,1] \,|\, \|y\| < r\},$$

$$\Omega_2 = \{y \in C[0,1] \,|\, \|y\| < \frac{2R}{a}\}.$$

For $y \in P$, define

$$(T_\varepsilon y)(t) = \alpha[[y-w]^*] + \int_0^1 k(t,s)q(s)F(s,\max\{\varepsilon,[y(s)-w(s)]^*\})ds, \ \ t \in [0,1],$$

where $k(t,s)$ is defined in (3.1) and

$$[y(t)-w(t)]^* = \begin{cases} y(t)-w(t), & \text{if } y(t)-w(t) > 0, \\ 0, & \text{if } y(t)-w(t) \leqslant 0. \end{cases}$$

Lemma 3.1 guarantees that $T_\varepsilon : P \to P$ is continuous and compact.

Now we show that

$$y \neq \lambda T_\varepsilon y, \forall y \in \partial\Omega_1 \cap P, \lambda \in [0,1]. \tag{4.8}$$

Suppose that there is a $y_0 \in \partial\Omega_1 \cap P$ and $\lambda_0 \in [0,1]$ with $y_0 = \lambda_0 T_\varepsilon y_0$. Since $y_0(t) \geqslant (1-t)\|y_0\| \geqslant (1-t)2c_1$ and

$$w(t) = \int_0^1 k(t,s)\gamma(s)ds \leqslant (1-t)\int_0^1 \gamma(s)ds$$
$$= c_1(1-t) = \frac{c_1}{\|y_0\|}(1-t)\|y_0\| \leqslant \frac{1}{2}y_0(t),$$

we have

$$y_0(t) - w(t) \geqslant \frac{1}{2}y_0(t), \ \ t \in [0,1].$$

Since y_0 satisfies

$$\begin{cases} y_0'' + \lambda_0 q(t)F(t,\max\{\varepsilon,[y_0(t)-w(t)]^*\}) = 0, \ \ 0 < t < 1, \\ y_0'(0) = 0, y_0(1) = \alpha[[y_0-w]^*]. \end{cases} \tag{4.9}$$

we get $y_0''(t) \leqslant 0$ on $(0,1)$ and $y_0'(0)=0$, $y_0(1)=\alpha[[y_0-w]^*] \leqslant r^\beta \int_0^1 dA(s) < r^\beta$. For $t \in (0,1)$ it is easy o see that

$$\begin{aligned} -y_0''(t) &\leqslant g(\max\{\varepsilon,[y_0(t)-w(t)]^*\})\left\{1+\frac{h(\max\{\varepsilon,[y_0(t)-w(t)]^*\})}{g(\max\{\varepsilon,[y_0(t)-w(t)]^*\})}\right\} q(t) \\ &\leqslant g(\max\{\varepsilon,[y_0(t)-w(t)]^*\})\left\{1+\frac{h(r)}{g(r)}\right\} q(t),\ t\in(0,1). \end{aligned} \tag{4.10}$$

Integrate from 0 to t to obtain

$$\begin{aligned} -y_0'(t) &\leqslant g(\max\{\varepsilon,[y_0(t)-w(t)]^*\})\left\{1+\frac{h(r)}{g(r)}\right\}\int_0^t q(s)\,ds \\ &\leqslant g(\max\{\varepsilon,\frac{1}{2}y_0(t)\})\left\{1+\frac{h(r)}{g(r)}\right\}\int_0^t q(s)\,ds \\ &\leqslant g(\frac{1}{2}y_0(t))\left\{1+\frac{h(r)}{g(r)}\right\}\int_0^t q(s)\,ds \end{aligned}$$

and then integrate from 0 to 1 to obtain

$$\int_{\alpha[[y_0-w]^*]}^{y_0(0)} \frac{dy}{g(\frac{1}{2}y)} \leqslant \left\{1+\frac{h(r)}{g(r)}\right\}\int_0^1 (1-s)q(s)\,ds,$$

which together with $\alpha[[y_0-w]^*] \leqslant \alpha[y_0] \leqslant c_0 r^\beta$ yields

$$\int_{c_0 r^\beta}^{r} \frac{dy}{g(\frac{1}{2}y)} \leqslant \int_{\alpha[[y_0-w]^*]}^{r} \frac{dy}{g(\frac{1}{2}y)} \leqslant \left\{1+\frac{h(r)}{g(r)}\right\} b_0.$$

