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FORCED OSCILLATION OF SECOND–ORDER NONLINEAR FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we establish some new oscillation criteria for the second-order nonlinear functional dynamic equation with forced term

$$(r(t)x^{\Delta}(t))^{\Delta} \pm p(t)f(x(\tau(t))) = e(t),$$

on a time scale $\mathbb{T}$. No restriction is imposed on the forcing term $e(t)$ to satisfy the Kartsatos condition. $p(t)$ and $r(t)$ are real-valued rd-continuous functions defined on $\mathbb{T}$. There are many cases have been taken into consideration: (i) $p(t) > 0$, $\tau(t) \leqslant t(\geqslant t)$ and $\tau(t) \leqslant \sigma(t)(\geqslant \sigma(t))$ (ii) $p(t)$ changes its sign, $\tau(t) \leqslant t(\geqslant t)$, $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing differentiable function and $\lim_{t\to\infty} \tau(t) = \infty$. Our results not only generalize and extend some existing results but also can be applied to the oscillation problems that are not covered in literature. Finally, we give some examples to illustrate our main results.

1. Introduction

The theory of time scales was introduced by Hilger [6] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus, where a time scale is an arbitrary closed subset of the reals. The cases when time scale is equal to the reals or to the integers represent the classical theories of differential and difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}^n$ the space of the harmonic numbers. For an introduction to time scale calculus and dynamic equations, we refer to the seminal books by Bohner and Peterson [1, 2].

In recent years, there has been much research activity concerning the oscillation of solutions of various forced second-order dynamic equations on time scales, we refer the reader to the articles [3, 4, 7, 8, 9, 10, 11, 13] and references cited thereien. Bohner and Tisdell [3] examined oscillation and non oscillation for

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x(\sigma(t)) = e(t), \qquad t \in \mathbb{T},$$

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Huang and Feng [7] considered the following second-order forced nonlinear dynamic equations

$$x^{\Delta\Delta}(t)+p(t)f(x(\sigma(t)))=e(t), \qquad t\in\mathbb{T},$$

and in [8], the authors studied the oscillation of the forced dynamic equations

$$x^{\Delta\Delta}(t)+p(t)f(x(t))=e(t), \qquad t\in\mathbb{T},$$

Tongxin li *et al.* [9] studied the oscillation of the forced dynamic equations

$$x^{\Delta\Delta}(t)+p(t)f(x(q(t)))=e(t), \qquad t\in\mathbb{T},$$

Peiguang Wang *et al.* [13] studied the oscillation of the forced dynamic equations of the form

$$x^{\Delta\Delta}(t)\pm p(t)f(x(\sigma(t)))=e(t), \qquad t\in\mathbb{T},$$

Yuangang Sun [11] investigated the second-order forced dynamic equations of the form

$$x^{\Delta\Delta}(t)-p(t)|x(q(t)|^{\lambda-1}x(q(t))=e(t), \qquad t\in\mathbb{T},$$

Oscillatory criteria for the forced dynamic equations

$$(a(t)x^{\Delta}(t))^{\Delta}+p(t)f(x(\sigma(t)))=r(t), \qquad t\in\mathbb{T},$$

where $\int_{t_0}^{\infty}|r(s)|\Delta s<\infty$ are analyzed in [10].

We are concerning here with the following second-order forced nonlinear functional dynamic equations of the form

$$(r(t)x^{\Delta}(t))^{\Delta}+p(t)f(x(\tau(t)))=e(t),\ t\in\mathbb{T},\ t\geqslant t_0, \tag{1.1}$$

and

$$(r(t)x^{\Delta}(t))^{\Delta}-p(t)f(x(\tau(t)))=e(t),\ t\in\mathbb{T},\ t\geqslant t_0, \tag{1.2}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0\in\mathbb{T}$; $r(t)$, $p(t)$ and $e(t)$ are real-valued right continuous functions on $\mathbb{T}$ with $p(t)>0$. The function $\tau(t)$ also satisfies $\tau:\mathbb{T}\to\mathbb{T}$, $\tau(t)\to\infty$ as $t\to\infty$ and $f\in C(\mathbb{R},\mathbb{R})$, $xf(x)>0$ whenever $x\neq 0$.

The main purpose of this paper is to extend and generalize the forementioned equations and some existing results. Our results not only include existing results as special cases, but also can be used to answer the oscillation problem of Eq. (1.1) and Eq. (1.2) that are not covered by existing results.

By a solution of (1.1) or (1.2), we mean that a nontrivial real-valued function x satisfies (1.1) or (1.2) for $t\in\mathbb{T}$. A solution x of (1.1) or (1.2) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Eq. (1.1) or (1.2) is said to be oscillatory if all of its nonconstant solutions defined for all large t are oscillatory.

2. Main results

In this section, we establish some sufficient conditions for the oscillation of equations (1.1) and (1.2). Our approach is based largely on the application of $H(t,s)$ and the following lemma. Let

$$\mathbb{D}=\{(t,s)\in\mathbb{T}\times\mathbb{T}: t>s\geqslant t_0\},\ \mathbb{D}_0=\{(t,s)\in\mathbb{T}\times\mathbb{T}: t\geqslant s\geqslant t_0\}.$$

We say that the function $H\in C_{rd}(\mathbb{D},\mathbb{R})$ belongs to the class $\Im$, if

(H_1) $H(t,t)=0$, $t\geqslant t_0$, $H(t,s)>0$ on $\mathbb{D}_0$,

(H_2) H has a non positive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ and a non negative continuous second-order Δ-partial derivative $H^{\Delta^2_{s^2}}(t,s)$ with respect to the second variable,

(H_3) $H^{\Delta_s}(t,t)=0$, $\lim_{t\to\infty}\frac{H^{\Delta_s}(t,t_0)}{H(t,t_0)}=O(1)$.

