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GLOBAL ATTRACTIVITY RESULTS FOR COMPARABLE SOLUTIONS OF NONLINEAR HYBRID FRACTIONAL INTEGRAL EQUATIONS

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Abstract. We present a couple of global attractivity and asymptotic stability results for the comparable solutions of a certain hybrid functional nonlinear fractional integral equation with a linear perturbation of first kind on the unbounded intervals of real line under some weaker partially Lipschitz and partially compactness type conditions. We employ a new partially measure theoretic fixed point theorem in our analysis and develop an algorithm for the solutions. We claim that the results are new to the literature.

1. Introduction

The topic of fractional calculus and fractional differential and integral equations is of current interest and have received significant attention of many mathematicians all over the world because of their occurrence in several areas of physical sciences (cf. Podlubny [13] and the references therein). The object of this paper is to discuss attractivity and stability results for comparable solutions and develop an algorithm for the following functional nonlinear fractional integral equation (in short HFIE)

$$x(t) = f(t,x(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} g(s,x(\gamma(s)))\,ds, \quad t \in \mathbb{R}_+, \tag{1.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, $1 \leqslant q < 2$ and Γ is the Euler gamma function.

By a *solution* of the HFIE (1.1) we mean a function $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies the equation (1.1), where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions on $\mathbb{R}_+$.

The above nonlinear fractional integral equation in question has rather general form and includes several classes of functional, integral and functional integral equations considered in the literature (cf. [1, 3, 6, 10, 11] and references therein). Let us also mention that the functional integral equation considered in [3, 6] is a special case of the equation (1.1), when $\alpha(t) = \beta(t) = \gamma(t) = t$ and $q = 1$. Note that the existence

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theorem for the HFIE (1.1) may be proved via a hybrid fixed point theorem on the lines of Kranoselskii [10] under the mixed Lipschitz and compactness type conditions. But in this case we do not get any information about the behavior of the solutions as well as there is no way to approximate the solutions by successive iterations. It is with this motivation, we present some qualitative analysis such as attractivity and stability of comparable solutions along with an algorithm for the solutions of HFIE (1.1) defined on unbounded interval of the real line.

Our investigations will be carried out in the partially ordered Banach space of real functions which are defined, continuous and bounded on the right half real axis $\mathbb{R}_+$. The partially measure of noncompactness used in this paper allows us not only to obtain the existence of comparable solutions of the mentioned functional integral equation but also to characterize the comparable solutions in terms of uniform global ultimate attractivity and to develop an algorithm for the solutions. See Appell [1], Banas and Goebel [2] and Banas and Dhage [3]. This assertion means that all the possible comparable solutions of the nonlinear fractional integral equation in question are globally uniformly attractive in the sense of notion defined in the following section.

2. Auxiliary Results

Let $(E,\preceq,\|\cdot\|)$ be a partially ordered normed linear space. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Nieto and Lopez [12] and Heikkilä and Lakshmikantham [9] and the references cited therein. The following definitions have been introduced in Dhage [5] and are frequently used in the subsequent part of this paper.

A subset S of E is called **partially bounded** if every chain C in S is bounded. Again S is called **uniformly partially bounded** if all chains in S are bounded with a unique constant.

Note that every bounded subset of a partially ordered normed linear space is uniformly partially bounded and uniformly partially bounded set in E is partially bounded, but the converse implications may not be held.

DEFINITION 2.1. A mapping $\mathscr{T} : E \to E$ is called **isotonic** or **monotonic** if it is either monotone nondecreasing (resp. non-increasing), that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ (resp. $\mathscr{T}x \succeq \mathscr{T}y$) for all $x,y \in E$.

DEFINITION 2.2. (Dhage [7, 8]) A mapping $\mathscr{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{T}x - \mathscr{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x-a\| < \delta$. $\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E. $\mathscr{T}$ is called **partially bounded** if $\mathscr{T}(C)$ is a bounded subset of E for all totally ordered sets or chains C in

E. Finally, $\mathscr{T}$ is called **uniformly partially bounded** if $\mathscr{T}(C)$ is a uniformly bounded subset of E for all totally ordered sets or chains C in E.

2.1. Partially measure of noncompactness

If C is a chain in E then the symbol $\overline{C}$ stands for the order-closure of C in E defined by $\overline{C} = \inf C \cup C \cup \sup C$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z \in E$ such that for every $\varepsilon > 0$ there exists a $c \in C$ such that $d(c,z) < \varepsilon$ and $x \leqslant z$ for all $x \in C$. Similarly, $\inf C$ is defined in the same way. Then $\overline{C}$ is again a chain, called the closed chain in E. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Moreover, we denote by $\mathscr{P}_{cl}(E)$, $\mathscr{P}_{bd}(E)$, $\mathscr{P}_{rcp}(E)$, $\mathscr{P}_{ch}(E)$, $\mathscr{P}_{bd,ch}(E)$, $\mathscr{P}_{rcp,ch}(E)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of E respectively.

We accept the following definition of partially measure of noncompactness in partially ordered normed linear space given in Dhage [7].

DEFINITION 2.3. A mapping $\mu^p : \mathscr{P}_{bd,ch}(E) \to \mathbb{R}_+ = [0,\infty)$ is said to be a partially measure of noncompactness in E if it satisfies the following conditions:

1^o $\emptyset \neq (\mu^p)^{-1}(\{0\}) \subset \mathscr{P}_{rcp,ch}(E)$,

2^o $\mu^p(\overline{C}) = \mu^p(C)$,

3^o μ^p is nondecreasing, i.e., if $C_1 \subset C_2 \Rightarrow \mu^p(C_1) \leqslant \mu^p(C_2)$, and

4^o If $\{C_n\}$ is a sequence of closed chains from $\mathscr{P}_{bd,ch}(E)$ such that $C_{n+1} \subset C_n$ $(n = 1,2,\ldots)$ and if $\lim_{n\to\infty} \mu^p(C_n) = 0$, then the intersection set $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty.

The partially measure μ^p of noncompactness is called **sublinear** if it satisfies

5^o $\mu^p(C_1 + C_2) \leqslant \mu^p(C_1) + \mu^p(C_2)$ for all $C_1, C_2 \in \mathscr{P}_{bd,ch}(E)$, and

6^o $\mu^p(\lambda C) = |\lambda| \mu^p(C)$ for $\lambda \in \mathbb{R}$.

REMARK 2.1. The family of sets described in 1^o is said to be the *kernel of the measure of noncompactness* μ^p and is defined as $\ker \mu^p = \{C \in \mathscr{P}_{bd,ch}(E) \mid \mu^p(C) = 0\}$. Clearly, $\ker \mu^p \subset \mathscr{P}_{rcp,ch}(E)$. Observe that the intersection set C_∞ from condition 4^o is a member of the family $\ker \mu^p$. In fact, since $\mu^p(C_\infty) \leqslant \mu^p(C_n)$ for any n, we infer that $\mu^p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu^p$. This simple observation will be essential in our further investigation of measure theoretic fixed point theorems in partially ordered normed linear spaces.

EXAMPLE 2.1. Define the functions $\alpha^p, \beta^p : \mathscr{P}_{bd,ch}(E) \to \mathbb{R}_+$ by

$$\alpha^p(C) = \inf\left\{ r > 0 \,\middle|\, C = \bigcup_{i=1}^{n} C_i,\ \operatorname{diam}(C_i) \leqslant r \forall i \right\},$$

where $C \in \mathscr{P}_{bd,ch}(E)$ and $\operatorname{diam}(C_i) = \sup\{\|x-y\| : x,y \in C_i\}$; and

$$\beta^p(C) = \inf\left\{ r > 0 \;\middle|\; C \subset \bigcup_{i=1}^{n} \mathscr{B}(x_i,r) \text{ for some } x_i \in E \right\},$$

where $\mathscr{B}(x_i,r) = \{x \in E : \|x_i - x\| < r\}$. It is easy to prove that α^p and β^p are partially measures of noncompactness called the partially Kuratowskii and ball or Hausdorff measures of noncompactness in E respectively. The partially measures of noncompactness α^p and β^p are **full** or **complete** in the sense that $(\mu^p)^{-1}(\{0\}) = \mathscr{P}_{rcp,ch}(E)$.