This contradicts (4.6), which means that (4.8) is true. Lemma 2.1 implies that

$$i(T_\varepsilon,\Omega_1\cap P,P)=1. \tag{4.11}$$

Next we show

$$T_\varepsilon y \not\leqslant y \ \text{ for } \ y\in P\cap\partial\Omega_2. \tag{4.12}$$

Suppose that there exists a $y_0 \in P\cap\partial\Omega_2$ with $T_\varepsilon y_0 \leqslant y_0$. Then, $\|y_0\| = \frac{2R}{a}$. Also since $y_0(t)$ is concave and nonincreaing on $[0,1]$ (since $y_0\in P$) we have from Lemma 2.4 that $y_0(t) \geqslant (1-t)\|y_0\| \geqslant (1-t)\frac{2R}{a}$ for $t\in[0,1]$. Also for $s\in[a,1-a]$ we have

$$[y_0(t)-w(t)]^* \geqslant \frac{1}{2}y_0(t) \geqslant \frac{1}{2}\frac{2R}{a} = R, \forall\ t\in[a,1-a],$$

which together with (4.7) yields that

$$F(t,\max\{\varepsilon,[y_0(t)-w(t)]^*) \geqslant N^*[y_0(t)-w(t)]^* \geqslant N^*R, \forall t\in[a,1-a]. \tag{4.13}$$

Then we have using (4.13),

$$y_0(0) \geqslant T_\varepsilon y_0(0) = \alpha[y_0] + \int_0^1 (1-s)q(s)F(s,\max\{\varepsilon,[y_0(s)-w(s)]^*\})\,ds$$

$$\begin{aligned}
&\geqslant \int_a^{1-a}(1-s)\,q(s)\,F(s,\max\{\varepsilon,[y_0(s)-w(s)]^*\})\,ds\\
&=\int_a^{1-a}(1-s)\,q(s)\,F(s,[y_0(s)-w(s)]^*)\,ds\\
&\geqslant N^*R\int_a^{1-a}(1-s)\,q(s)\,ds\\
&=N^*\frac{2R}{a}\frac{a}{2}\int_a^{1-a}(1-s)\,q(s)\,ds\\
&>\frac{2R}{a}=\|y_0\|,
\end{aligned}$$

which is a contradiction. Hence (4.12) is true. Then Lemma 2.2 guarantees that

$$i(T_\varepsilon,\Omega_2\cap P,P)=0,$$

and so

$$i(T_\varepsilon,(\Omega_2-\overline{\Omega}_1)\cap P,P)=-1. \tag{4.14}$$

Now (4.11) and (4.14) imply that there exists a $y_1\in\Omega_1\cap P$ and a $y_2\in(\Omega_2-\overline{\Omega}_1)\cap P$ such that

$$T_\varepsilon y_1=y_1,\ T_\varepsilon y_2=y_2.$$

Define

$$\begin{aligned}
H=\Big\{&x\in C([0,1],\mathbb{R})\cap C^1([0,1),\mathbb{R})\cap C((0,1),(0,+\infty))\cap C^2((0,1),\mathbb{R})\\
&|x \text{ satisfies } x''(t)+q(t)F(t,\max\{\varepsilon,[x(t)-w(t)]^*\})=0,\ 0<t<1,\\
&x'(0)=0, x(1)=\alpha[[x-w]^*]=\int_0^1([x(s)-w(s)]^*)^\beta dA(s),\ \forall\varepsilon>0\Big\}.
\end{aligned}$$

