EXISTENCE OF POSITIVE SOLUTIONS FOR A QUASILINEAR ELLIPTIC SYSTEM OF \( p \)-KIRCHHOFF TYPE

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Abstract. In this paper, we consider the existence of positive solutions to the following \( p \)-Kirchhoff-type system

\[
\begin{align*}
- M \left( \int_{\Omega} |\nabla u|^p \, dx \right) \Delta_p u &= g(x) |u|^{q-2} u + \frac{\alpha}{\alpha + \beta} |u|^{\alpha-2} |v|^\beta, \quad x \in \Omega, \\
- M \left( \int_{\Omega} |\nabla v|^p \, dx \right) \Delta_p v &= h(x) |v|^{q-2} v + \frac{\beta}{\alpha + \beta} |u|^{\alpha} |v|^{\beta-2} v, \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( M(s) = a + bs^k \), \( \Delta_p u = \text{div}(\nabla u |\nabla u|^{p-2}) \) is the \( p \)-Laplacian operator, \( \alpha > 1, \beta > 1 \), and \( 1 < p < q < \alpha + \beta < p^* = \frac{Np}{N-p} \).

1. Introduction

In this paper, we deal with the nonlocal elliptic system of the \( p \)-Kirchhoff type given by

\[
\begin{align*}
- M (\int_{\Omega} |\nabla u|^p \, dx) \Delta_p u &= g(x) |u|^{q-2} u + \frac{\alpha}{\alpha + \beta} |u|^{\alpha-2} |v|^\beta, \quad x \in \Omega, \\
- M (\int_{\Omega} |\nabla v|^p \, dx) \Delta_p v &= h(x) |v|^{q-2} v + \frac{\beta}{\alpha + \beta} |u|^{\alpha} |v|^{\beta-2} v, \quad x \in \Omega, \\
u &= v = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \Delta_p u = \text{div}(\nabla u |\nabla u|^{p-2}) \) is the \( p \)-Laplacian operator, \( \alpha > 1, \beta > 1 \) and \( 1 < p < q < \alpha + \beta < p^* = \frac{Np}{N-p} \).

In recent years, there have been many papers concerned with the existence of positive solutions for Kirchhoff equation

\[
\begin{align*}
- M (\int_{\Omega} |\nabla u|^2 \, dx) \Delta u &= f(x, u) \quad x \in \Omega, \\
u &= 0 \quad x \in \partial \Omega,
\end{align*}
\]

which is related to the stationary analogue of the Kirchhoff equation

\[
u_{tt} - M (\int_{\Omega} |\nabla u|^2 \, dx) \Delta u = f(x, t)
\]


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where $M(s) = a + bs$, $a > 0$, $b > 0$. It was proposed by Kirchhoff [2] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings.

Some interesting studies of problem (1.2) by variational methods can be found in [1,3,6,7,9,10]. As for quasilinear problems, [3] studied the following equation of the classical D’Alembert’s wave equation for free vibrations of elastic strings.

\[
\begin{cases}
-M \left( \int_\Omega |\nabla u|^p dx \right) \Delta_p u = f(x,u) & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}
\]  

(1.3)

where $M(s) = a + bs^k$, $f(x,u) = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u + h_3(x)$, $1 < q < p < r < p^*$, $0 \leq k < \frac{p}{N-p}$, $p(k+1) < r$. The authors proved (1.3) has at least two nontrivial weak solutions when $\|h_3\|_\mu < m_0$.

In [4], the authors established the existence of a weak solution for the following system

\[
\begin{cases}
-\left[ M_1 \left( \frac{1}{\Omega} |\nabla u|^p dx \right)^{p-1} \Delta_p u \right] = f(u,v) + \rho_1(x) & x \in \Omega, \\
d_{\frac{\partial u}{\partial \eta}} = \frac{\partial v}{\partial \eta} = 0 & x \in \partial \Omega,
\end{cases}
\]  

(1.4)

where $M_1(t), M_2(t) \geq m_0 > 0$.

Motivated by the results of the above cited papers, we shall attempt to treat problem (1.1) and extend the results of the literature [4].

In this paper, we make the following assumptions:

(A1) $M(s) = a + bs^k$, $a, b > 0$, $k > 0$;

(A2) $1 < p < q < \alpha + \beta < p^*$, $p(k+1) < q$;

(A3) $h(x), H(x) \in L^{\delta+\gamma}(\Omega)$ are nonnegative with $\delta = \frac{p^*}{p^*-q}$ and $\gamma$ is a small positive number.

Our main result is given as follows.

**Theorem 1.** Under assumptions (A1) – (A3), problem (1.1) admits at least one positive solution $(u_0, v_0) \in W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega)$.