LEMMA 2.1. *(see [5]). If A and B are positive constants, then*

$$A^{\lambda}+(\lambda-1)B^{\lambda}-\lambda AB^{\lambda-1}\geqslant 0,\qquad \lambda>1.$$

I- Oscillatory behavior of solutions of Eq. (1.1):

THEOREM 2.1. *Assume that $f\in C(\mathbb{R},\mathbb{R})$ satisfies $xf(x)>0$ for $x\neq 0$. If there exists a function $H\in\Im$, such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\geqslant 0, \tag{2.1}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s=\infty, \tag{2.2}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s=-\infty. \tag{2.3}$$

Then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Suppose that $x(t)>0$ for $t\geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Multiplying Eq. (1.1) by $H(t,\sigma(s))$ for $t\geqslant t_0$ and integrating from t_0 to t, we get

$$\int_{t_0}^{t}H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s+\int_{t_0}^{t}H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s =\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s. \tag{2.4}$$

Using the integration by parts two times, we have

$$\begin{aligned}
&\int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s \\
&\quad = -H(t,t_0)r(t_0)x^{\Delta}(t_0) - \int_{t_0}^{t} H^{\Delta_s}(t,s)r(s)x^{\Delta}(s)\Delta s \\
&\quad = -H(t,t_0)r(t_0)x^{\Delta}(t_0) + H^{\Delta_s}(t,t_0)r(t_0)x(t_0) \\
&\qquad + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s \\
&\quad = M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s, \qquad (2.5)
\end{aligned}$$

where $M(t,t_0) = -H(t,t_0)r(t_0)x^{\Delta}(t_0) + H^{\Delta_s}(t,t_0)r(t_0)x(t_0)$. Substituting (2.5) into (2.4), we have

$$\begin{aligned}
\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = M(t,t_0) &+ \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s \\
&+ \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s. \qquad (2.6)
\end{aligned}$$

Dividing through by $H(t,t_0)$, we have

$$\begin{aligned}
\frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = \frac{M(t,t_0)}{H(t,t_0)} &+ \frac{1}{H(t,t_0)} \int_{t_0}^{t} [(H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s)) \\
&+ H(t,\sigma(s))p(s)f(x(\tau(s)))]\Delta s. \qquad (2.7)
\end{aligned}$$

Taking $\liminf$ as $t \to \infty$, we derive a contradiction. The proof is completed.

REMARK 2.1. 1. Theorem 2.1 is delay-independent. 2. This Theorem is true for $\tau(t) = \sigma(t)$, for $\tau(t) \leqslant \sigma(t)$ and for $\tau(t) \geqslant \sigma(t)$.

EXAMPLE 2.1. Consider the equation $(\mathbb{T} = \mathbb{R})$

$$\left(\frac{1}{t}x'(t)\right)' + p(t)f(x(\tau(t))) = t^{\alpha}\sin t, \qquad (2.8)$$

where $\alpha > 0$ and $xf(x) > 0$. Here, $r(t) = \frac{1}{t}$ and $e(t) = t^{\alpha}\sin t$. To apply Theorem 2.1, let us take $H(t,s) = (t-s)^{\beta}$, $\beta > 1$. Therefore, we have

$$(H'(t,s)r(s))' = \frac{\beta(t-s)^{\beta-1}}{s^2} + \frac{\beta(\beta-1)(t-s)^{\beta-2}}{s} > 0,$$

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^\beta}\int_{t_0}^{t}(t-s)^\beta e(s)ds=\infty,$$

$$\liminf_{t\to\infty}\frac{1}{(t-t_0)^\beta}\int_{t_0}^{t}(t-s)^\beta e(s)ds=-\infty.$$

Hence, by Theorem 2.1, Eq. (2.8) is oscillatory.

REMARK 2.2. The results of [9, 13] can not be applied to Eq. (2.8) for $r(t)=\frac{1}{t}$. But, according to Theorem 2.1, when $(\mathbb{T}=\mathbb{R})$ and $H(t,s)=(t-s)^\beta$, $\beta>1$, this equation is oscillatory for $\alpha>0$.

THEOREM 2.2. *Assume that*

(i) $\tau(t)\geqslant\sigma(t)$,

(ii) *there exist two positive constants c and ν such that*

$$|f(x)|\geqslant c|x|^\nu,\qquad \nu>1,$$

(iii) *if there exist a kernel function $H(t,s)$ satisfying (H_1)-(H_3) such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\leqslant 0, \tag{2.9}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s-\int_{\sigma(t_0)}^{\tau(t)}P(t,s)\Delta s\Big]=\infty, \tag{2.10}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s+\int_{\sigma(t_0)}^{\tau(t)}P(t,s)\Delta s\Big]=-\infty, \tag{2.11}$$

where

$$P(t,s)=(\nu-1)\nu^{\frac{\nu}{1-\nu}}\big[|(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^\Delta\big]^{\frac{\nu}{\nu-1}}$$
$$\times\big[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^\Delta\big]^{\frac{1}{1-\nu}},$$

$\tau^*(t)$ and $\sigma^*(t)$ are the inverse functions of $\tau(t)$ and $\sigma(t)$ respectively, then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Suppose that $x(t)>0$ for $t\geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 2.1 to get (2.6), i.e.,

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s=M(t,t_0)+\int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s$$
$$+\int_{t_0}^{t}H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s.$$

Since $|f(x)| \geqslant c|x|^{\nu}$, $\nu > 1$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s \\ + c\int_{t_0}^{t} H(t,\sigma(s))p(s)x^{\nu}(\tau(s))\Delta s. \quad (2.12)$$

Since

$$\int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s \\ = \int_{\sigma(t_0)}^{\sigma(t)} (H^{\Delta}(t,\sigma^*(\theta))r(\sigma^*(\theta)))^{\Delta}(\sigma^*(\theta))^{\Delta} x(\theta)\Delta\theta, \quad (2.13)$$

and

$$\int_{t_0}^{t} H(t,\sigma(s))p(s)x^{\nu}(\tau(s))\Delta s \\ = \int_{\tau(t_0)}^{\tau(t)} cH(t,,\sigma(\tau^*(\xi)))p(\tau^*(\xi))(\tau^*(\xi))^{\Delta} x^{\nu}(\xi)\Delta\xi, \quad (2.14)$$

then (2.12), becomes

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) \\ + \int_{\sigma(t_0)}^{\sigma(t)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s \\ + c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s))p(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s. \quad (2.15)$$

Since $\tau(t) \geqslant \sigma(t)$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) \\ - \int_{\sigma(t_0)}^{\tau(t)} |(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta} x(s)\Delta s \\ + c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s,$$

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = M(t,t_0) \\ - c\int_{\sigma(t_0)}^{\tau(t_0)} H(t,\sigma(\tau^*(s))p(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s$$

$$+\int_{\sigma(t_0)}^{\tau(t)}[cH(t,\sigma(\tau^*(s))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s) \\ -|(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta}x(s)]\Delta s,$$