Since $y_1,\ y_2\in H$, we know that $H\neq\emptyset$. Let $\overline{c}=\inf_{x\in H}\min_{t\in[0,1]}[x(t)-w(t)]^*$. Now we show that

$$\overline{c}>0. \tag{4.15}$$

In fact, if $x\in H$, there are two cases to consider:

(1) $\|x\|\geqslant 4c_1$. Since

$$w(t)\leqslant(1-t)\int_0^1\gamma(s)ds=c_1(1-t), \tag{4.16}$$

we have

$$w(t)\leqslant\frac{c_1}{4c_1}4c_1(1-t)\leqslant\frac{1}{4}\|x\|(1-t).$$

Lemma 2.4 implies that

$$[x(t)-w(t)]^*\geqslant\frac{3}{4}x(t)\geqslant\frac{3}{4}4c_1(1-t)=3c_1(1-t),\ t\in[0,1]$$

and so

$$\begin{aligned}x(t) &= \alpha[[x-w]^*] + \int_0^1 k(t,s)q(s)F(s, \max\{\varepsilon, [x(s)-w(s)]^*\})ds \\ &\geqslant \int_0^1 (3c_1(1-s))^\beta dA(s),\ t \in [0,1].\end{aligned}$$

Then

$$[x(t)-w(t)]^* \geqslant \frac{3}{4}x(t) \geqslant \frac{3}{4}\int_0^1 (3c_1(1-s))^\beta dA(s),\ t \in [0,1]. \tag{4.17}$$

(2) $0 < \|x_\Psi\| \leqslant 4c_1$. Then (4.2) means that

$$\begin{aligned}x(0) &= \alpha[[x-w]^*] + \int_0^1 (1-s)q(s)F(s, \max\{\varepsilon, [x(s)-w(s)]^*\})ds \\ &\geqslant \int_0^1 (1-s)q(s)\psi_{4c_1}(s)ds > 2c_1,\ \ t \in [0,1],\end{aligned}$$

which together with $x \in P$ implies that

$$x(t) \geqslant (1-t)\|x\| \geqslant 2c_1(1-t),\ t \in [0,1]. \tag{4.18}$$

From (4.16) and (4.18), we have

$$w(t) \leqslant c_1(1-t) = \frac{1}{2}2c_1(1-t) \leqslant \frac{1}{2}x(t),\ t \in [0,1]$$

and so

$$[x(t)-w(t)]^* \geqslant \frac{1}{2}x(t) \geqslant c_1(1-t),\ t \in [0,1].$$

Then

$$\begin{aligned}x(t) &= \alpha[[x-w]^*] + \int_0^1 k(t,s)q(s)F(s, \max\{\varepsilon, [x(s)-w(s)]^*\})ds \\ &\geqslant \int_0^1 (c_1(1-s))^\beta dA(s),\ \ t \in [0,1],\end{aligned}$$

which implies

$$[x(t)-w(t)]^* \geqslant \frac{1}{2}x(t) \geqslant \frac{1}{2}\int_0^1 (c_1(1-s))^\beta dA(s). \tag{4.19}$$

(4.17) and (4.19) guarantee that

$$\overline{c} \geqslant \frac{1}{2}\int_0^1 (c_1(1-s))^\beta dA(s) > 0.$$

Then (4.15) is true.