The above two partially Kuratowskii and Hausdorff measures of noncompactness α^p and β^p are sublinear and enjoy the maximum property in E. The verification of this claim is same as classical Kuratowskii and Hausdorff measures of noncompactness and so we omit the details. In the following we prove some hybrid measure theoretic fixed point theorems (in short FPTs) in partially ordered normed linear spaces for their further use in the subsequent sections of the paper.

2.2. Measure theoretic FPTs

DEFINITION 2.4. A mapping $\mathscr{T} : E \to E$ is called a partially nonlinear $\mathscr{D}$-set-contraction if there exists a upper semi-continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any bounded chain C, $\mathscr{T}(C)$ is a bounded chain and $\mu^p(\mathscr{T}(C)) \leqslant \psi(\mu^p(C))$, where $\psi(r) < r$, for $r > 0$. The function ψ is called a $\mathscr{D}$-function of the operator $\mathscr{T}$ on E.

It is proved in Dhage [6] that if ψ is a $\mathscr{D}$-function, then $\psi^n(t) = 0$ for all $t \in \mathbb{R}_+$. We need the following definition in what follows.

DEFINITION 2.5. (Dhage [6]) The order relation $\preceq$ and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges monotonically to x^* implies that the whole sequence $\{x_n\}$ converges monotonically to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

The following applicable hybrid fixed point theorem for monotone mappings is the key tool for proving the main existence and attractivity result for the HFIE (1.1).

THEOREM 2.1. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible. Let $\mathscr{T} : S \to S$ be a partially continuous, nondecreasing and partially nonlinear $\mathscr{D}$-set-contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq Tx_0$, then $\mathscr{T}$ has a fixed point x^* and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is given in Dhage [7]. Since the proof is not well-known, we give the details of it. Assume first that there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$. Define a sequence $\{x_n\}$ of points in E by

$$x_{n+1} = \mathscr{T}x_n, \quad n = 0,1,2,\dots. \tag{2.1}$$

Since $\mathscr{T}$ is nondecreasing and $x_0 \preceq \mathscr{T}x_0$, we have that

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{2.2}$$

Denote $C_n = \overline{\{x_n, x_{n+1}, \dots\}}$ for $n = 0,1,2,\dots$. By construction, each C_n is a bounded and closed chain in E and $C_n = \mathscr{T}(C_{n-1}), \quad n = 0,1,2,\dots$. Moreover,

$$C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots. \tag{2.3}$$

Therefore, by nondecreasing nature of μ^p we obtain

$$\mu^p(C_n) = \mu^p(\mathscr{T}(C_{n-1})) \leqslant \psi(\mu^p(C_{n-1})) \leqslant \psi^2(\mu^p(C_{n-2})) \leqslant \cdots \leqslant \psi^n(\mu^p(C_0)). \tag{2.4}$$

Taking the limit superior as $n \to \infty$ in the above equality (2.4), in view of Lemma 3.1 we obtain that

$$\lim_{n\to\infty} \mu^p(C_n) \leqslant \limsup_{n\to\infty} \psi^n(\mu^p(C_0)) = \lim_{n\to\infty} \psi^n(\mu^p(C_0)) = 0. \tag{2.5}$$

Hence, by condition (4^o) of μ^p, $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$ and $C_\infty \in \mathscr{P}_{rcp,ch}(E)$. From (2.5) it follows that for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\mu^p(C_n) < \varepsilon$, $\forall\, n \geqslant n_0$.

This shows that $\overline{C}_{n_0}$ and consequently $\overline{C}_0$ is a compact chain in E. Hence, $\{x_n\}$ has a convergent subsequence. Further since the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible, the whole sequence $\{x_n\} = \{\mathscr{T}^n x_0\}$ is convergent and converges monotonically to a point, say $x^* \in \overline{C}_0$. Since the ordered space E is regular, we have that $x_n \leqslant x^*$ for all $n \in \mathbb{N}$. Finally, from the partial continuity of $\mathscr{T}$, we get

$$\mathscr{T}x^* = \mathscr{T}\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} \mathscr{T}x_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Similarly, if the condition $x_0 \succeq \mathscr{T}x_0$ holds, then following the above arguments it is shown that $\mathscr{T}$ has a fixed point. This completes the proof. □

REMARK 2.2. The regularity of E and the partial continuity of $\mathscr{T}$ in above Theorem 2.1 may be replaced with the stronger continuity condition of the operator $\mathscr{T}$ on E.

COROLLARY 2.1. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $(E, \preceq, \|\cdot\|)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible. Let $\mathscr{T} : S \to S$ be a partially continuous, nondecreasing and partially nonlinear k-set-contraction with $k < 1$. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$, then $\mathscr{T}$ has a fixed point x^* and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

REMARK 2.3. If the set $S_{\mathscr{T}}$ of all solutions to the operator equation $\mathscr{T}x = x$ is a chain in partially ordered Banach space E, then all solutions belonging to S are comparable. Further, if $\mu^p(S_{\mathscr{T}}) > 0$, then $\mu^p(S_{\mathscr{T}}) = \mu^p(\mathscr{T}S_{\mathscr{T}}) \leqslant \psi(\mu^p(S_{\mathscr{T}})) < \mu^p(S_{\mathscr{T}})$ which is a contradiction. Consequently, $S_{\mathscr{T}} \in \ker \mu^p$. This simple fact has been utilized in the study of qualitative properties of dynamic systems under consideration.

REMARK 2.4. Suppose that the order relation $\preceq$ is introduced in E with the help of an order cone $\mathscr{K}$ in the Banach space E by $x \preceq y \iff y - x \in \mathscr{K}$ (see Heikilla and Lakshmikantham [9]). The element $x_0 \in E$ satisfying $x_0 \preceq \mathscr{T}x_0$ in above Theorem 2.1 is called a lower solution of the operator equation $x = \mathscr{T}x$. If the operator equation $x = \mathscr{T}x$ has more than one lower solution and set of all these lower solutions are comparable, then the corresponding set S of solutions to above operator equation is a chain and hence all solutions in S are comparable (cf. Dhage [7]).

Before giving a further generalization of Theorem 3.1, we state a useful definition.

DEFINITION 2.6. A nondecreasing mapping $\mathscr{T} : E \to E$ is called partially μ^p-**condensing** if for any bounded chain C in E, $\mu^p(\mathscr{T}(C)) < \mu^p(C)$ for $\mu^p(C) > 0$.

We remark that every partially compact and partially nonlinear $\mathscr{D}$-set-contraction mappings are partially condensing, however the reverse implications may not hold.

THEOREM 2.2. *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ are compatible. Let S be a non-empty, closed and partially bounded subset of E and let $\mathscr{T} : S \to S$ be a nondecreasing, partially continuous and partially condensing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$, then $\mathscr{T}$ has a fixed point x^* and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Proof. The proof is standard and hence we omit the details.

REMARK 2.5. We note that the proof of Theorems 2.1 and 2.2 do not make any use of linear structure of the underlined normed linear space E, and therefore, Theorems 2.1 and 2.2 also remain true in the setting of the partially ordered metric space E. Thus, in view of this fact, we obtain the following fixed point results in partially ordered metric spaces.