2. Preliminaries

Let $X = W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega)$ be the Sobolev space endowed with the norm

$$
\|(u,v)\| = \left( \int_\Omega (|\nabla u|^p + |\nabla v|^p) dx \right)^{\frac{1}{p}}
$$

and $|u|_r$ denotes the norm in $L^r(\Omega)$, i.e.

$$
|u|_r = \left( \int_\Omega |u|^r dx \right)^{\frac{1}{r}}
$$

We shall look for solutions of (1.1) by finding critical points of the energy functional $I : X \rightarrow \mathbb{R}$ given by

...
\[
I(u,v) = \frac{1}{p} \hat{M}(\int_{\Omega} |\nabla u|^p \, dx) + \frac{1}{p} \hat{M}(\int_{\Omega} |\nabla v|^p \, dx) - \frac{1}{\alpha + \beta} \int_{\Omega} |u|^\alpha |v|^\beta \, dx - \frac{1}{q} \int_{\Omega} (g(x)|u|^q + h(x)|v|^q) \, dx,
\]
where \( \hat{M}(t) = \int_0^t M(s) \, ds = at + \frac{b}{k+1} t^{k+1} \). It is well known that the functional \( I(u,v) \in C^1(X, \mathbb{R}) \). For any \((\varphi_1, \varphi_2) \in X\), there holds
\[
\langle I'(u,v), (\varphi_1, \varphi_2) \rangle = M\left(\int_{\Omega} |\nabla u|^p \, dx\right) \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi_1 \, dx
+ M\left(\int_{\Omega} |\nabla v|^p \, dx\right) \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \varphi_2 \, dx
- \int_{\Omega} (g(x)|u|^{q-2} u \varphi_1 + h(x)|v|^{q-2} v \varphi_2) \, dx
- \frac{\alpha}{\alpha + \beta} \int_{\Omega} |u|^{\alpha-2} u |v|^\beta \varphi_1 \, dx
- \frac{\beta}{\alpha + \beta} \int_{\Omega} |u|^{\alpha} |v|^{\beta-2} v \varphi_2 \, dx.
\]
Consider the Nehari manifold
\[
N = \{(u,v) \in X \setminus \{(0,0)\} | \langle I'(u,v), (u,v) \rangle = 0\}.
\]
Thus, \((u,v) \in N\) if and only if
\[
M\left(\int_{\Omega} |\nabla u|^p \, dx\right) \int_{\Omega} |\nabla u|^p \, dx + M\left(\int_{\Omega} |\nabla v|^p \, dx\right) \int_{\Omega} |\nabla v|^p \, dx
- \int_{\Omega} (g(x)|u|^q + h(x)|v|^q) \, dx - \int_{\Omega} |u|^{\alpha} |v|^{\beta} \, dx = 0.
\]
Note that the Nehari manifold \(N\) contains all nontrivial weak solutions of \((1.1)\).
Denote
\[
S_{\alpha,\beta} = \inf_{u,v \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{||I(u,v)||^p}{(\int_{\Omega} |u|^{\alpha} |v|^{\beta} \, dx)^{\frac{p}{\alpha+\beta}}},
\]
\(S\) is the best Sobolev constant defined by
\[
S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p \, dx}{(\int_{\Omega} |u|^p \, dx)^{\frac{p}{p}}} > 0.
\]
Note that \(I\) is not bounded from below on \(X\). But from the following lemma, we have that \(I\) is bounded from below on the Nehari manifold \(N\).

**Lemma 1.** The energy functional \(I\) is bounded from below on \(N\).

**Proof.** For any \((u,v) \in N\), we have
\[
I(u,v) = \frac{a}{p} ||(u,v)||^p + \frac{b}{p(k+1)} \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{k+1} + \left( \int_{\Omega} |\nabla v|^p \, dx \right)^{k+1}
\]
Thus, \( I \) is bounded from below on \( N \). □

Then, we define

\[
\theta = \inf_{(u,v) \in N} I(u,v).
\]

**Lemma 2.** (i) There are \( \rho, r_0 > 0 \) such that \( I(u,v) \geq r_0 \) for \( \| (u,v) \| = \rho \).

(ii) There exists \( (\overline{\pi}, \overline{\tau}) \in X \setminus \{(0,0)\} \) such that \( \| (\overline{\pi}, \overline{\tau}) \| > \rho \) and \( I(\overline{\pi}, \overline{\tau}) < 0 \).