Let $0 < \varepsilon < \overline{c}$ and $\varepsilon < r$(r is defined in (4.6)). (4.11) and (4.14) guarantee that there exists a $y_1 \in \Omega_1 \cap P$ and a $y_2 \in (\Omega_2 - \overline{\Omega_1}) \cap P$ such that

$$T_\varepsilon y_1 = y_1,\ T_\varepsilon y_2 = y_2,$$

i.e., y_1 and y_2 satisfy

$$\begin{cases} y_1'' + q(t)F(t, \max\{\varepsilon, [y_1(t) - w(t)]^*\}) = 0, \ 0 < t < 1, \\ y_1'(0) = 0, y_1(1) = \alpha[[y_1 - w]^*], \end{cases} \tag{4.20}$$

$$\begin{cases} y_2'' + q(t)F(t, \max\{\varepsilon, [y_2(t) - w(t)]^*\}) = 0, \ 0 < t < 1, \\ y_2'(0) = 0, y_2(1) = \alpha[[y_2 - w]^*], \end{cases} \tag{4.21}$$

and

$$[y_1(t) - w(t)]^* > \varepsilon, \ [y_2(t) - w(t)]^* > \varepsilon, \ t \in [0,1].$$

Hence, y_1 and y_2 satisfy

$$\begin{cases} y_1'' + q(t)F(t, y_1(t) - w(t)) = 0, \ 0 < t < 1, \\ y_1'(0) = 0, y_1(1) = \alpha[y_1 - w] \end{cases} \tag{4.22}$$

and

$$\begin{cases} y_2'' + q(t)F(t, y_2(t) - w(t)) = 0, \ 0 < t < 1, \\ y_2'(0) = 0, y_2(1) = \alpha[y_2 - w]. \end{cases} \tag{4.23}$$

Let $x_1(t) = y_1(t) - w(t)$, $x_2(t) = y_2(t) - w(t)$. It is easy to see that x_1 and x_2 are two positive solutions of BVP(1.1)-(1.2). □

EXAMPLE 4.2. Consider

$$y''(t) + \mu\left(y^{-2}(t) + y^{\delta_2}(t) - \frac{1}{1000}\frac{1}{1-t}\right) = 0, \ 0 < t < 1, \tag{4.24}$$

$$y'(0) = 0, \ y(1) = \int_0^1 (y(s))^{\frac{1}{2}} dA(s), dA(s) = \frac{1}{2} ds, \tag{4.25}$$

where $\delta_2 > 1$. Let

$$q(t) = \mu, \ F(t,y) = y^{-2} + y^{\delta_2}, \ g(y) = y^{-2}, \ h(y) = y^{\delta_2},$$
$$c_0 = \int_0^1 dA(s) = \frac{1}{2}, \ b_0 = \frac{1}{2}\mu,$$
$$\gamma(t) = \frac{1}{1000}\frac{1}{1-t}, \ c_1 = \int_0^1 (1-s)\gamma(s) ds = \frac{1}{1000}.$$

It is easy to see that (C_1), (C_3)-(C_4) and (4.1) hold, and since $F(t,y) \geqslant \frac{1}{(4c_1)^2} = (250)^2$ and

$$\int_0^1 (1-s)q(s)(250)^2 ds > 2c_1.$$

Then (4.2) is true.

Since

$$\frac{1}{1 + \frac{h(1)}{g(1)}} \int_{c_0 1^{\frac{1}{2}}}^1 \frac{1}{g(\frac{1}{2}y)} dy = \frac{1 - \frac{1}{8}}{24},$$

letting $\mu_0 < 2\frac{1-\frac{1}{8}}{24}$, we have

$$\sup_{r\in(2c_1,+\infty)} \frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 r^{\frac{1}{2}}}^{r} \frac{1}{g(\frac{1}{2}y)} dy > \int_0^1 (1-t)q(t)dt = b_0$$

for all $\mu \leqslant \mu_0$, which means that (4.4) is true. Moreover, since

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$$

uniformly on $[0,1]$, (4.5) is true. Then, Theorem 4.1 implies that (4.24)-(4.25) has at least two positive solutions.

5. Positive solutions for singular boundary value problems with sign-changing nonlinearities

(H1) $f(t,y) \in C([0,1]\times(0,+\infty),(-\infty,+\infty))$,

(H2) $a(t) \in C([0,1],(0,+\infty))$, $(1-t)q(t) \in L(0,1]$,

(H3) there exist $F(y) \in C((0,+\infty),(0,+\infty))$, $G(y) \in C([0,+\infty),[0,+\infty))$ such that $f(t,y) \leqslant F(y)+G(y)$.