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geqslant N(t,t_0) \\ +\int_{\sigma(t_0)}^{\tau(t)}[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s) \\ -|(H^{\Delta_s}(t,\sigma^*(s)r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta}x(s)]\Delta s,$$

where

$$N(t,t_0)=M(t,t_0)-\int_{\sigma(t_0)}^{\tau(t_0)}cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s.$$

Set

$$A^{\nu}=[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}]x^{\nu}(s),$$
$$B^{\nu-1}=\frac{|(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta}}{\nu[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}]^{\frac{1}{\nu}}}.$$

Applying Lemma 2.1, we get

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\geqslant N(t,t_0)-\int_{\sigma(t_0)}^{\tau(t)}P(t,s)\Delta s,$$

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s+\int_{\sigma(t_0)}^{\tau(t)}P(t,s)\Delta s\geqslant N(t,t_0). \tag{2.16}$$

Thus, multiplying (2.16) by $H^{-1}(t,t_0)$ and taking the lower limit of (2.16), we get a contradiction with (2.11). This completes the proof.

THEOREM 2.3. *Assume that:*

(i) $\tau(t)\leqslant\sigma(t)$,

(ii) *there exist two positive constants c and* ν *such that*

$$|f(x)|\leqslant c|x|^{\nu},\qquad 0<\nu<1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* $(H_1)-(H_3)$ *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\leqslant 0, \tag{2.17}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}[\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s+\int_{\tau(t_0)}^{\sigma(t)}P(t,s)\Delta s]=\infty, \tag{2.18}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s-\int_{\tau(t_0)}^{\sigma(t)}P(t,s)\Delta s\Big]=-\infty, \tag{2.19}$$

where $P(t,s)$, $\tau^*(t)$ and $\sigma^*(t)$ are the same as in Theorem 2.2, then Eq. (1.1) is oscillatory.

THEOREM 2.4. *Assume that:*

(i) $\tau(t)\leqslant t$,

(ii) *there exist two positive constants c and ν such that*

$$|f(x)|\leqslant c|x|^{\nu}, \qquad 0<\nu<1,$$

(iii) *if there exist a kernel function $H(t,s)$ satisfying (H_1)-(H_3) such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\leqslant 0, \tag{2.20}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s+\int_{t_0}^{\tau(t)}P(t,s)\Delta s\Big]=\infty, \tag{2.21}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s-\int_{t_0}^{\tau(t)}P(t,s)\Delta s\Big]=-\infty, \tag{2.22}$$

where $P(t,s)$, $\tau^*(t)$ and $\sigma^*(t)$ are the same as in Theorem 2.2, then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.1). Suppose that $x(t)>0$ for $t\geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 2.1 to get (2.6), i.e.,

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s=M(t,t_0)+\int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s + \int_{t_0}^{t}H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s.$$

Since $|f(x)|\leqslant c|x|^{\nu}$, $0<\nu<1$, we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\leqslant M(t,t_0)+\int_{t_0}^{t}(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s))\Delta s + c\int_{t_0}^{t}H(t,\sigma(s))p(s)x^{\nu}(\tau(s))\Delta s. \tag{2.23}$$

Substituting from (2.13) and (2.14) into (2.23), we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\leqslant M(t,t_0)$$

$$+\int_{\sigma(t_0)}^{\sigma(t)}(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}x(s)\Delta s$$
$$+c\int_{\tau(t_0)}^{\tau(t)}H(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s. \quad (2.24)$$

Since $\tau(t)\leqslant t$, we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\leqslant M(t,t_0)$$
$$-\int_{\sigma(t_0)}^{\tau(t)}|(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta}x(s)\Delta s$$
$$+c\int_{\tau(t_0)}^{\tau(t)}H(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s,$$

and

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\leqslant L(t,t_0)$$
$$-\int_{t_0}^{\tau(t)}[|(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta}x(s)$$
$$-cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)]\Delta s, \quad (2.25)$$

where

$$L(t,t_0)=M(t,t_0)+\int_{t_0}^{\sigma(t_0)}|(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}|(\sigma^*(s))^{\Delta}x(s)\Delta s+$$
$$\int_{\tau(t_0)}^{t_0}cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)]\Delta s.$$

Set $F(x)=ax-bx^{\nu}$, for $x>0,\ a\geqslant 0,\ b>0$. If $0<\nu<1$, then $F(x)$ has the minimum $F_{min}=(\nu-1)\nu^{\frac{\nu}{1-\nu}}a^{\frac{\nu}{\nu-1}}b^{\frac{1}{1-\nu}}$. From (2.25), we have

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s\leqslant L(t,t_0)-\int_{t_0}^{\tau(t)}P(t,s)\Delta s.$$

$$\int_{t_0}^{t}H(t,\sigma(s))e(s)\Delta s+\int_{t_0}^{\tau(t)}P(t,s)\Delta s\leqslant L(t,t_0). \quad (2.26)$$

Thus, multiplying (2.26) by $H^{-1}(t,t_0)$ and taking the upper limit of (2.26), we get a contradiction with (2.21). This completes the proof.

THEOREM 2.5. *Assume that*

(i) $\tau(t)\geqslant t$,

(ii) *there exist two positive constants* c *and* ν *such that*

$$|f(x)| \leqslant c|x|^{\nu}, \qquad 0 < \nu < 1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* $(H_1)-(H_3)$ *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \leqslant 0, \tag{2.27}$$

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_t^{\tau(t)} cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta} s^{\nu k} \Delta s < \infty, \tag{2.28}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} [\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{\tau(t_0)}^{t} P(t,s)\Delta s] = \infty, \tag{2.29}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\tau(t_0)}^{t} P(t,s)\Delta s] = -\infty, \tag{2.30}$$

where $P(t,s)$, $\tau^*(t)$ *and* $\sigma^*(t)$ *are the same as in Theorem 2.2, then all solutions of Eq. (1.1) satisfying* $x(t) = O(t^k)$ *are oscillatory.*

EXAMPLE 2.2. Consider the equation $(\mathbb{T} = \mathbb{R})$

$$(tx^{'}(t))^{'} + t^m x^{\nu}(t) = t^{\alpha}\cos t, \tag{2.31}$$

where $m \geqslant 0$, $\alpha > 0$ and $0 < \nu < 1$. Here,

$$r(t) = t,\ p(t) = t^m,\ f(x) = x^{\nu},\ 0 < \nu < 1$$

with $c = 1$ and $e(t) = t^{\alpha}\cos t$. To apply Theorem 2.4, let us take $H(t,s) = (t-s)$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'} = -1 < 0.$$