THEOREM 2.3. *Let $(E, \preceq, d)$ be a regular partially ordered complete metric space such that the order relation $\preceq$ and the metric d are compatible. Let S be a non-empty, closed and partially bounded subset of E and let $\mathscr{T} : S \to S$ be a nondecreasing, partially continuous and partially condensing mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathscr{T}x_0$ or $x_0 \succeq \mathscr{T}x_0$, then $\mathscr{T}$ has a fixed point x^* and the sequence $\{\mathscr{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Note that Theorem 2.3 unified the following known hybrid fixed point principles in partially ordered metric spaces which are also useful from the point of view of applications to several nonlinear equations of dynamic systems of nonlinear analysis.

COROLLARY 2.2. (Dhage [6]) *Let* $(E, \preceq, d)$ *be a regular partially ordered complete metric space such that the order relation* $\preceq$ *and the metric* d *are compatible. Let* S *be a non-empty, closed and partially bounded subset of* E *and let* $\mathscr{T} : S \to S$ *be a nondecreasing, partially completely continuous mapping. If there exists an element* $x_0 \in S$ *such that* $x_0 \preceq \mathscr{T}x_0$ *or* $x_0 \succeq \mathscr{T}x_0$, *then* $\mathscr{T}$ *has a fixed point* x^* *and the sequence* $\{\mathscr{T}^n x_0\}$ *of successive iterations converges monotonically to* x^*.

COROLLARY 2.3. (Dhage [6]) *Let* $(E, \preceq, d)$ *be a partially ordered complete metric space such that the order relation* $\preceq$ *and the metric* d *are compatible. Let* S *be a non-empty, closed and partially bounded subset of* E *and let* $\mathscr{T} : S \to S$ *be a nondecreasing, partially nonlinear* $\mathscr{D}$*-contraction mapping. If there exists an element* $x_0 \in S$ *such that* $x_0 \preceq \mathscr{T}x_0$ *or* $x_0 \succeq \mathscr{T}x_0$, *then* $\mathscr{T}$ *has a fixed point* x^* *and the sequence* $\{\mathscr{T}^n x_0\}$ *of successive iterations converges monotonically to* x^*.

2.3. FPTs of Krasnoselskii and Dhage type

Before stating the main results, we need the following definitions in what follows..

DEFINITION 2.7. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{T} : E \to E$ is called partially nonlinear $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathscr{T}x - \mathscr{T}y\| \leqslant \psi(\|x - y\|) \tag{2.6}$$

for all comparable elements $x, y \in E$, where $\psi(0) = 0$. If $\psi(r) = kr$, $k > 0$, then $\mathscr{T}$ is called a partially Lipschitz with a Lipschitz constant k. If $k < 1$, $\mathscr{T}$ is called a partially contraction with contraction constant k. Finally, $\mathscr{T}$ is called nonlinear $\mathscr{D}$-contraction if it is a nonlinear $\mathscr{D}$-Lipschitz with $\psi(r) < r$ for $r > 0$.

Before going to the main fixed point result we prove a useful lemma which we need in what follows.

LEMMA 2.1. *Let* $(E, \preceq, \|\cdot\|)$ *be a partially ordered complete normed linear space. If* $\mathscr{T} : E \to E$ *is a nondecreasing and partially nonlinear* $\mathscr{D}$*-Lipschitz mapping, then for any bounded chain* C *in* E, *we have*

$$\alpha^p(\mathscr{T}C) \leqslant \psi(\alpha^p(C)) \tag{2.7}$$

where α^p *is a partially Kurotowskii measure of noncompactness and* ψ *is an associated* $\mathscr{D}$*-function of* $\mathscr{T}$ *on* E.

Proof. The proof is similar to standard result for usual nonlinear $\mathscr{D}$-Lipschitz mappings with Kurotowskii measure of noncomactness α in the Banach space E. We omit the details.

DEFINITION 2.8. An operator $\mathscr{T}$ on a partially normed linear space E into itself is called **partially compact** if $\mathscr{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathscr{T}$ is called **partially totally bounded** if for any totally ordered and bounded subset C of E, $\mathscr{T}(C)$ is a relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

REMARK 2.6. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

THEOREM 2.4. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear space $\big(E,\preceq,\|\cdot\|\big)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible. Let $\mathscr{A}_i : E \to E$ and $\mathscr{B}_j : S \to E$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$, be two systems of nondecreasing operators such that for each i and j,*

(a) $\mathscr{A}_i$ is partially nonlinear $\mathscr{D}$-contraction,

(b) $\mathscr{B}_j$ is partially completely continuous,

(c) $\sum_{i=1}^{k} \mathscr{A}_i x + \sum_{j=1}^{l} \mathscr{B}_j x \in S$ for all $x \in S$,

(d) $\sum_{i=1}^{k} \psi_{\mathscr{A}_i}(r) < r$ for $r > 0$, and

(e) there exists an element $x_0 \in S$ such that
$$x_0 \preceq \sum_{i=1}^{k} \mathscr{A}_i x_0 + \sum_{j=1}^{l} \mathscr{B}_j x_0 \quad or \quad x_0 \succeq \sum_{i=1}^{k} \mathscr{A}_i x_0 + \sum_{j=1}^{l} \mathscr{B}_j x_0.$$

Then the operator equation
$$\sum_{i=1}^{k} \mathscr{A}_i x + \sum_{j=1}^{l} \mathscr{B}_j x = x \tag{2.8}$$
has a solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by
$$x_{n+1} = \sum_{i=1}^{k} \mathscr{A}_i x_n + \sum_{j=1}^{l} \mathscr{B}_j x_n,$$
for $n = 0, 1, \ldots$; converges monotonically to x^*.

Proof. Define the operator $\mathscr{T} : S \to E$ by

$$\mathscr{T}x = \sum_{i=1}^{k} \mathscr{A}_i x + \sum_{j=1}^{l} \mathscr{B}_j x.$$

Then, using Lemma 2.1 it is proved that $\mathscr{T}$ is a α-condensing mapping on S into itself. Now the desired result follows by a direct application of Theorem 2.1. □

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta\} \quad \text{and} \quad \mathscr{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\},$$

where θ is the zero element of E.

LEMMA 2.2. (Dhage [5]) *If $u_1, u_2, v_1, v_2 \in \mathscr{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1 u_2 \preceq v_1 v_2$.*

DEFINITION 2.9. An operator $\mathscr{T} : E \to E$ is said to be positive if the range $R(\mathscr{T})$ of $\mathscr{T}$ is such that $R(\mathscr{T}) \subseteq \mathscr{K}$.

For any two chains C_1 and C_2 in E, denote

$$C_1 C_2 = \{x \in E \mid x = c_1 c_2,\ c_1 \in C_1 \text{ and } c_2 \in C_2\}.$$

Then we have the following lemma.

LEMMA 2.3. *If C_1 and C_2 are two bounded chains in a partially ordered normed linear algebra E, then*

$$\alpha^p(C_1 C_2) \leqslant \|C_2\| \alpha^p(C_1), + \|C_1\| \alpha^p(C_2) \tag{2.9}$$

where $\|C\| = \sup\{\|c\| \mid c \in C\}$.