(S1) $f(t,y) \geqslant a(t)$ hold for $0 < y < b$,

(S2) $F(y)$ is decreasing in $(0,+\infty)$,

(S3) there exist $R > 1$, such that $\int_{c_0 R^\beta}^{R} \frac{dy}{F(y)} \cdot (1+\frac{\bar{G}(R)}{F(R)})^{-1} > \int_0^1 (1-s)q(s)ds$, where $\bar{G}(R) = \max_{s\in[0,R]} G(s)$.

For $y \in C[0,1]$, we define T as

$$(Ty)(t) = \int_0^1 (y(s))^\beta dA(s) + \int_0^1 k(t,s)q(s)f(s,\max\{\frac{b}{k_0},y(s)\})ds,\ t \in [0,1],$$

where $\frac{b}{k_0} < \min\{R,\frac{b}{2}\}$.

From a standard argument, we have the following result.

LEMMA 5.1. *Suppose (C_1), $(H1)$-$(H3)$ and $(S1)$-$(S2)$ hold. Then the operator T is continuous and compact from $C[0,1]$ to $C[0,1]$.*

THEOREM 5.2. *label5.1 Suppose (C_1), $(H1)$-$(H3)$ and $(S1)$-$(S3)$ hold. Then BVP(1.1)-(1.2) has at least one positive solution $y \in C[0,1]$ with*

$$\frac{b}{k_0} \leqslant y(t) \leqslant R,\ t \in [0,1]. \tag{5.1}$$

Proof. Let $\Omega=\{y\in C|\|y\|<R\}$. For $y\in\partial\Omega$, we now prove that

$$y(t)\neq\lambda(Ty)(t)=\lambda\int_0^1(y(s))^\beta dA(s)+\lambda\int_0^1 k(t,s)q(s)f(s,\max\{\frac{b}{k_0},y(s)\})ds \quad (5.2)$$

for $t\in[0,1]$ any $\lambda\in(0,1)$.

Suppose (5.2) is not true. Then there exists $y\in C[0,1]$ with $\|y\|=R$ and $0<\lambda<1$ such that for $t\in[0,1]$,

$$y(t)=\lambda(Ty)(t)=\lambda\int_0^1(y(s))^\beta dA(s)+\lambda\int_0^1 k(t,s)q(s)f(s,\max\{\frac{b}{k_0},y(s)\})ds. \quad (5.3)$$

It is easy to see that $y'(0)=0$, $y(1)=\lambda\int_0^1 y^\beta(s)dA(s)$.

We first claim that $y(t)\geqslant\lambda\frac{b}{k_0}$ for any $t\in[0,1]$.

Suppose there exists an $\eta\in(0,1)$ with $y(\eta)<\lambda\frac{b}{k_0}$. Let

$$\gamma_0=\inf\{t_1:\ y(s)<\lambda\frac{b}{k_0},\forall s\in[t_1,\eta]\},$$
$$\gamma_1=\sup\{t_1:\ y(s)<\lambda\frac{b}{k_0},\forall s\in[\eta,t_1]\}.$$

There are two cases to consider:

(1) $\gamma_0=0$ and $\gamma_1=1$. Then, $y(0)\leqslant\lambda\frac{b}{k_0}$, $y(1)\leqslant\lambda\frac{b}{k_0}$ and $y(t)<\lambda\frac{b}{k_0}$ for all $t\in(0,1)$, which implies

$$y''(t)=-\lambda f(t,\frac{b}{k_0})<0,t\in(0,1)$$

and so $y(t)$ is concave down on $[0,1]$. Since $y'(0)=0$, we have $y'(t)<0$ for all $t\in(0,1)$, which implies that $|y(1)|=R$. However (C_1) guarantees that

$$|y(1)|=|\lambda\int_0^1(y(s))^\beta dA(s)|\leqslant R^\beta\int_0^1|dA(s)|<R.$$

This is a contradiction.