Since

$$P(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}(t-s)^{\frac{1}{1-\nu}} s^{\frac{m}{1-\nu}},$$

then

$$\begin{aligned}\int_0^t P(t,s)ds &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} \int_0^t (t-s)^{\frac{1}{1-\nu}} s^{\frac{m}{1-\nu}} ds \\ &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} t^{\frac{m+1}{1-\nu}+1} \int_0^1 (1-u)^{\frac{1}{1-\nu}} u^{\frac{m}{1-\nu}} du \\ &= (\nu-1)\nu^{\frac{\nu}{1-\nu}} B(\frac{1}{1-\nu}+1, \frac{m}{1-\nu}+1) t^{\frac{m+1}{1-\nu}+1},\end{aligned}$$

where the beta function $B(\frac{1}{1-\nu}+1, \frac{m}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t (t-s)s^{\alpha}\cos s\, ds = t^{\alpha+2} \int_0^1 (1-u)u^{\alpha}\cos ut\, du = t^{\alpha+2} I_{1,\alpha}(t),$$

where $I_{1,\alpha}(t)$ has the asymptotic formula

$$I_{1,\alpha}(t) = \Gamma(2)t^{-2}\cos(t-\pi) + o(t^{-2}) \quad \text{as } t \to \infty.$$

Consequently, Eq. (2.31) is oscillatory if $\alpha > \frac{m+1}{1-\nu} + 1$.

EXAMPLE 2.3. Consider the equation $(\mathbb{T} = \mathbb{R})$

$$-x''(t) + t^m x^\nu(t) = t^\alpha \cos t, \tag{2.32}$$

where $m \geqslant 0$, $\alpha > 0$ and $0 < \nu < 1$. Here,

$$r(t) = -1,\ p(t) = t^m,\ f(x) = x^\nu,\ 0 < \nu < 1$$

with $c = 1$ and $e(t) = t^\alpha \cos t$. To apply Theorem 2.3, let us take $H(t,s) = (t-s)^\beta$, $\beta > 1$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'} = -\beta(\beta-1)(t-s)^{(\beta-2)} < 0.$$

Since

$$P(t,s) = (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}(t-s)^{\frac{(2-\beta)\nu+\beta}{1-\nu}} s^{\frac{m}{1-\nu}},$$

then

$$\begin{aligned}
\int_0^t P(t,s)ds &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}} \int_0^t (t-s)^{\frac{(2-\beta)\nu+\beta}{1-\nu}} s^{\frac{m}{1-\nu}} ds \\
&= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}} t^{\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1} \int_0^1 (1-u)^{\frac{(2-\beta)\nu+\beta}{1-\nu}} u^{\frac{m}{1-\nu}} du \\
&= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}} B\Big(\frac{m}{1-\nu}+1, \frac{(2-\beta)\nu+\beta}{1-\nu}+1\Big) t^{\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1},
\end{aligned}$$

where the beta function $B(\frac{m}{1-\nu}+1, \frac{(2-\beta)\nu+\beta}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t (t-s)^\beta s^\alpha \cos s\, ds = t^{\beta+\alpha+1} \int_0^1 (1-u)^\beta u^\alpha \cos ut\, du = t^{\beta+\alpha+1} I_{\beta,\alpha}(t),$$

where $I_{\beta,\alpha}(t)$ has the asymptotic formula

$$I_{\beta,\alpha}(t) = \Gamma(\beta+1)t^{-\beta-1}\cos(t - \frac{(\beta+1)\pi}{2}) + o(t^{-\beta-1}) \text{ as } t \to \infty.$$

Consequently, Eq. (2.32) is oscillatory if $\alpha > \frac{(2-\beta)\nu+\beta+m}{1-\nu} + 1$.

REMARK 2.3. The results of [10] can not be applied to Eq. (2.32) for $r(t) = -1 < 0$. But, according to Theorem 2.3, when $(\mathbb{T} = \mathbb{R})$ and $H(t,s) = (t-s)^\beta$, $\beta > 1$, this equation is oscillatory.

EXAMPLE 2.4. Consider the equation $(\mathbb{T}=\mathbb{R})$

$$-x''(t)+t^m\,|x(t-\tau)|^{\nu} sgn\, x(t-\tau)=t^{\alpha}\cos t, \tag{2.33}$$

where $m \geqslant 0$, $\alpha>0$ and $0<\nu<1$. Here,

$$r(t)=-1,\ p(t)=t^m,\ f(x)=x^{\nu},\ 0<\nu<1$$

with $c=1$ and $e(t)=t^{\alpha}\cos t$. To apply Theorem 2.3, let us take $H(t,s)=(t-s)^{\beta}$, $\beta>1$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'}=-\beta(\beta-1)(t-s)^{(\beta-2)}<0.$$

Since $P(t,s)=(\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}(t-s)^{\frac{(2-\beta)\nu}{1-\nu}}(t-s-\tau)^{\frac{\beta}{1-\nu}}(s+\tau)^{\frac{m}{1-\nu}}$, then

$$\begin{aligned}\int_0^{t-\tau}P(t,s)ds &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}\int_{\tau}^{t}(t-s+\tau)^{\frac{(2-\beta)\nu}{1-\nu}}(t-s)^{\frac{\beta}{1-\nu}}s^{\frac{m}{1-\nu}}ds\\ &\geqslant (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}t^{\frac{(2-\beta)\nu}{1-\nu}}\int_0^t(t-s)^{\frac{\beta}{1-\nu}}s^{\frac{m}{1-\nu}}ds\\ &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}t^{\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1}\int_0^1(1-u)^{\frac{\beta}{1-\nu}}u^{\frac{m}{1-\nu}}du\\ &= (\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}B(\frac{m}{1-\nu}+1,\frac{\beta}{1-\nu}+1)t^{\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1},\end{aligned}$$

where the beta function $B(\frac{m}{1-\nu}+1,\frac{\beta}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t(t-s)^{\beta}s^{\alpha}\cos s\,ds=t^{\beta+\alpha+1}\int_0^1(1-u)^{\beta}u^{\alpha}\cos ut\,du=t^{\beta+\alpha+1}I_{\beta,\alpha}(t),$$

where $I_{\beta,\alpha}(t)$ has the asymptotic formula

$$I_{\beta,\alpha}(t)=\Gamma(\beta+1)t^{-\beta-1}\cos(t-\frac{(\beta+1)\pi}{2})+o(t^{-\beta-1})\ \text{ as } t\to\infty.$$

Consequently, Eq. (2.33) is oscillatory if $\alpha>\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1$.