THEOREM 2.5. *Let S be a non-empty, closed and partially bounded subset of a regular partially ordered complete normed linear algebra $\big(E, \preceq, \|\cdot\|\big)$ such that the order relation $\preceq$ and the norm $\|\cdot\|$ in X are compatible. Let $\mathscr{A}_i : E \to \mathscr{K}$, $\mathscr{B}_i : S \to \mathscr{K}$ and $\mathscr{C}_j : E \to E$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$, be three systems of nondecreasing operators such that for each i and j,*

(a) $\mathscr{A}_i$ and $\mathscr{C}_j$ are partially nonlinear $\mathscr{D}$-Lipschitz with $\mathscr{D}$-functions $\psi_{\mathscr{A}_i}$ and $\psi_{\mathscr{C}_i}$ respectively,

(b) $\mathscr{B}_i$ is partially completely continuous,

(d) $\displaystyle\sum_{i=1}^{k} \mathscr{A}_i x \mathscr{B}_i x + \sum_{j=1}^{l} \mathscr{C}_j x \in S$ for all $x \in S$,

(c) $\displaystyle\sum_{i=1}^{k} M_i \psi_{\mathscr{A}_i}(r) + \sum_{j=1}^{l} \psi_{\mathscr{C}_j}(r) < r,\ r > 0$, where $M_i = \|\mathscr{B}_i(S)\|$, and

(e) there exists an element $x_0 \in S$ *such that*

$$x_0 \preceq \sum_{i=1}^{k} \mathscr{A}_i x_0 \,\mathscr{B}_i x_0 + \sum_{j=1}^{l} \mathscr{C}_j x_0 \quad or \quad x_0 \succeq \sum_{i=1}^{k} \mathscr{A}_i x_0 \,\mathscr{B}_i x_0 + \sum_{j=1}^{l} \mathscr{C}_j x_0 .$$

Then the operator equation

$$\sum_{i=1}^{k} \mathscr{A}_i x \,\mathscr{B}_i x + \sum_{j=1}^{l} \mathscr{C}_j x = x \tag{2.10}$$

has a solution x^* *in* S *and the sequence* $\{x_n\}$ *of successive approximations defined by*

$$x_{n+1} = \sum_{i=1}^{k} \mathscr{A}_i x_n \,\mathscr{B}_i x_n + \sum_{j=1}^{l} \mathscr{C}_j x_n,$$

for $n = 0, 1, \ldots;$ *converges monotonically to* x^*.

Proof. Define the operator $\mathscr{T} : S \to E$ by

$$\mathscr{T} x = \sum_{i=1}^{k} \mathscr{A}_i x \,\mathscr{B}_i x + \sum_{j=1}^{l} \mathscr{C}_j x.$$

Then, using Lemmas 2.1, 2.2 and 2.3 it is proved that $\mathscr{T}$ is a partially continuous and α-condensing mapping on S into itself. Now the desired result follows by a direct application of Theorem 2.1. □

3. Attractivity and Stability Results

Our considerations will be placed in the Banach space $BC(\mathbb{R}_+, \mathbb{R})$ consisting of all real functions $x = x(t)$ defined, continuous and bounded on $\mathbb{R}_+$. This space is equipped with the standard supremum norm

$$||x|| = \sup\{|x(t)| : t \in \mathbb{R}_+\}. \tag{3.1}$$

Define the order relation $\leqslant$ in $BC(\mathbb{R}_+, \mathbb{R})$ as follows. Let $x, y \in BC(\mathbb{R}_+, \mathbb{R})$. Then by $x \leqslant y$ we mean $x(t) \leqslant y(t)$ for all $t \in \mathbb{R}_+$. It is clear that $(BC(\mathbb{R}_+, \mathbb{R}), \leqslant, \|\cdot\|)$ is regular and the order relation $\leqslant$ and the norm $\|\cdot\|$ are compatible in $BC(\mathbb{R}_+, \mathbb{R})$. Further $(BC(\mathbb{R}_+, \mathbb{R}), \leqslant)$ is also a lattice so that every pair of elements in it has a least lower bound and a greatest upper bound. (cf. Nieto and Lopez [12]). See also Carl and Heikkilä [4] and the references therein.

For our purpose we introduce a handy tool for the partial measure of noncompactness in the space $BC(\mathbb{R}_+, \mathbb{R})$ which is useful in the study of the solutions of certain nonlinear integral equations. To define this partial measure, let us fix a nonempty and bounded chain X of the space $BC(\mathbb{R}_+, \mathbb{R})$ and a positive number T. For $x \in X$ and

$\varepsilon \geqslant 0$ denote by $\omega^T(x,\varepsilon)$ *the modulus of continuity* of the function x on the interval $[0,T]$ defined by

$$\omega^T(x,\varepsilon) = \sup\{|x(t)-x(s)| : \ t,s \in [0,T], \ |t-s| \leqslant \varepsilon\}.$$

Next, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : \ x \in X\}, \ \ \omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X,\varepsilon), \ \ \omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

The partially Hausdorff measure of noncompactness β^p is very much useful in applications to nonlinear differential and integral equations and it can be shown that $\beta^p(X) = \frac{1}{2}\omega_0(X)$ for all bounded chain X in $BC(\mathbb{R}_+,\mathbb{R})$. Thus ω_0 is a handy tool for β^p in $BC(\mathbb{R}_+,\mathbb{R})$. See Dhage [5] and the references therein for the details.

Now, for a fixed number $t \in \mathbb{R}_+$ and a fixed chain X in $BC(\mathbb{R}_+,\mathbb{R})$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Again, for a fixed real number c, denote

$$X(t) - c = \{x(t) - c \ : \ x \in X\}$$

and

$$\delta_b(X(t)) = |X(t)-c| = \sup\{|x(t)-c| : \ x \in X\} \ .$$

Denote

$$\delta_b^T(X(t)) = \sup_{t \geqslant T} \delta_b(X(t)) = \sup_{t\geqslant T} |X(t)-c|$$

and

$$\delta_b(X) = \lim_{T\to\infty} \delta_b^T(X(t)) = \limsup_{t\to\infty} |X(t)-c|.$$

Similarly, let

$$\delta(X(t)) = \operatorname{diam} X(t) = \sup\{|x(t)-y(t)| : \ x,y \in X\}.$$

Let us denote

$$\delta^T(X(t)) = \sup_{t\geqslant T} \delta(X(t)) = \sup_{t\geqslant T} \operatorname{diam} X(t)$$

and

$$\delta_c(X) = \lim_{T\to\infty} \delta^T(X(t)) = \limsup_{t\to\infty} \operatorname{diam} \ X(t).$$

Finally, consider the functions μ_b^p and μ_c^p defined on the family of bounded chains in $BC(\mathbb{R}_+,\mathbb{R})$ by the formula

$$\mu_b^p(X) = \max\{\omega_0(X), \delta_b(X)\} \tag{3.2}$$

and

$$\mu_c^p(X) = \max\{\omega_0(X), \delta_c(X)\}. \tag{3.3}$$

It can be shown that the functions μ_b^p and μ_c^p are a partially measures of noncompactness in the space $BC(\mathbb{R}_+,\mathbb{R})$. The components ω_0, δ_b and ω_0, δ_c are called the characteristic values of the partially measures of noncompactness μ_b^p and μ_c^p respectively in $BC(\mathbb{R}_+,\mathbb{R})$.

REMARK 3.1. The kernel $\ker \mu_c^p$ of the measure μ_c^p consists of nonempty and bounded chains X in $BC(\mathbb{R}_+,\mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity. This particular characteristic of $\ker \mu_c^p$ has been useful in establishing the global attractivity of the comparable solutions. Similarly, the kernel $\ker \mu_b^p$ of the measure μ_b^p consists of nonempty and bounded chains X in $BC(\mathbb{R}_+,\mathbb{R})$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X around the line $x(t)=c$ tends to zero at infinity. This particular characteristic of $\ker \mu_b^p$ has been useful in establishing the global asymptotic attractivity and stability of the comparable solutions for the considered functional fractional integral equations.

In order to introduce further concepts used in the paper let us assume that Ω is a nonempty chain of the space $BC(\mathbb{R}_+,\mathbb{R})$. Moreover, let Q be an operator defined on Ω with values in $BC(\mathbb{R}_+,\mathbb{R})$. Consider the operator equation of the form

$$x(t) = Qx(t),\ t \in \mathbb{R}_+. \tag{3.4}$$

DEFINITION 3.1. We say that comparable solutions of the equation (3.2) are **globally attractive** if for arbitrary comparable solutions $x=x(t)$ and $y=y(t)$ of the equation (3.2) in the space $BC(\mathbb{R}_+,\mathbb{R})$ we have that

$$\lim_{t\to\infty} \big[x(t)-y(t)\big] = 0. \tag{3.5}$$

In the case when limit (3.2) is uniform with respect to the set of comparable solutions, i.e., when for each $\varepsilon>0$ there exists $T>0$ such that

$$|x(t)-y(t)| \leqslant \varepsilon \tag{3.6}$$

for all $x,y\in BC(\mathbb{R}_+,\mathbb{R})$ being the comparable solutions of (3.2) and for $t \geqslant T$, we will say that the comparable solutions of the operator equation (3.2) are **uniformly globally ultimately attractive** defined on $\mathbb{R}_+$.