(2) $\gamma_0>0$ or $\gamma_1<1$. We assume that $\gamma_0>0$. Then, $y(\gamma_0)=\lambda\frac{b}{k_0}$, $y(\gamma_1)\leqslant\lambda\frac{b}{k_0}$ and $y(t)<\lambda\frac{b}{k_0}$ for all $t\in(\gamma_0,\gamma_1)$, which implies

$$y''(t)=-\lambda f(t,\frac{b}{k_0})<0,t\in(\gamma_0,\gamma_1)$$

and so $y(t)$ is concave down on $[\gamma_0,\gamma_1]$. If $\gamma_1=1$, from $\|y\|=R$, the concavity of $y(t)$ implies that $|y(1)|=R$. However (C_1) guarantees that

$$|y(1)|=|\lambda\int_0^1 y^\beta(s)dA(s)|\leqslant R^\beta\int_0^1|dA(s)|<R.$$

This is a contradiction. If $\gamma_1 < 1$, we have $y(\gamma_1) = \lambda \frac{b}{k_0}$. Then the concavity of $y(t)$ implies that $y(t) \geqslant \lambda \frac{b}{k_0}$. This is a contradiction also.

Next we claim that

$$\int_{c_0 R^\beta}^{R} \frac{dy}{F(y)} \leqslant \big(1 + \frac{\bar{G}(R)}{F(R)}\big) \int_0^1 (1-s)q(s)\,ds. \tag{5.4}$$

Since $y(1) = \lambda[\int_0^1 y^\beta(s)dA(s)] \leqslant c_0 R^\beta < R$ and $y'(0) = 0$, there exist $t^* \in [0,1)$ such that $y(t^*) = R$, $y'(t^*) = 0$. Setting $t' = \sup\{t^* : t^* \in [0,1), y'(t^*) = 0, y(t^*) = \|y\| = R\}$, we obtain $t' \in [0,1), y'(t') = 0, y(t') = \|y\| = R$. Let $t_1 = \inf\{0 < t \leqslant 1 | y(t) = y(1)\}$. It is easy to see that $t' < t_1 \leqslant 1$ and $y(t) > y(t_1)$ for all $t \in (t', t_1)$. Furthermore we get a countable set $\{t_i\}$ of $(0,1]$ such that

1. $t' > \cdots \geqslant t_{2m} > t_{2m-1} > \cdots > t_5 \geqslant t_4 > t_3 \geqslant t_2 > t_1$, $t_{2m} \to t'$,
2. $y(t_{2i}) = y(t_{2i+1})$, $y'(t_{2i}) = 0$, $i = 1,2,3\cdots$,
3. $y(t)$ is strictly decreasing in $[t_{2i}, t_{2i-1}]$, $i = 1,2,3\cdots$ (if $y(t)$ is strictly decreasing in $[t', t_1]$, put $m = 1$; i.e, $[t_2, t_1] = [t', t_1]$).

Differentiating (5.3) and using the assumptions $(H3)$, $(S1)$-$(S3)$, we obtain

$$\begin{aligned}
-y''(t) &= \lambda q(t) f(t, \max\{\frac{b}{k_0}, y(t)\}) \\
&\leqslant \lambda q(t)(F(\max\{\frac{b}{k_0}, y(t)\}) + G(\max\{\frac{b}{k_0}, y(t)\})) \\
&= \lambda q(t) F(\max\{\frac{b}{k_0}, y(t)\})(1 + \frac{G(\max\{\frac{b}{k_0}, y(t)\})}{F(\max\{\frac{b}{k_0}, y(t)\})}) \\
&< q(t) F(\max\{\frac{b}{k_0}, y(t)\})(1 + \frac{\bar{G}(R)}{F(R)}) \\
&\leqslant q(t) F(y(t))(1 + \frac{\bar{G}(R)}{F(R)}), \quad t \in [t_{2i}, t_{2i-1}),\ i = 1,2,3\cdots.
\end{aligned} \tag{5.5}$$