REMARK 2.4. The results of [9, 10, 13] can not be applied to Eq. (2.33) for $\tau(t)\neq\sigma(t)(\tau(t)\leqslant\sigma(t))$. But, according to Theorem 2.3, when $(\mathbb{T}=\mathbb{R})$ and $H(t,s)=(t-s)^{\beta}$, $\beta>1$, this equation is oscillatory.

Next, we have illustrative examples for the difference equation and 2-delay difference equation cases.

EXAMPLE 2.5. Consider the equation $(\mathbb{T} = 2^{\overline{\mathbb{Z}}})$

$$\Delta_2(r(t)\Delta_2 x(t)) - p(t)f(x(\tau((t)))) = e(t), \qquad t \geqslant t_0 := 2, \tag{2.34}$$

where $f^{\Delta}(s) = \Delta_2 f(s) = [f(2s) - f(s)]/(2s - s)$ and $\sigma(s) = 2s$. Here,

$$r(t) = 1,\ p(t) = \frac{t}{\sigma(t)},\ e(t) = t^2 \sin t,\ f(x) = x^{\nu},\ 0 < \nu < 1$$

with $c = 1$ and $\tau(t) = t$. To apply Theorem 2.4, let us take $H(t,s) = (t - s)$. Therefore, we have

$$\Delta_2(\Delta_2 H(t,s)r(s)) = 0,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,\sigma(s))e(s) + P(t,s)]\, \Delta_2\, s$$
$$= \limsup_{t\to\infty} \frac{1}{(t-2)} \int_{2}^{t} (t - 2s)\, s^2 \sin s\, \Delta_2\, s = \infty,$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,\sigma(s))e(s) - P(t,s)]\, \Delta_2\, s$$
$$= \liminf_{t\to\infty} \frac{1}{(t-2)} \int_{2}^{t} (t - 2s)\, s^2 \sin s\, \Delta_2\, s = -\infty,$$

where

$$P(t,s) = 0,\ \sigma^*(s) = \frac{s}{2} \quad \text{and} \quad \tau^*(s) = s.$$

Then, by Theorem 2.4, every solution of Eq. (2.34) is oscillatory.

EXAMPLE 2.6. Consider the equation $(\mathbb{T} = \mathbb{N})$

$$\Delta(r(t)\Delta x(t)) - p(t)f(x(\tau((t)))) = e(t), \tag{2.35}$$

where $f^{\Delta}(s) = \Delta f(s) = f(s+1) - f(s)$ and $\sigma(s) = s + 1$. Here,

$$r(t) = -1,\ p(t) = t\, \sigma(t),\ e(t) = (-1)^t\, t^2,\ f(x) = x^{\nu},\ 0 < \nu < 1$$

with $c = 1$ and $\tau(t) = t$. To apply Theorem 2.4, let us take $H(t,s) = (t - s)$. Therefore, we have

$$\Delta_s(\Delta_s H(t,s)r(s)) = 0,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,\sigma(s))e(s) + P(t,s)]\, \Delta s$$
$$= \limsup_{t\to\infty} \frac{1}{(t-1)} \sum_{s=1}^{t-1} (t - s - 1)(-1)^s\, s^2 = \infty,$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,\sigma(s))e(s) - P(t,s)]\, \Delta s$$

$$= \liminf_{t\to\infty} \frac{1}{(t-1)} \sum_{s=1}^{t-1} (t-s-1)(-1)^s s^2 = -\infty,$$

where

$$P(t,s) = 0,\ \sigma^*(s) = s-1 \quad \text{and} \quad \tau^*(s) = s.$$

Then, by Theorem 2.4, every solution of Eq. (2.35) is oscillatory.

II - Oscillatory behavior of solutions of Eq. (1.2):

In the following we establish oscillation criteria for Eq. (1.2). However, when $\tau(t) \neq \sigma(t)$ (e.g. $\tau(t) \geqslant \sigma(t)$ or $\tau(t) \leqslant \sigma(t)$), the oscillation of this equation is not discussed before except for $\tau(t) = \sigma(t)$ with $r(t) = 1$ (see [9, 13] and references cited therein)

THEOREM 2.6. *Assume that $f \in C(\mathbb{R},\mathbb{R})$ satisfying $xf(x) > 0$ for $x \neq 0$. If there exists a function $H \in \Im$, such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \leqslant 0, \tag{2.36}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = \infty, \tag{2.37}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = -\infty. \tag{2.38}$$

Then Eq. (1.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.2). Suppose that $x(t) > 0$ for $t \geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Multiplying Eq. (1.2) by $H(t,\sigma(s))$ for $t \geqslant t_0$ and integrating from t_0 to t, we get

$$\int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s - \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s = \int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s. \tag{2.39}$$

Using the integration by parts two times, we have

$$\int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s = M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s. \tag{2.40}$$

Substituting from (2.40) into (2.39), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s - \int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s. \tag{2.41}$$

Dividing through by $H(t,t_0)$, we get

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = \frac{M(t,t_0)}{H(t,t_0)} + \frac{1}{H(t,t_0)}\int_{t_0}^{t}[(H^{\Delta_s}(t,s)r(s))^{\Delta_s}x(\sigma(s)) - H(t,\sigma(s))p(s)f(x(\tau(s)))]\Delta s. \quad (2.42)$$

Taking lim sup as $t\to\infty$, we derive a contradiction. The proof is completed.

REMARK 2.5. Theorem 2.6 is true for $\tau(t)\leqslant t$ or $\tau(t)\geqslant t$ and $\tau(t)\leqslant\sigma(t)$ or $\tau(t)\geqslant\sigma(t)$.

EXAMPLE 2.7. Consider the equation $(\mathbb{T}=\mathbb{R})$

$$(tx^{'}(t))^{'} - p(t)f(x(\tau(t))) = t^{\alpha}\cos t, \quad (2.43)$$

where $\alpha>0$ and $xf(x)>0$. Here, $r(t)=t$ and $e(t)=t^{\alpha}\cos t$. To apply Theorem 2.6, let us take $H(t,s)=(t-s)^{\beta}$, $0<\beta<1$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'} = -\beta(t-s)^{\beta-2}(t-\beta s)<0,$$

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}e(s)ds=\infty,$$

$$\liminf_{t\to\infty}\frac{1}{(t-t_0)^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}e(s)ds=-\infty.$$

Hence, by Theorem 2.6, Eq. (2.43) is oscillatory.