The equation (1.1) will be considered under the following assumptions:

(H_0) The functions $\alpha,\beta,\gamma:\mathbb{R}_+\to\mathbb{R}_+$ are continuous and satisfy $\alpha(t)\geqslant t$ and $\beta(t)\leqslant t$ for all $t\in\mathbb{R}_+$.

(H_1) The function k is continuous and nonnegative on $\mathbb{R}_+\times\mathbb{R}_+$.

(H_2) The function $F:\mathbb{R}_+\to\mathbb{R}_+$ defined by $F(t)=|f(t,0)|$ is bounded on $\mathbb{R}_+$ with $F_0=\sup\limits_{t\geqslant 0} F(t)$.

(H_3) There exist constants $L > 0$ and $K > 0$ such that

$$0 \leqslant f(t,x) - f(t,y) \leqslant \frac{L(x-y)}{K+(x-y)}$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}$ with $x \geqslant y$. Moreover $L \leqslant K$.

(H_4) There exists a number $c \in \mathbb{R}$ such that $f(t,c) = c$ for all $t \in \mathbb{R}_+$.

(H_5) $g(t,x)$ is nondecreasing in x for each $t \in \mathbb{R}_+$.

(H_6) There exists an element $u \in C(J,\mathbb{R})$ such that

$$u(t) \leqslant f(t,u(\alpha(t))) + \frac{1}{\Gamma(q)} \int_{t_0}^{\beta(t))} \frac{k(t,s)}{(t-s)^{1-q}} g(s,u(\gamma(s)))) \, ds,$$

for all $t \in \mathbb{R}_+$.

(H_7) There exists a function $b: \mathbb{R}_+ \to \mathbb{R}_+$ such that $|g(t,x)| \leqslant b(t)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Moreover, we assume that $\lim_{t\to\infty} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} b(s) ds = 0$.

The hypotheses (H_0) through (H_7) are standard and have been widely used in the literature on nonlinear differential and integral equations. The special case of the hypothesis (H_3) with $L < K$ is considered recently in Nieto and Lopez [12]. Now we formulate the main existence result for the integral equation (1.1) under above mentioned natural conditions.

THEOREM 3.1. *Assume that the hypotheses (H_0)-(H_3) and (H_5)-(H_7) hold. Then the functional fractional integral equation (1.1) has at least one solution x^* in the space $BC(\mathbb{R}_+,\mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_n(t) = f(t,x_{n-1}(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} g(s,x_{n-1}(\gamma(s))) \, ds, \; t \in \mathbb{R}_+, \quad (3.7)$$

for each $n \in \mathbb{N}$ with $x_0 = u$ converges monotonically to x^*. Moreover, the comparable solutions of the equation (1.1) are uniformly globally ultimately attractive defined on $\mathbb{R}_+$.

Proof. We seek the solutions of the HFIE (1.1) in the space $BC(\mathbb{R}_+,\mathbb{R})$. Consider the operator Q defined on the space E by the formula

$$Qx(t) = f(t,x(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} g(s,x(\gamma(s))) \, ds, \; t \in \mathbb{R}_+. \quad (3.8)$$

Observe that in view of our assumptions, for any function $x \in E$ the function Qx is continuous on $\mathbb{R}_+$. As a result, Q defines a mapping $Q : E \to E$. We show that Q

satisfies all the conditions of Theorem 3.1 on E. This will be achieved in a series of following steps:

Step I: *Q is nondecreasing on E.*

Let $x,y \in E$ be such that $x \leqslant y$. Then by hypothesis (H_3) and (H_5), we obtain

$$\begin{aligned} Qx(t) &= f(t,x(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} g(s,x(\gamma(s)))\,ds \\ &\leqslant f(t,y(\alpha(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} g(s,y(\gamma(s)))\,ds \\ &= Qy(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$. This shows that Q is a nondecreasing operator on E.

Step II: *Q maps a closed and partially bounded set into itself.*

Define an open ball $\mathscr{B}(x_0,r)$, where $r = \|x_0\| + L + F_0 + \frac{V}{\Gamma(q)}$. Let X be a chain in E and let $x \in X$ be arbitrary. Since the function $v : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$v(t) = \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} b(s)\,ds \tag{3.9}$$

is continuous and in view of hypothesis (H_7), the number $V = \sup_{t\geqslant 0} v(t)$ exists. Moreover if $x \geqslant \theta$, then for arbitrarily fixed $t \in \mathbb{R}_+$ we obtain:

$$\begin{aligned} |x_0(t) - Qx(t)| &\leqslant |x_0(t)| + |f(t,x(\alpha(t)))| + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} |g(s,x(s))|\,ds \\ &\leqslant |x_0(t)| + |f(t,x(\alpha(t))) - f(t,0)| + |f(t,0)| + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} b(s)\,ds \\ &\leqslant |x_0(t)| + \frac{L|x(\alpha(t))|}{K + |x(\alpha(t))|} + F(t) + \frac{v(t)}{\Gamma(q)} \\ &\leqslant \|x_0\| + L + F_0 + \frac{V}{\Gamma(q)} \\ &= r. \end{aligned} \tag{3.10}$$

Similarly, if $x \leqslant \theta$, then it can be shown that $|x_0(t) - Qx(t)| \leqslant r$ for all $t \in \mathbb{R}_+$. Taking the supremum over t, we obtain $\|x_0 - Qx\| \leqslant r$ for all $x \in X$. This means that the operator Q transforms any bounded chain X in E into a bounded chain in E. More precisely, we infer that the operator Q transforms a chain X belonging to E into the chain $Q(X)$ contained in the ball $\overline{\mathscr{B}}(x_0,r)$. In particular, Q defines a mapping $Q : \mathscr{P}_{ch}(\overline{\mathscr{B}}(x_0,r))) \to \mathscr{P}_{ch}(\overline{\mathscr{B}}(x_0,r)))$ and that Q is partially bounded on $S = \overline{\mathscr{B}}(x_0,r)$ into itself. Therefore, if Q has any fixed point x^* in E, then it must belong to $S = \overline{\mathscr{B}}(x_0,r)$. As a result, any solution of the HFIE (1.1) existing in S is a global solution defined on $\mathbb{R}_+$.

Step III: *Q is partially continuous on S.*

Now we show that the operator Q is partially continuous on the ball $\overline{\mathscr{B}}(x_0,r)$. To do this, let us fix arbitrarily $\varepsilon > 0$ and let X be a chain in $\overline{\mathscr{B}}(x_0,r)$. Take $x,y \in X \subset \overline{\mathscr{B}}(x_0,r)$ such that $x \geqslant y$ and $\|x-y\| \leqslant \varepsilon$. Then,

$$\begin{aligned}
|Qx(t)-Qy(t)| &\leqslant \big|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))\big| \\
&\quad + \frac{1}{\Gamma(q)}\left|\int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}}\, g(s,x(\gamma(s)))\,ds - \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}}\, g(s,y(\gamma(s)))\,ds\right| \\
&\leqslant \big|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))\big| + \frac{1}{\Gamma(q)}\int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}}\,|g(s,x(\gamma(s)))|\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}}\,|g(s,y(\gamma(s)))|\,ds \\
&\leqslant \frac{L|x(\alpha(t))-y(\alpha(t))|}{K+|x(\alpha(t))-y(\alpha(t))|} + \frac{2}{\Gamma(q)}\int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}}\, b(s)\,ds \\
&\leqslant \frac{L\|x-y\|}{K+\|x-y\|} + \frac{2}{\Gamma(q)} v(t) \\
&< L\varepsilon + \frac{2}{\Gamma(q)} v(t).
\end{aligned}$$