**Proof.** (i) By Hölder’s inequality \( (q_1 = \frac{p^*}{p^*-q}, q_2 = \frac{p^*}{q}, \frac{1}{q_1} + \frac{1}{q_2} = 1) \) and the Sobolev embedding theorem, we have

\[
I(u,v) = \frac{a}{p} \| (u,v) \|^p + \frac{b}{p(k+1)} \left( \int_{\Omega} |\nabla u|^p dx \right)^{k+1} + \frac{1}{q} \int_{\Omega} \left( g(x) |u|^q + h(x) |v|^q \right) dx
\]

\[
\geq \frac{a}{p} \| (u,v) \|^p + \frac{b}{2kp(k+1)} \left( \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla v|^p dx \right)^{k+1}
\]

\[
- \frac{1}{\alpha + \beta} \left( \int_{\Omega} |u|^\alpha |v|^\beta dx \right) - \frac{1}{q} \int_{\Omega} \left( g(x) |u|^q + h(x) |v|^q \right) dx
\]

\[
\geq \frac{a}{p} \| (u,v) \|^p + \frac{b}{2kp(k+1)} \| (u,v) \|^p(k+1) - \frac{1}{\alpha + \beta} S \frac{\alpha + \beta}{p^*} \| (u,v) \|^{\alpha + \beta}
\]

\[
- \frac{1}{q} \max \{ \| g \|_\infty, \| h \|_\infty \} |\Omega| \frac{p^*}{p^* - q} S^{-\frac{q}{p}} \| (u,v) \|^q.
\]

Since \( p < p(k+1) < q < \alpha + \beta \), there are \( \rho, r_0 > 0 \) sufficiently small such that \( I(u,v) \geq r_0 \) for \( \| (u,v) \| = \rho \).

(ii) Let \( (u,v) \in X \setminus \{(0,0)\} \), we have

\[
I(tu, tv) = \frac{at^p}{p} \| (u,v) \|^p + \frac{bt^{p(k+1)}}{p(k+1)} \left( \int_{\Omega} |\nabla u|^p dx \right)^{k+1} + \frac{t^q}{q} \int_{\Omega} \left( g(x) |u|^q + h(x) |v|^q \right) dx
\]

\[
\leq \frac{at^p}{p} \| (u,v) \|^p + \frac{bt^{p(k+1)}}{p(k+1)} \| (u,v) \|^{p(k+1)} - \frac{t^{\alpha + \beta}}{\alpha + \beta} \int_{\Omega} |u|^\alpha |v|^\beta dx
\]
\[
- \frac{t^q}{q} \int_{\Omega} (g(x)|u|^q + h(x)|v|^q)dx
\]

Since \( p < p(k + 1) < q < \alpha + \beta \), we have \( I(tu, tv) \rightarrow -\infty \) as \( t \rightarrow +\infty \). Then, for fixed \((u, v) \in X \setminus \{(0, 0)\}\), there exists \( T > 0 \) such that \( ||(tu, tv)|| > \rho \) and \( I(tu, tv) < 0 \). Let \((\tilde{u}, \tilde{v}) = (tu, tv)\), then we finish the proof. \( \square \)

**Lemma 3.** There exists a \((PS)_\theta\)-sequence \(\{(u_n, v_n)\} \subset N\) for \(I\).

**Proof.** By Lemma 2, \(I\) satisfies the conditions of the Mountain Pass Lemma. Thus, by applying Ekeland’s variational principle and using the same argument as in Cao and Zhou [8] or Tarantello [5], we can easily find a \((PS)_\theta\)-sequence \(\{(u_n, v_n)\} \subset N\) for the functional \(I\). \( \square \)

Next, we will show that \(I\) satisfies the \((PS)_\theta\)-condition in \(X\).

**Lemma 4.** Let \(\{(u_n, v_n)\} \subset X\) be an arbitrary \((PS)_\theta\)-sequence for \(I\). That is, \(I(u_n, v_n) \rightarrow \theta\) and \(I'(u_n, v_n) \rightarrow 0\) in \(X^{-1}\). Then \(\{(u_n, v_n)\}\) has a convergent subsequence.