REMARK 2.6. The results of [9, 13] can not be applied to Eq. (2.43) for $r(t)=t$. But, according to Theorem 2.6, when $(\mathbb{T}=\mathbb{R})$ and $H(t,s)=(t-s)^{\beta}$, $0<\beta<1$, this equation is oscillatory for $\alpha>0$.

THEOREM 2.7. *Assume that*

(i) $\tau(t)\geqslant\sigma(t)$,

(ii) *there exist two positive constants c and* ν *such that*

$$|f(x)|\geqslant c|x|^{\nu}, \qquad \nu>1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* $(H_1)-(H_3)$ *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s}\geqslant 0, \quad (2.44)$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\sigma(t_0)}^{\tau(t)} Q(t,s)\Delta s\Big] = \infty, \tag{2.45}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{\sigma(t_0)}^{\tau(t)} Q(t,s)\Delta s\Big] = -\infty, \tag{2.46}$$

where

$$Q(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}\big[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}\big]^{\frac{\nu}{\nu-1}}$$
$$\times\big[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}\big]^{\frac{1}{1-\nu}},$$

$\tau^*(s)$ and $\sigma^*(s)$ are the inverse functions of $\tau(s)$ and $\sigma(s)$ respectively, then Eq. (1.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1.2). Suppose that $x(t) > 0$ for $t \geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 2.6 to get (2.41), i.e.,

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s = M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s -$$
$$\int_{t_0}^{t} H(t,\sigma(s))p(s)f(x(\tau(s)))\Delta s. \tag{2.47}$$

Since $|f(x)| \geqslant c|x|^{\nu}$, $\nu > 1$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \leqslant M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s -$$
$$c\int_{t_0}^{t} H(t,\sigma(s))p(s)x^{\nu}(\tau(s))\Delta s. \tag{2.48}$$

Substituting from (2.13) and (2.14) into (2.48), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \leqslant M(t,t_0)$$
$$+\int_{\sigma(t_0)}^{\sigma(t)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s$$
$$-c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s.$$

Since $\tau(t) \geqslant \sigma(t)$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \leqslant M(t,t_0)$$
$$+\int_{\sigma(t_0)}^{\tau(t)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s$$

$$- c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s,$$

$$\begin{aligned}\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s &= M(t,t_0)\\ &\quad + c\int_{\sigma(t_0)}^{\tau(t_0)} H(t,\sigma(\tau^*(s)))p(\tau^*(s))((\tau^*(s)))^{\Delta}x^{\nu}(s)\Delta s\\ &\quad + \int_{\sigma(t_0)}^{\tau(t)}[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}x(s)\\ &\qquad - cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)]\Delta s,\end{aligned}$$

and

$$\begin{aligned}\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s &\leqslant K(t,t_0)\\ &\quad - \int_{\sigma(t_0)}^{\tau(t)}[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\\ &\qquad - (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}x(s)]\Delta s,\end{aligned}$$

where

$$K(t,t_0) = M(t,t_0) + c\int_{\sigma(t_0)}^{\tau(t_0)} H(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s.$$

Set

$$\begin{aligned}A^{\nu} &= [cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}]x^{\nu}(s),\\ B^{\nu-1} &= \frac{(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}}{\nu[cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}]^{\frac{1}{\nu}}}.\end{aligned}$$

Applying Lemma 2.1, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \leqslant K(t,t_0) + \int_{\sigma(t_0)}^{\tau(t)} Q(t,s)\Delta s,$$

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\sigma(t_0)}^{\tau(t)} Q(t,s)\Delta s \leqslant K(t,t_0). \tag{2.49}$$

Thus, multiplying (2.49) by $H^{-1}(t,t_0)$ and taking the upper limit of (2.49), we get a contradiction with (2.45). This completes the proof.

THEOREM 2.8. *Assume that*

(i) $\tau(t) \leqslant \sigma(t)$,

(ii) *there exist two positive constants c and* ν *such that*

$$|f(x)| \leqslant c|x|^{\nu}, \qquad 0 < \nu < 1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* $(H_1)-(H_3)$ *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geqslant 0, \tag{2.50}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{\tau(t_0)}^{\sigma(t)} Q(t,s)\Delta s\Big] = \infty, \tag{2.51}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\tau(t_0)}^{\sigma(t)} Q(t,s)\Delta s\Big] = -\infty, \tag{2.52}$$

where $Q(t,s)$, $\tau^*(t)$ *and* $\sigma^*(t)$ *are the same as in Theorem 2.7, then Eq. (1.2) is oscillatory.*

THEOREM 2.9. *Assume that*

(i) $\tau(t) \leqslant t$,

(ii) *there exist two positive constants c and* ν *such that*

$$|f(x)| \leqslant c|x|^{\nu}, \qquad 0 < \nu < 1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* (H_1)-(H_3) *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geqslant 0, \tag{2.53}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{t_0}^{\tau(t)} Q(t,s)\Delta s\Big] = \infty, \tag{2.54}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{t_0}^{\tau(t)} Q(t,s)\Delta s\Big] = -\infty, \tag{2.55}$$

where $Q(t,s)$, $\tau^*(t)$ *and* $\sigma^*(t)$ *are the same as in Theorem 2.7, then Eq. (1.2) is oscillatory.*

THEOREM 2.10. *Assume that*

(i) $\tau(t) \geqslant t$,

(ii) *there exist two positive constants c and* ν *such that*

$$|f(x)| \leqslant c|x|^{\nu}, \qquad 0 < \nu < 1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* (H_1)-(H_3) *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geqslant 0, \tag{2.56}$$

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_t^{\tau(t)} cH(t,\sigma(\tau^*(s)))p(\tau^*(s))(\tau^*(s))^{\Delta}s^{\nu k}\Delta s<\infty, \tag{2.57}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}[\int_{t_0}^t H(t,\sigma(s))e(s)\Delta s+\int_{\tau(t_0)}^t Q(t,s)\Delta s]=\infty, \tag{2.58}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}[\int_{t_0}^t H(t,\sigma(s))e(s)\Delta s-\int_{\tau(t_0)}^t Q(t,s)\Delta s]=-\infty, \tag{2.59}$$

where $Q(t,s)$, $\tau^*(s)$ and $\sigma^*(s)$ are the same as in Theorem 2.7, then all solutions of Eq. (1.2) satisfying $x(t)=O(t^k)$ are oscillatory.