Hence, in virtue of hypothesis (H_7) we infer that there exists $T > 0$ such tha $v(t) \leqslant \dfrac{\varepsilon}{2/\Gamma(q)}$ for $t \geqslant T$. Thus, for $t \geqslant T$ we derive that

$$|Qx(t)-Qy(t)| < (L+1)\varepsilon\,. \tag{3.11}$$

Further, let us assume that $t \in [0,T]$. Then, evaluating as above we get:

$$\begin{aligned}
|Qx(t)-Qy(t)| &\leqslant |f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|\Big| \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}}\,[|g(s,x(\gamma(s)))-g(s,y(\gamma(s)))|]\,ds \\
&\leqslant \frac{L|x(\alpha(t))-y(\alpha(t))|}{K+|x(\alpha(t))-y(\alpha(t))|} \\
&\quad + \frac{1}{\Gamma(q)}\int_0^{t} \frac{k(t,s)}{(t-s)^{1-q}}\,\big[|g(s,x(\gamma(s)))-g(s,y(\gamma(s)))|\big]\,ds \\
&< \varepsilon + \frac{CT^q}{\Gamma(q+1)}\,\omega_r^T(g,\varepsilon)\,,
\end{aligned} \tag{3.12}$$

where we have denoted

$$C = \sup\{k(t,s)\ :\ t,s \in [0,T]\},$$

and

$$\omega_r^T(g,\varepsilon) = \sup\{|g(s,x)-g(s,y)| : t,s \in [0,T],\ x,y \in [-r,r],\ |x-y| \leqslant \varepsilon\}.$$

Obviously, in view of continuity of β, we have that $\beta_T < \infty$. Moreover, from the uniform continuity of the function $g(s,x)$ on the set $[0,T]\times[-r,r]$ we derive that $\omega_r^T(g,\varepsilon)\to 0$ as $\varepsilon\to 0$. Now, linking (3.10), (3.12) and the above established facts we conclude that the operator Q maps partially continuously the ball $\overline{\mathscr{B}}(x_0,r)$ into itself.

Step IV: *Q is a nonlinear $\mathscr{D}$-set-contraction w.r.t. the characteristic value* ω_0.

Further on let us take a chain X belonging to the ball $\mathscr{B}(x_0,r)$. Next, fix arbitrarily $T>0$ and $\varepsilon>0$. Let us choose $x\in X$ and $t_1,t_2\in[0,T]$ with $|t_2-t_1|\leqslant\varepsilon$. Without loss of generality we may assume that $x(\alpha(t_1))\geqslant x(\alpha(t_2))$. Then, taking into account our assumptions, we get:

$$
\begin{aligned}
|Qx(t_1)-Qx(t_2)| &\leqslant \big|f(t_1,x(\alpha(t_1)))-f(t_2,x(\alpha(t_2)))\big| \\
&\quad+\frac{1}{\Gamma(q)}\left|\int_0^{\beta(t_1)}\frac{k(t_1,s)}{(t_1-s)^{1-q}}g(s,x(\gamma(s)))\,ds-\int_0^{\beta(t_2)}\frac{k(t_2,s)}{(t_2-s)^{1-q}}g(s,x(\gamma(s)))\,ds\right| \\
&\leqslant \big|f(t_1,x(\alpha(t_1)))-f(t_2,x(\alpha(t_2)))\big| \\
&\quad+\frac{1}{\Gamma(q)}\left|\int_0^{\beta(t_1)}\frac{k(t_1,s)}{(t_1-s)^{1-q}}g(s,x(\gamma(s)))\,ds-\int_0^{\beta(t_1)}\frac{k(t_2,s)}{(t_2-s)^{1-q}}g(s,x(\gamma(s)))\,ds\right| \\
&\quad+\frac{1}{\Gamma(q)}\left|\int_0^{\beta(t_1)}\frac{k(t_2,s)}{(t_2-s)^{1-q}}g(s,x(\gamma(s)))\,ds-\int_0^{\beta(t_2)}\frac{k(t_2,s)}{(t_2-s)^{1-q}}g(s,x(\gamma(s)))\,ds\right| \\
&\leqslant \big|f(t_1,x(\alpha(t_1)))-f(t_2,x(\alpha(t_2)))\big| \\
&\quad+\frac{1}{\Gamma(q)}\int_0^{\beta(t_1)}\left|\frac{k(t_1,s)}{(t_1-s)^{1-q}}-\frac{k(t_2,s)}{(t_2-s)^{1-q}}\right||g(s,x(\gamma(s)))|\,ds \\
&\quad+\frac{1}{\Gamma(q)}\left|\int_{\beta(t_2)}^{\beta(t_1)}\frac{k(t_2,s)}{(t_2-s)^{1-q}}|g(s,x(\gamma(s)))|\,ds\right| \\
&\leqslant \big|f(t_1,x(\alpha(t_1)))-f(t_2,x(\alpha(t_2)))\big| \\
&\quad+\frac{1}{\Gamma(q)}\int_0^{\beta_T}\left|\frac{k(t_1,s)}{(t_1-s)^{1-q}}-\frac{k(t_2,s)}{(t_2-s)^{1-q}}\right|b(s)\,ds+\frac{G_r^T}{\Gamma(q)}|\beta(t_1)-\beta(t_2)|.
\end{aligned} \tag{3.13}
$$

where $G_r^T=\sup\{|g(t,s,x)|:t\in[0,T],s\in[0,\beta_T],x\in[-r,r]\}$ which does exists in view of the fact that the function $g(t,s,x)=\frac{k(t,s)}{(t-s)^{1-q}}g(s,x)$ is continuous on compact $[0,T]\times[0,\beta_T]\times[-r,r]$. Now from (3.13) we obtain,

$$
\begin{aligned}
|Qx(t_2)-Qx(t_1)| &\leqslant \big|f(t_1,x(\alpha(t_1)))-f(t_2,x(\alpha(t_1)))\big|+\frac{L|x(\alpha(t_1))-x(\alpha(t_2))|}{K+|x(\alpha(t_1))-x(\alpha(t_2))|} \\
&\quad+\frac{1}{\Gamma(q)}\int_0^{\beta_T}\left|\frac{k(t_1,s)}{(t_1-s)^{1-q}}-\frac{k(t_2,s)}{(t_2-s)^{1-q}}\right|b(s)\,ds \\
&\quad+\frac{G_r^T}{\Gamma(q)}|\beta(t_1)-\beta(t_2)| \\
&\leqslant \frac{L\,\omega^T(x,\omega^T(\alpha,\varepsilon))}{K+\omega^T(x,\omega^T(\alpha,\varepsilon))}+\omega_r^T(f,\varepsilon)
\end{aligned}
$$

$$+\frac{1}{\Gamma(q)}\int_0^{\beta_T}\left|\frac{k(t_1,s)}{(t_1-s)^{1-q}}-\frac{k(t_2,s)}{(t_2-s)^{1-q}}\right|b(s)\,ds+\frac{G_r^T}{\Gamma(q)}\omega^T(\beta,\varepsilon) \tag{3.14}$$

where we have denoted

$$\omega^T(\alpha,\varepsilon)=\sup\{|\alpha(t_2)-\alpha(t_1)|:\ t_1,t_2\in[0,T],\ |t_2-t_1|\leqslant\varepsilon\},$$

$$\omega^T(v,\varepsilon)=\sup\{|\beta(t_2)-\beta(t_1)|:\ t_1,t_2\in[0,T],\ |t_2-t_1|\leqslant\varepsilon\},$$

and

$$\omega_r^T(f,\varepsilon)=\sup\{|f(t_2,x)-f(t_1,x)|:\ t_1,t_2\in[0,T],\ |t_2-t_1|\leqslant\varepsilon,\ x\in[-r,r]\}.$$