Integrating (5.5) from t_{2i} to t, we have by the decreasing property of $F(y)$,

$$-\int_{t_{2i}}^{t} y''(s)ds \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i}}^{t} q(s)F(y(s))ds \leqslant F(y(t))(1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i}}^{t} q(s)ds,$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1,2,3\cdots$; that is to say

$$-y'(t) \leqslant F(y(t))(1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i}}^{t} q(s)ds, \quad t \in [t_{2i}, t_{2i-1}),\ i = 1,2,3\cdots. \tag{5.6}$$

It follows from (5.6) that

$$-\frac{y'(t)}{F(y(t))} \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i}}^{t} q(s)ds \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^t q(s)ds, \tag{5.7}$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1,2,3\cdots$.

On the other hand, for any $z \in (0,1)$ with $y(z) > c_0 R^\beta$, we can choose i_0 and $z' \in (t', t_1)$ such that $z' \in [t_{2i_0}, t_{2i_0-1})$, $y(z') = y(z)$ and $z \leqslant z'$. Integrating (5.7) from t_{2i} to t_{2i-1}, $i = 1,2,3...i_0 - 1$ and from t_{2i_0} to z', we have

$$\int_{y(t_{2i-1})}^{y(t_{2i})} \frac{dy}{F(y)} \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{t_{2i}}^{t_{2i-1}} \int_0^t q(s)\,ds\,dt, \quad i = 1,2,3\cdots i_0 - 1, \tag{5.8}$$

and

$$\int_{y(t_{2i_0})}^{y(z')} \frac{dy}{F(y)} \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{z'}^{t_{2i_0}} \int_0^t q(s)\,ds\,dt. \tag{5.9}$$

Summing (5.8) from 1 to $i_0 - 1$, we have by (5.9) and $y(t_{2i}) = y(t_{2i+1})$, that

$$\int_{y(t_1)}^{y(z')} \frac{dy}{F(y)} \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{z'}^{t_1} \int_0^t q(s)\,ds\,dt \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{z}^{t_1} \int_0^t q(s)\,ds\,dt.$$

Since $y(z) = y(z')$,

$$\int_{y(t_1)}^{y(z)} \frac{dy}{F(y)} \leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{z}^{t_1} \int_0^t q(s)\,ds\,dt. \tag{5.10}$$

Letting $z \to t'$ in (5.10), we have

$$\begin{aligned} \int_{c_0 R^\beta}^{R} \frac{dy}{F(y)} \leqslant \int_{y(t_1)}^{R} \frac{dy}{F(y)} &\leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_{t'}^{t_1} \int_t^1 q(s)\,ds\,dt \\ &\leqslant (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^1 \int_0^t q(s)\,ds\,dt \\ &= (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^1 (1-s) q(s) ds. \end{aligned}$$

Then, (5.4) is true, which contradicts $\int_{c_0 R^\beta}^{R} \frac{dy}{F(y)} > (1 + \frac{\bar{G}(R)}{F(R)}) \int_0^1 (1-s) q(s) ds$. Hence (5.2) holds.

It follows from Lemma 2.3 that T has a fixed point y in C. Using y and 1 in place of y and λ in (5.3), we obtain easily $\frac{b}{k_0} \leqslant y(t) \leqslant R, t \in [0,1]$. And y satisfies that

$$y(t) = \int_0^1 (y(s))^\beta dA(s) + \int_0^1 k(t,s) q(s) f(s, y(s)) ds, \quad t \in [0,1],$$

i.e., y is a positive solution for BVP(1.1)-(1.2). The proof is complete. □

COROLLARY 5.3. *Suppose the assumptions of Theorem 5.1 hold. If further $f(t,\cdot)$ is non-increasing in $(0,+\infty)$ for each $t \in (0,1)$ and $\beta = 1$, the solution of BVP(1.1)-(1.2) is unique.*

Proof. Suppose $y_1(t)$ and $y_2(t)$ are two solutions of BVP(1.1)-(1.2). We need to prove that $y_1(t) \equiv y_2(t), t \in [0,1]$. Let $z(t) = y_1(t) - y_2(t), t \in [0,1]$.