EXAMPLE 2.8. Consider the equation $(\mathbb{T}=\mathbb{R})$

$$\Big(\frac{1}{t}x'(t)\Big)'-t^m x^{\nu}(t)=t^{\alpha}\cos t, \tag{2.60}$$

where $m\geqslant 0$, $\alpha>0$ and $0<\nu<1$. Here,

$$r(t)=\frac{1}{t},\ p(t)=t^m,\ f(x)=x^{\nu},\ 0<\nu<1$$

with $c=1$ and $e(t)=t^{\alpha}\cos t$. To apply Theorem 2.8, let us take $H(t,s)=(t-s)$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'}=\frac{1}{s^2}>0.$$

Since

$$Q(t,s)=(\nu-1)\nu^{\frac{\nu}{1-\nu}}(t-s)^{\frac{1}{1-\nu}}s^{\frac{2\nu+m}{1-\nu}},$$

then

$$\begin{aligned}\int_0^t Q(t,s)ds&=(\nu-1)\nu^{\frac{\nu}{1-\nu}}\int_0^t(t-s)^{\frac{1}{1-\nu}}s^{\frac{2\nu+m}{1-\nu}}ds\\&=(\nu-1)\nu^{\frac{\nu}{1-\nu}}t^{\frac{2\nu+m+1}{1-\nu}+1}\int_0^1(1-u)^{\frac{1}{1-\nu}}u^{\frac{2\nu+m}{1-\nu}}du\\&=(\nu-1)\nu^{\frac{\nu}{1-\nu}}B(\frac{1}{1-\nu}+1,\frac{2\nu+m}{1-\nu}+1)t^{\frac{2\nu+m+1}{1-\nu}+1},\end{aligned}$$

where the beta function $B(\frac{1}{1-\nu}+1,\frac{2\nu+m}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t(t-s)s^{\alpha}\cos s\,ds=t^{\alpha+2}\int_0^1(1-u)u^{\alpha}\cos ut\,du=t^{\alpha+2}I_{1,\alpha}(t),$$

where $I_{1,\alpha}(t)$ has the asymptotic formula

$$I_{1,\alpha}(t)=\Gamma(2)t^{-2}\cos(t-\pi)+o(t^{-2})\quad\text{as } t\to\infty.$$

Consequently, Eq. (2.60) is oscillatory if $\alpha>\frac{2\nu+m+1}{1-\nu}+1$.

REMARK 2.7. The results of [9, 13] can not be applied to Eq. (2.60) for $r(t) = \frac{1}{t}$. But, according to Theorem 2.7, when $(\mathbb{T} = \mathbb{R})$ and $H(t,s) = (t-s)$, this equation is oscillatory.

III - Oscillation criteria of the Eq:

$$(r(t)x^{\Delta}(t))^{\Delta} \pm p(t)f(x(\tau(t))) = e(t). \tag{2.61}$$

When $p(t)$ changes its sign.

THEOREM 2.11. *Assume that*

(i) $\tau(t) \leqslant t$,

(ii) *there exist two positive constants c and ν such that*

$$|f(x)| \leqslant c|x|^{\nu}, \qquad 0 < \nu < 1,$$

(iii) *if there exist a kernel function $H(t,s)$ satisfying (H_1)-(H_3) such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geqslant 0, \tag{2.62}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{t_0}^{\tau(t)} G(t,s)\Delta s] = \infty, \tag{2.63}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{t_0}^{\tau(t)} G(t,s)\Delta s] = -\infty, \tag{2.64}$$

where

$$G(t,s) = (\nu-1)\nu^{\frac{\nu}{1-\nu}}[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}]^{\frac{\nu}{\nu-1}} \times [cH(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}]^{\frac{1}{1-\nu}},$$

$\tau^*(s)$ and $\sigma^*(s)$ are the inverse functions of $\tau(s)$ and $\sigma(s)$ respectively and $p^*(s) = max\{\pm p(s), 0\}$, then Eq. (1.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2.61). Suppose that $x(t) > 0$ for $t \geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Multiplying Eq. (2.61) by $H(t,\sigma(s))$ for $t \geqslant t_0$ and integrating from t_0 to t, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant \int_{t_0}^{t} H(t,\sigma(s))(r(s)x^{\Delta}(s))^{\Delta}\Delta s - \int_{t_0}^{t} H(t,\sigma(s))p^*(s)f(x(\tau(s)))\Delta s.$$

From (2.5), we get

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s \\ - \int_{t_0}^{t} H(t,\sigma(s))p^*(s)f(x(\tau(s)))\Delta s.$$

Since $|f(x)| \leqslant c|x|^{\nu}$, $0<\nu<1$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) + \int_{t_0}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} x(\sigma(s))\Delta s \\ - c\int_{t_0}^{t} H(t,\sigma(s))p^*(s)x^{\nu}(\tau(s))\Delta s. \quad (2.65)$$

Substituting from (2.13) and (2.14) into (2.65), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) \\ + \int_{\sigma(t_0)}^{\sigma(t)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s \\ - c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s. \quad (2.66)$$

Since $\tau(t) \leqslant t$, we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) \\ + \int_{\sigma(t_0)}^{\tau(t)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s \\ - c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s,$$

and

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant K(t,t_0) \\ + \int_{t_0}^{\tau(t)} \Big[(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s) \\ - cH(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Big]\Delta s, \quad (2.67)$$

where

$$K(t,t_0) = M(t,t_0) + \int_{\sigma(t_0)}^{t_0} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s \\ - c\int_{\tau(t_0)}^{t_0} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s.$$

Set $F(x) = ax - bx^{\nu}$, for $x > 0$, $a \geqslant 0$, $b > 0$. If $0 < \nu < 1$, then $F(x)$ has the minimum $F_{min} = (\nu - 1)\nu^{\frac{\nu}{1-\nu}} a^{\frac{\nu}{\nu-1}} b^{\frac{1}{1-\nu}}$. From (2.67), we have

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant K(t,t_0) + \int_{t_0}^{\tau(t)} G(t,s)\Delta s,$$

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{t_0}^{\tau(t)} G(t,s)\Delta s \geqslant K(t,t_0). \tag{2.68}$$

Thus, multiplying (2.68) by $H^{-1}(t,t_0)$ and taking the lower limit of (2.68), we get a contradiction with (2.64). This completes the proof.