From the above estimate we derive the following one:

$$\omega^T(Q(X),\varepsilon)\leqslant\frac{L\omega^T(X,\omega^T(\alpha,\varepsilon))}{K+\omega^T(X,\omega^T(\alpha,\varepsilon))}+\omega_r^T(f,\varepsilon)$$
$$+\frac{1}{\Gamma(q)}\int_0^{\beta_T}\left|\frac{k(t_1,s)}{(t_1-s)^{1-q}}-\frac{k(t_2,s)}{(t_2-s)^{1-q}}\right|b(s)\,ds+\frac{G_r^T}{\Gamma(q)}\omega^T(\beta,\varepsilon). \tag{3.15}$$

Observe that $\omega_r^T(f,\varepsilon)\to 0$ and $\left|\frac{k(t_1,s)}{(t_1-s)^{1-q}}-\frac{k(t_2,s)}{(t_2-s)^{1-q}}\right|\to 0$ as $\varepsilon\to 0$, which is a simple consequence of the uniform continuity of the functions f and $(t-s)^{q-1}$ on the sets $[0,T]\times[-r,r]$ and $[0,T]\times[0,\beta_T]$ respectively. Moreover, from the uniform continuity of α,β on $[0,T]$, it follows that $\omega^T(\alpha,\varepsilon)\to 0$, $\omega^T(\beta,\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus, linking the established facts with the estimate (3.13), we get $\omega_0^T(Q(X))\leqslant\dfrac{L\,\omega_0^T(X)}{K+\omega_0^T(X)}$. Consequently, we obtain

$$\omega_0(Q(X))\leqslant\frac{L\,\omega_0(X)}{K+\omega_0(X)}\,. \tag{3.16}$$

Step V: *Q is a nonlinear $\mathscr{D}$-set-contraction w.r.t. characteristic value δ_c.*

Now, taking into account our assumptions, for arbitrarily fixed $t\in\mathbb{R}_+$ and for $x,y\in X$ with $x\geqslant y$, we deduce the following estimate (cf. the estimate in Step III):

$$|(Qx)(t)-(Qy)(t)|\leqslant|f(t,x(\alpha(t)))-f(t,y(\alpha(t)))|+2\frac{1}{\Gamma(q)}\int_0^{\beta(t)}\frac{k(t,s)}{(t-s)^{1-q}}b(s)\,ds$$
$$\leqslant\frac{L|x(\alpha(t))-y(\alpha(t))|}{K+|x(\alpha(t))-y(\alpha(t))|}+\frac{2v(t)}{\Gamma(q)}.$$

From the above inequality it follows that

$$\operatorname{diam}(QX(t))\leqslant\frac{L\operatorname{diam}(X(\alpha(t)))}{K+\operatorname{diam}(X(\alpha(t)))}+\frac{2v(t)}{\Gamma(q)}.$$

for each $t\in\mathbb{R}_+$. Therefore, taking the limit superior over $t\to\infty$, we obtain

$$\delta_c(QX) \leqslant \frac{L \limsup_{t\to\infty} \operatorname{diam}(X(\alpha(t)))}{K + \limsup_{t\to\infty} \operatorname{diam}(X(\alpha(t)))} \leqslant \frac{L\,\delta_c(X)}{K+\delta_c(X)}. \tag{3.17}$$

Step VI: *Q is a partially nonlinear $\mathscr{D}$-set-contraction on S.*

Further, using the measure of noncompactness μ_c^p defined by the formula (3.3) and keeping in mind the estimates (3.16) and (3.17), we obtain

$$\begin{aligned}\mu_c^p(QX) &= \max\{\omega_0(QX),\, \delta_c(QX)\}\\ &\leqslant \max\left\{\frac{L\,\omega_0(X)}{K+\omega_0(X)}, \frac{L\,\delta_c(X)}{K+\delta_c(X)}\right\}\\ &\leqslant \frac{L\mu_c^p(X)}{K+\mu_c^p(X)}.\end{aligned} \tag{3.18}$$

This shows that Q is a partially nonlinear $\mathscr{D}$-set-contraction on S with $\mathscr{D}$-function $\psi(r) = \dfrac{Lr}{K+r}$. Again, by hypothesis (H$_6$), there exists an element $x_0 = u \in S$ such that $x_0 \leqslant Qx_0$, that is, x_0 is a lower solution of the HFIE (1.1) defined on $\mathbb{R}_+$.

Thus Q satisfies all the conditions of Theorem 2.1 on S. Hence we apply it to the operator equation $Qx = x$ and deduce that the operator Q has a fixed point x^* in the ball $\overline{\mathscr{B}}(x_0,r)$. Obviously x^* is a solution of the functional integral equation (1.1) and the sequence $\{x_n\}$ defined by (3.7) converges monotonically to x^*. Moreover, taking into account that the image of every chain X under the operator Q is again a chain $Q(X)$ contained in the ball $\overline{\mathscr{B}}(x_0,r)$ we infer that the set $\mathscr{F}(Q)$ of all fixed points of Q is contained in $\overline{\mathscr{B}}(x_0,r)$. If the set $\mathscr{F}(Q)$ contains all comparable solutions of the equation (1.1), then we conclude from Remark 2.3 that the set $\mathscr{F}(Q)$ also belongs to the family $\ker \mu_c^p$. Now, taking into account the description of sets belonging to $\ker \mu_c^p$ (given in Section 3) we deduce that all comparable solutions of the equation (1.1) are uniformly globally ultimately attractive on $\mathbb{R}_+$. This completes the proof. □

Similarly, we can prove the following result concerning the asymptotic stability of the comparable solutions.

THEOREM 3.2. *Assume that the hypotheses (H$_0$) through (H$_7$) hold. Then the functional HFIE (1.1) has at least one solution x^* in the space $BC(\mathbb{R}_+,\mathbb{R})$ and the sequence $\{x_n\}$ of successive approximations defined by (3.7) converges monotonically to x^*. Moreover, the comparable solutions of the equation (1.1) are uniformly globally ultimately asymptotically stable to the line $x(t) = c$ defined on $\mathbb{R}_+$.*

Proof. As in Theorem 3.1, we seek the solutions of the HFIE (1.1) in the space $E = BC(\mathbb{R}_+,\mathbb{R})$. Define the closed bounded set $S = \overline{\mathscr{B}}(x_0,r)$ and define the operator Q on S into itself by (3.8). Then proceeding as in the Step IV of the proof of Theorem 3.1 it can be proved that the inequality (3.16) is held. Next, we show that Q is a nonlinear $\mathscr{D}$-set-contraction with respect to the characteristic value δ_a. Let X be a set in S. Now, taking into account our assumptions, for arbitrarily fixed $t \in \mathbb{R}_+$ and for $x \in X$ with

$x \geqslant c$, we deduce the following estimate:

$$\begin{aligned} |(Qx)(t)-c| &\leqslant |f(t,x(\alpha(t)))-f(t,c)| + \frac{1}{\Gamma(q)}\int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} b(s)\,ds \\ &\leqslant \frac{L|x(\alpha(t))-c|}{K+|x(\alpha(t))-c|} + \frac{v(t)}{\Gamma(q)}. \end{aligned}$$

From the above inequality it follows that

$$|QX(t)-c| \leqslant \frac{L|X(\alpha(t))-c|}{K+|X(\alpha(t))-c|} + \frac{v(t)}{\Gamma(q)}$$

for each $t \in \mathbb{R}_+$. Therefore, taking limit superior over $t \to \infty$, we obtain

$$\begin{aligned} \delta_b(QX) &= \limsup_{t\to\infty} \frac{L|X(\alpha(t))-c|}{K+|X(\alpha(t))-c|} \\ &\leqslant \frac{L \limsup_{t\to\infty} |X(t)-c|}{K+\limsup_{t\to\infty} |X(t)-c|} \\ &= \frac{L\delta_b(X)}{K+\delta_b(X)}. \end{aligned} \tag{3.19}$$