**Proof.** First, we prove that \(\{(u_n, v_n)\}\) is bounded in \(X\). In fact, we have

\[
\theta + c_n + \frac{d_n ||(u_n, v_n)||}{q} \geq I(u_n, v_n) - \frac{1}{q} \langle I'(u_n, v_n), (u_n, v_n) \rangle
\]

\[
= a \frac{||(u_n, v_n)||^p}{p} + \frac{b}{p(k + 1)} \left[ \left( \int_{\Omega} |\nabla u_n|^p dx \right)^{k+1} + \left( \int_{\Omega} |\nabla v_n|^p dx \right)^{k+1} \right]
\]

\[
- \frac{1}{\alpha + \beta} \int_{\Omega} |u_n|^\alpha |v_n|^\beta dx - \frac{1}{q} \int_{\Omega} (g(x)|u_n|^q + h(x)|v_n|^q)dx
\]

\[
- \frac{a}{q} ||(u_n, v_n)||^p - \frac{b}{q} \left[ \left( \int_{\Omega} |\nabla u_n|^p dx \right)^{k+1} + \left( \int_{\Omega} |\nabla v_n|^p dx \right)^{k+1} \right]
\]

\[
+ \frac{a}{q} \int_{\Omega} |u_n|^\alpha |v_n|^\beta dx + \frac{1}{q} \int_{\Omega} (g(x)|u_n|^q + h(x)|v_n|^q)dx
\]

\[
= \left( \frac{a}{p} - \frac{a}{q} \right) ||(u_n, v_n)||^p + \left( \frac{b}{p(k + 1)} - \frac{b}{q} \right) \left[ \left( \int_{\Omega} |\nabla u_n|^p dx \right)^{k+1} + \left( \int_{\Omega} |\nabla v_n|^p dx \right)^{k+1} \right]
\]

\[
+ \left[ \frac{1}{q} - \frac{1}{\alpha + \beta} \right] \int_{\Omega} |u_n|^\alpha |v_n|^\beta dx
\]

\[
\geq \frac{aq - ap}{pq} ||(u_n, v_n)||^p
\]

where \(c_n = o_n(1)\), \(d_n = o_n(1)\) as \(n \rightarrow \infty\). From where we get \(\{(u_n, v_n)\}\) is bounded in \(X\). Then, there exist a subsequence (still denoted by \(\{(u_n, v_n)\}\)) and \((u, v) \in X\) such that

\[
u_n \rightharpoonup u, v_n \rightharpoonup v \quad \text{weakly in } W^{1,p}_0(\Omega);
\]
\[ u_n \to u, v_n \to v \quad \text{a.e in } \Omega; \]
\[ u_n \to u, v_n \to v \quad \text{strongly in } L^s(\Omega), \ 1 \leq s < p^* \]

and \( l'(u, v) = 0 \) in \( X^{-1} \).

Next, we prove that
\[
\int_\Omega g(x)|u_n|^q - 2 u_n(u_n - u)dx \to 0, \quad n \to \infty \tag{2.1}
\]
\[
\int_\Omega h(x)|v_n|^q - 2 v_n(v_n - v)dx \to 0, \quad n \to \infty \tag{2.2}
\]
and
\[
\int_\Omega |u_n|^p - 2 u_n |u_n|^p_t dx \to 0, \quad n \to \infty \tag{2.3}
\]
\[
\int_\Omega |v_n|^p - 2 v_n |v_n|^p_t dx \to 0, \quad n \to \infty \tag{2.4}
\]

By Hölder’s Inequality, we have
\[
\int_\Omega g(x)|u_n|^q - 1 |u_n - u| dx \leq C |g|_{\gamma + \delta} |u_n|^q - 1 |u_n - u| \mu
\]
where \( \frac{1}{\gamma + \delta} + \frac{q - 1}{p^*} + \frac{1}{p} = 1 \). It is easy to show that \( \mu < p^* \) and so \( |u_n - u|_\mu \to 0 \) as \( n \to \infty \).

Since \( \{(u_n, v_n)\} \subset X \) is bounded, there exists \( M_1 > 0 \) such that
\[
|u_n|^q - 1 \leq S^{-q - 1/p} \left( \int_\Omega |\nabla u_n|^p dx \right)^{q - 1/p} \leq S^{-q - 1/p} \|(u_n, v_n)\|^{q - 1} \leq M_1
\]

Then, we can get (2.1). (2.2) can be proved similarly.

By Hölder’s Inequality again, we get
\[
\int_\Omega |u_n|^{\alpha - 1} |u_n - u| |v_n|^\beta dx \leq |u_n|^{\alpha - 1} |v_n|^{\beta} |u_n - u| \eta
\]
here \( \frac{\alpha - 1}{p^*} + \frac{\beta}{p} + \frac{1}{\eta} = 1 \), then \( \eta < p^* \) and \( |u_n - u|_\eta \to 0 \) as \( n \to \infty \). Similarly, there exists \( M_2, M_3 > 0 \) such that \( |u_n|^{\alpha - 1} \leq M_2, \ |v_n|^{\beta} \leq M_3 \) Then,
\[
\int_\Omega |u_n|^{\alpha - 2} u_n(u_n - u) |v_n|^\beta dx \to 0
\]