THEOREM 2.12. *Assume that*

(i) $\tau(t) \geqslant t$,

(ii) *there exist two positive constants* c *and* ν *such that*

$$|f(x)| \leqslant c|x|^{\nu}, \qquad 0 < \nu < 1,$$

(iii) *if there exist a kernel function* $H(t,s)$ *satisfying* (H_1)-(H_3) *such that*

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geqslant 0, \tag{2.69}$$

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t}^{\tau(t)} cH(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta} s^{\nu k}\Delta s < \infty, \tag{2.70}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s + \int_{\tau(t_0)}^{t} G(t,s)\Delta s\Big] = \infty, \tag{2.71}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}\Big[\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\tau(t_0)}^{t} G(t,s)\Delta s\Big] = -\infty, \tag{2.72}$$

where $G(t,s)$, $\tau^*(s)$, $\sigma^*(s)$ *and* $p^*(t)$ *are the same as in Theorem 2.11, then all solutions of Eq. (2.59) satisfying* $x(t) = O(t^k)$ *are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2.61). Suppose that $x(t) > 0$ for $t \geqslant t_0$ (when $x(t)$ is eventually negative, the proof follows the same argument). Proceeding as in the proof of Theorem 2.11 to get (2.66), i.e.,

$$\begin{aligned}\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s &\geqslant M(t,t_0) \\ &\quad + \int_{\sigma(t_0)}^{\sigma(t)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta} x(s)\Delta s \\ &\quad - c\int_{\tau(t_0)}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta} x^{\nu}(s)\Delta s.\end{aligned}$$

Since $\tau(t) \geqslant t$, we have

$$\begin{aligned}\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant M(t,t_0) \\ + \int_{\sigma(t_0)}^{t} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}x(s)\Delta s \\ - c\int_{\tau(t_0)}^{t} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s \\ - c\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s,\end{aligned}$$

and

$$\begin{aligned}\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s \geqslant F(t,t_0) \\ - c\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)\Delta s \\ + \int_{\tau(t_0)}^{t} [(H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta_s}(\sigma^*(s))^{\Delta}x(s) \\ - cH(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}x^{\nu}(s)]\Delta s,\end{aligned}$$

where

$$F(t,t_0) = M(t,t_0) + \int_{\sigma(t_0)}^{\tau(t_0)} (H^{\Delta_s}(t,\sigma^*(s))r(\sigma^*(s)))^{\Delta}(\sigma^*(s))^{\Delta}x(s)\Delta s.$$

Since $x(t) \leqslant Mt^k$ for some constant $M > 0$, we get from the last theorem, that

$$\int_{t_0}^{t} H(t,\sigma(s))e(s)\Delta s - \int_{\tau(t_0)}^{t} G(t,s)\Delta s \geqslant F(t,t_0) - cM^{\nu}\int_{t}^{\tau(t)} H(t,\sigma(\tau^*(s)))p^*(\tau^*(s))(\tau^*(s))^{\Delta}s^{\nu k}\Delta s. \quad (2.73)$$

Thus, multiplying (2.73) by $H^{-1}(t,t_0)$ and taking the lower limit of (2.73), we get a contradiction with (2.72). This completes the proof.

EXAMPLE 2.9. Consider the equation $(\mathbb{T} = \mathbb{R})$

$$x''(t) \pm t^m \sin t\, |x(t-\tau)|^{\nu} sgn\, x(t-\tau) = t^{\alpha}\cos t, \qquad t \geqslant 0, \quad (2.74)$$

where $m \geqslant 0$, $\alpha > 0$ and $0 < \nu < 1$ are constants. Here,

$$r(t) = t,\ p(t) = t^m \sin t,\ f(x) = x^{\nu},\ 0 < \nu < 1$$

with $c=1$ and $e(t)=t^{\alpha}\cos t$. To apply Theorem 2.11, let us take $H(t,s)=(t-s)^{\beta}$, $\beta>1$. Therefore, we have

$$(H^{'}(t,s)r(s))^{'}=\beta(\beta-1)(t-s)^{\beta-2}>0.$$

Since

$$G(t,s)\geqslant(\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}(t-s)^{\frac{(2-\beta)\nu}{1-\nu}}(t-s-\tau)^{\frac{\beta}{1-\nu}}(s+\tau)^{\frac{m}{1-\nu}},$$

then

$$\begin{aligned}\int_0^{t-\tau}G(t,s)ds&=(\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}\int_{\tau}^{t}(t-s+\tau)^{\frac{(2-\beta)\nu}{1-\nu}}(t-s)^{\frac{\beta}{1-\nu}}s^{\frac{m}{1-\nu}}ds\\&\geqslant(\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}t^{\frac{(2-\beta)\nu}{1-\nu}}\int_0^t(t-s)^{\frac{\beta}{1-\nu}}s^{\frac{m}{1-\nu}}ds\\&=(\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}t^{\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1}\int_0^1(1-u)^{\frac{\beta}{1-\nu}}u^{\frac{m}{1-\nu}}du\\&=(\nu-1)(\frac{\nu}{\beta(\beta-1)})^{\frac{\nu}{1-\nu}}B(\frac{\beta}{1-\nu}+1,\frac{m}{1-\nu}+1)t^{\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1},\end{aligned}$$

where the beta function $B(\frac{\beta}{1-\nu}+1,\frac{m}{1-\nu}+1)$ is positive constant. On the other hand,

$$\int_0^t(t-s)^{\beta}s^{\alpha}\cos s\,ds=t^{\beta+\alpha+1}\int_0^1(1-u)^{\beta}u^{\alpha}\cos ut\,du=t^{\beta+\alpha+1}I_{\beta,\alpha}(t),$$

where $I_{\beta,\alpha}(t)$ has the asymptotic formula

$$I_{\beta,\alpha}(t)=\Gamma(\beta+1)t^{-\beta-1}\cos\left(t-\frac{(\beta+1)\pi}{2}\right)+o(t^{-\beta-1})\quad\text{as } t\to\infty.$$

Consequently, Eq. (2.74) is oscillatory if $\alpha>\frac{(2-\beta)\nu+\beta+m}{1-\nu}+1$.

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