Further, using the measure of noncompactness μ_b^p defined by the formula (3.2) and keeping in mind the estimates (3.16) and (3.19), we obtain

$$\begin{aligned} \mu_b^p(QX) &= \max\{\omega_0(QX), \delta_b(QX)\} \\ &\leqslant \max\left\{ \frac{L\,\omega_0(X)}{K+\omega_0(X)}, \frac{L\delta_b(X)}{K+\delta_b(X)} \right\} \\ &\leqslant \frac{L\mu_b^p(X)}{K+\mu_b^p(X)}. \end{aligned} \tag{3.20}$$

This shows that Q is a nonlinear $\mathscr{D}$-set-contraction on S with a $\mathscr{D}$-function $\psi(r) = \dfrac{Lr}{K+r}$. Again, by hypothesis (H$_6$), there exists an element $x_0 = u \in S$ such that $x_0 \leqslant Qx_0$, that is, x_0 is a lower solution of the HFIE (1.1) defined on $\mathbb{R}_+$. The rest of the proof is similar to Theorem 3.1 and now we conclude from Remark 2.3 that the set $\mathscr{F}(Q)$ belongs to the family $\ker \mu_b^p$. Now, taking into account the description of sets belonging to $\ker \mu_b^p$ (given in Section 3) we deduce that the equation (1.1) has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by (3.7) converges monotonically to x^*. Moreover, all comparable solutions of the HFIE (1.1) are uniformly globally asymptotically stable to the line $x(t) = c$ defined on $\mathbb{R}_+$. This completes the proof. □

If $c = 0$ in Theorem 3.2, it reduces to the existence result concerning the asymptotic stability of the solutions to zero and all comparable solutions if exist have the same property.

REMARK 3.2. The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H_3) with the following one:

(H'_3) There exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 \leqslant f(t,x) - f(t,y) \leqslant \phi(x-y)$$

for all $x, y \in \mathbb{R}$, $x \geqslant y$, where $\phi(r) < r$ for $r > 0$.

In the following we give a numerical example to illustrate the abstract theory developed in this paper.

EXAMPLE 3.1. Consider the nonlinear hybrid fractional integral equation with linear perturbation of first type,

$$x(t) = \tan^{-1} x(2t) + \frac{1}{\Gamma(3/2)} \int_0^t \frac{(t-s)^{1/2}}{t^2+1} g(s, x(3s))\, ds \tag{3.21}$$

for all $t \in \mathbb{R}_+$, where $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$g(t,x) = \begin{cases} 1, & \text{if} \quad x \leqslant 1, \\ \dfrac{2x}{x+1}, & \text{if} \quad x > 1. \end{cases}$$

We shall show that all the hypotheses of Theorem 3.1 are satisfied by the functions involved in HFIE (3.21). Here, $\alpha(t) = 2t$, $\beta(t) = t$ and $\gamma(t) = 3t$ and so, α, β, γ are continuous on $\mathbb{R}_+$ into itself and satisfy $\alpha(t) \geqslant t$ and $\beta(t) \leqslant t$ for all $t \in \mathbb{R}_+$. Thus, hypothesis (H_0) is satisfied.

Again, $f(t,x) = \tan^{-1} x$ so that f is nondecreasing in x and continuous on $\mathbb{R}_+ \times \mathbb{R}$. The kernel $k(t,s)$ is given by $k(t,s) = \frac{1}{t^2+1}$. Obviously k is continuous and nonnegative on $\mathbb{R}_+ \times \mathbb{R}_+$ and so (H_1) holds. Next, $g(t,x)$ is defines a continuous and nondecreasing function in x for each $t \in \mathbb{R}_+$. Moreover, $f(t,0) = 0$. So the hypotheses (H_2), (H_4) and (H_5) are held.

Now, we show that f is partially Lipschitz on $\mathbb{R}_+ \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ with $x \geqslant y$. Then,

$$0 \leqslant f(t,x) - f(t,y) = \tan^{-1} x - \tan^{-1} y = \frac{1}{1+\xi^2}(x-y)$$

for all $y < \xi < x$, and so hypothesis (H'_3) is satisfied with $\phi(r) = \dfrac{r}{1+\xi^2}$ for $0 < \xi < r$. Furthermore, $|g(t,x)| \leqslant 2$ for all $t \in \mathbb{R}_+$ and $\mathbb{R}$. Therefore,

$$v(t) = 2\int_0^t \frac{(t-s)^{\frac{1}{2}}}{t^2+1} \cdot 1\, ds = \frac{4}{3} \cdot \frac{t^{\frac{3}{2}}}{t^2+1}.$$

Therefore,

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{4}{3} \frac{t^{\frac{3}{2}}}{t^2+1} = 0,$$

and so hypothesis (H_7) of Theorem 3.1 is satisfied.

Finally, it is easy to prove that $u \equiv 0$ is a lower solution of the HFIE (3.21) on $\mathbb{R}_+$ and hence the hypothesis (H_6) is held. Thus all the conditions of Theorem 3.1 are satisfied and by a direct application, we conclude that the HFIE (3.21) has a solution x^* and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = \tan^{-1} x_n(2t) + \frac{1}{\Gamma(3/2)} \int_0^t \frac{(t-s)^{1/2}}{t^2+1}\, g(s, x_n(3s))\, ds, \tag{3.22}$$

for all $t \in \mathbb{R}_+$ converges monotonically to x^*, where $x_0 = 0$. Moreover, the comparable solutions of the HFIE (3.21) are uniformly asymptotically attractive and stable to zero defined on $\mathbb{R}_+$.

REMARK 3.3. The existence theorems proved in Section 3 may be extended with appropriate modifications to the linearly perturbed generalized nonlinear hybrid fractional integral equation,

$$x(t) = f(t, x(\alpha_1(t)), \ldots, x(\alpha_n(t))) + \frac{1}{\Gamma(q)} \int_0^{\beta(t)} \frac{k(t,s)}{(t-s)^{1-q}} g(s, x(\gamma_1(s)), \ldots, \gamma_n(s)))\, ds \tag{3.23}$$

for all $t \in \mathbb{R}_+$, where $\alpha_i, \beta, \gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, \ldots, n$, $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions and $1 \leqslant q < 2$.

REMARK 3.4. The existence theorem for the HFIE (1.1) may be proved using Theorem 2.4 under weaker Carathéodory condition. Finally, we remark that the study of the present paper may be extended to other types of nonlinear hybrid integral equations with different linear and quadratic perturbations of first and second type.

4. Conclusion

In this paper we have been able to weaken the Lipschitz condition to partially Lipschitz condition and continuity to partially continuity of the nonlinearities which otherwise are considered to be very strong conditions in the existence theory for nonlinear differential and integral equations. However, in this situation we needed an additional assumption of the monotonicity on the nonlinearities involved in the considered integral equation in order to guarantee the required characterization of asymptotic attractivity or asymptotic stability of the comparable solutions defined on unbounded interval of real line. The advantage of present approach over the previous ones lies in the fact that we have been able to develop an algorithm for the solutions of the considered integral equations which otherwise is not possible via classical approach of measure of noncompactness described in Banas and Goebel [2] and several related papers. Another interesting feature of our work is that we generally need the uniqueness of the solution for predicting the behavior of the dynamic systems related to the considered nonlinear fractional integral equation, however with the present approach it is possible for us to discuss the qualitative behavior even though there exist a number of solutions of the dynamic systems in question. Finally, while concluding this paper we mention that the results presented here are for a linearly perturbed Volterra fractional integral equation, however analogous study can also be made for any nonlinear fractional integral equation related to global asymptotic attractivity and stability using the similar arguments with appropriate modifications.

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