Similarly,
\[
\int_\Omega |v_n|^{\beta - 2} v_n(v_n - v) |u_n|^\alpha dx \to 0
\]

Finally, we prove \( ||(u_n - u, v_n - v)|| \to 0 \) in \( X \). In fact,
\[
M \left( \int_\Omega |\nabla u_n|^p dx \right) \int_\Omega |\nabla u_n|^{p - 2} \nabla u_n - |\nabla u|^{p - 2} \nabla u, \nabla u_n - \nabla u dx
\]
$$+ M \left( \int_\Omega |\nabla v_n|^p \, dx \right) \int_\Omega \langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v, \nabla v_n - \nabla v \rangle \, dx$$

$$= \langle l'(u_n, v_n), (u_n - u, v_n - v) \rangle - M \left( \int_\Omega |\nabla u_n|^p \, dx \right) \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla u_n - \nabla u \rangle \, dx$$

$$- M \left( \int_\Omega |\nabla v_n|^p \, dx \right) \int_\Omega |\nabla v|^{p-2} \langle \nabla v, \nabla v_n - \nabla v \rangle \, dx$$

$$+ \frac{\alpha}{\alpha + \beta} \int_\Omega |u_n|^\alpha (u_n - u) |v_n|^\beta \, dx$$

$$+ \frac{\beta}{\alpha + \beta} \int_\Omega |u_n|^\alpha |v_n|^\beta (v_n - v) \, dx$$

$$+ \int_\Omega \left[ g(x) |u_n|^{q-2} u_n (u_n - u) + h(x) |v_n|^{q-2} v_n (v_n - v) \right] \, dx.$$

Since $u_n \rightharpoonup u$, $v_n \rightharpoonup v$, we have

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla u_n - \nabla u \rangle \, dx \to 0, \quad n \to \infty$$

$$\int_\Omega |\nabla v|^{p-2} \langle \nabla v, \nabla v_n - \nabla v \rangle \, dx \to 0, \quad n \to \infty.$$

Thus,

$$M \left( \int_\Omega |\nabla u_n|^p \, dx \right) \int_\Omega \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla u_n - \nabla u \rangle \, dx$$

$$+ M \left( \int_\Omega |\nabla v_n|^p \, dx \right) \int_\Omega \langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v, \nabla v_n - \nabla v \rangle \, dx \to 0.$$

Using the standard inequality

$$\langle |x|^{p-2} x - |y|^{p-2} y, x - y \rangle \geq C_p |x - y|^p, \quad p \geq 2$$

or

$$\langle |x|^{p-2} x - |y|^{p-2} y, x - y \rangle \geq \frac{C_p |x - y|^2}{(|x| + |y|)^{2-p}}, \quad 2 > p > 1$$

We obtain

$$M \left( \int_\Omega |\nabla u_n|^p \, dx \right) \int_\Omega \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla u_n - \nabla u \rangle \, dx$$

$$\geq ac_p \int_\Omega |\nabla u_n - \nabla u|^p \, dx$$

and

$$M \left( \int_\Omega |\nabla v_n|^p \, dx \right) \int_\Omega \langle |\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v, \nabla v_n - \nabla v \rangle \, dx$$

$$\geq ac_p \int_\Omega |\nabla v_n - \nabla v|^p \, dx$$
which implies that
\[ \int_\Omega (|\nabla u_n - \nabla u|^p + |\nabla v_n - \nabla v|^p)dx \to 0, \quad n \to \infty \]
That is, \(|(u_n - u, v_n - v)|| \to 0 \) in \( X \). Then, the proof is finished. □

3. Proof of Theorem 1

By Lemma 3, there is a minimizing sequence \( \{(u_n, v_n)\} \subset N \) for \( I \) satisfying
\[ I(u_n, v_n) = \theta + o_n(1) \quad \text{and} \quad I'(u_n, v_n) = o_n(1) \quad \text{in} \quad X^{-1} \]. By Lemma 4, there exist a subsequence (still denoted by \( \{(u_n, v_n)\} \)) and \((u_0, v_0) \in X \) such that \((u_n, v_n) \to (u_0, v_0) \) in \( X \).

It is easy to show that \((u_0, v_0)\) is a nontrivial solution of (1.1) and \( I(u_0, v_0) = \theta \). Using the fact that \( I(u_0, v_0) = I(|u_0|, |v_0|) \) and \((|u_0|, |v_0|) \in N \), we may assume that \( u_0 \geq 0, \quad v_0 \geq 0 \). By the maximum principle, we can get that \( u_0 > 0, \quad v_0 > 0 \) in \( \Omega \). Then, (1.1) admits a positive solution.

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