We claim that $z(t) \equiv 0, t \in (0,1)$. In fact, if it is not true, without loss of generality, we assume $z(t_0) > 0$ for some $t_0 \in (0,1)$. Let $t_3 = \max\{t \in (0,t_0), z(t) = 0\}, t_4 = \min\{t \in (t_0,1), z(t) = 0\}$.

There are two cases to consider:

(1) $t_3 = 0$, $t_4 = 1$, which implies that $z(t) > 0$, $t \in (0,1)$ and

$$0 < z(1) = \int_0^1 z(s)dA(s) \leqslant \max_{t\in[0,1]} z(t) \int_0^1 dA(s) < \max_{t\in[0,1]} z(t).$$

Then

$$\begin{aligned} -z''(t) &= q(t)f(t,y_1(t)) - q(t)f(t,y_2(t)) \\ &\leqslant 0, \forall t \in (0,1), \end{aligned}$$

which yields that $z(t)$ is convex on $(0,1)$. Since $z'(0) = 0$, we have $z(t)$ is nondecreasing on $(0,1)$. And so $z(1) = \max_{t\in[0,1]} z(t)$. This contradicts $z(1) = \int_0^1 z(s)dA(s) < \max_{t\in[0,1]} z(t)$.

(2) $t_3 \geqslant 0$, $t_4 < 0$, which implies that $z(t) > 0$, $t \in (t_3,t_4)$, $z(t_4) = 0$. Then

$$\begin{aligned} -z''(t) &= q(t)f(t,y_1(t)) - q(t)f(t,y_2(t)) \\ &\leqslant 0, \forall t \in (t_3,t_4), \end{aligned}$$

which yields that $z(t)$ is convex on (t_3,t_4). If $t_3 > 0$, we have $z(t_3) = 0$. This contradicts $\max_{t\in[t_3,t_4]} z(t) > z(t_3) = z(t_4) = 0$. If $t_3 = 0$, we have $z'(t_3) = 0$ and $z(t)$ is nondecreasing on (t_3,t_4), which together with $z(t_4) = 0$ means that $z(t) \leqslant 0$ for all $t \in (t_3,t_4)$. This contradicts $z(t) > 0$ for all $t \in (t_3,t_3)$.

Hence we get $y_1(t) = y_2(t), t \in [0,1]$. Thus the result is proved. □

EXAMPLE 5.4. Consider

$$y''(t) + \frac{1}{4}(\cos^2 t + \frac{1}{y^2(t)} - y^2(t)) = 0,\ 0 < t < 1, \tag{5.11}$$

$$y'(0) = 0,\ y(1) = \int_0^1 y(s)dA(s), dA(s) = sds, \tag{5.12}$$

Let

$$q(t) = \frac{1}{8},\ f(t,y) = \cos^2 t + \frac{1}{y^2} - y^2,\ G(y) = 1 + y^2,\ F(y) = \frac{1}{y^2},\ b = \frac{1}{2},\ a(t) = \frac{11}{4}.$$

It is easy to see that $(H1)$-$(H3)$ and $(S1)$-$(S2)$ hold. Let $R = 2$. We have

$$\int_1^2 \frac{1}{F(y)}dy = \frac{8}{3},\ (1 + \frac{G(2)}{F(2)}) \int_0^1 (1-s)q(s)ds = \frac{21}{16},$$

so $(S3)$ holds. Corollary 5.1 implies that (5.13)-(5.14) has at least one positive solution.

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(Received July 17, 2013)

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