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SUBCRITICAL NONLINEAR PSEUDODIFFERENTIAL EQUATION OF SOBOLEV TYPE ON A HALF–LINE

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Abstract. We study the initial- boundary value problem for the complex pseudodifferential equation of Sobolev type on a half-line

$$\begin{cases} \partial_t u + \lambda\,|u|^{\sigma}\,u + \mathbb{K}u = 0,\; x \in \mathbb{R}^+,\; t > 0, \\ u(0,x) = u_0(x),\, x \in \mathbb{R}^+, \end{cases}$$

where $0 < \sigma < 1,\ \lambda \in \mathbb{C}$,

$$\mathbb{K}u = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p)\widehat{u}(t,p)dp.$$

the symbol $K(p)$ is defined as

$$K(p) = (-1)^{n+1}p^{2n}\prod_{j=1}^{n}(p^2 - a_j^2)^{-1},$$

$n \in N,\ \operatorname{Re} a_j > 0, j = 1,2...,n,\ \ \theta(x)..$ The aim of this paper is to prove the global existence of solutions to the inital-boundary value problem and to find the main term of the asymptotic representation of solutions in the subcritical case,when the nonlinear term of the equation has the time decay rate less then that of the linear terms.

1. Introduction

We consider the initial-boundary value problem on a half-line for the nonlinear pseudodifferential equation

$$\begin{cases} \partial_t u(t,x) + \mathscr{N}(u) + \mathbb{K}u = 0,\; x \in \mathbb{R}^+,\; t > 0, \\ u(0,x) = u_0(x),\, x \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

with a subcritical nonlinearity $\mathscr{N}(u) = \lambda\,|u|^{\sigma}\,u,\ 0 < \sigma < n,\ \lambda \in \mathbb{C}$. Here the pseudodifferential operator $\mathbb{K}$ is defined as

$$\mathbb{K}u = \frac{1}{2\pi i}\int_{-\mathbf{i}\infty}^{i\infty} e^{xp}K(p)\widehat{u}(t,p)dp,$$

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with a rational symbol

$$K(p)=(-1)^{n+1}p^{2n}\prod_{j=1}^{n}(p^2-a_j^2)^{-1},$$

$n\in N,$ $\operatorname{Re}a_j>0, j=1,...,n$ and $\hat{\phi}$ is the Laplace transform of ϕ defined by

$$\hat{\phi}\,(p)=\int_{\mathbb{R}^+}e^{-xp}\phi\,(x)\,dx.$$

Equation (1.1) is a pseudodifferential form of wave equations for media with a strong spatial dispersion, which appear in the nonlinear theory of the quasy-stationary processes in the electric media (see [12]). For example, the equation of the form

$$\left(1-\partial_x^2\right)u_t=u_{xx}-\lambda\left(1-\partial_x^2\right)|u|^2u \tag{1.2}$$

describes the creation, propagation, and collapse of so-called electric domains in semiconductors. In the case of the whole line we can invert the operator $\left(1-\partial_x^2\right)$, so that we arrive to equation (1.1) with a symbol $K(p)=\frac{p^2}{p^2-1}$. Note that in the case of the Cauchy problem equation (1.2) is equivalent to its pseudodifferential form (1.1). For results concerning the Cauchy problem for nonlinear pseudoparabolic type equations see [3], [4],[5], [14], [15]. The large time asymptotic of solutions to the Cauchy problem was obtained in papers [10], [13]. Recently much attention was drown to the study of the global existence and large time asymptotic behavior of solutions to the Cauchy problems for nonlinear equations in the subcritical case , when the nonlinearity has a slow time decay property comparing with the linear part of the equation (see papers [1], [2], [7], [11] and literature cited therein).

One of the most important developments in the theory of pseudodifferential operators is a generalization of the Cauchy problem to the case of the initial-boundary value problem on a half-line. The boundary value problems are more natural for applications, however their mathematical investigations are more complicated. It is necessary to answer the question: how many boundary values should be given in the problem for its solvability and the uniqueness of the solution?

For the general theory of nonlinear pseudodifferential equations on a half-line with analytic symbol we refer to the book [6]. As far as we know there are few results in the case of subcritical nonlinear pseudodifferential equations with analytic symbol on a half-line (see papers [8], [9] and literature cited therein). We give a review of these works. In the paper [8] it was studied subcritical nonlinear nonlocal equations on a half-line

$$\partial_t u+\beta\,|u|^{\rho}\,u+\mathbb{K}u=0,\ x>0,t>0, \tag{1.3}$$

where $\beta\in\mathbf{C}$ and the order of nonlinearity $\rho\in(0,\alpha)$. The linear operator $\mathbb{K}$ is a pseudodifferential operator defined by the inverse Laplace transform as follows

$$\mathbb{K}u=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{px}E_\alpha p^\alpha\left(\hat{u}(p,t)-\sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,t)}{p^j}\right)dp. \tag{1.4}$$

Note that the symbol $K(p) = E_\alpha p^\alpha$ is analytic in a right-half complex plane. It was obtained that for small initial data in $\mathbf{L}^1$ solution of Dirichlet problem for equation (1.3) has the same time decay in $\mathbf{L}^\infty$ norm comparing with case of the corresponding Cauchy problem. Also the main term of the asymptotic depend on the mean value of the solution . Explicit asymptotics for solution of Kuramoto-Sivashinsky-type equation on a half-line in the case of inhomogeneous symbol $K(p) = p^2 - p^4$ and nonlinearity $\mathscr{N}(u,u_x) = u^\sigma u_x^\rho, \rho \geqslant 10, \sigma \geqslant 0$ was obtained in paper [9]. It was proved that for small initial data in $\mathbf{L}^{1,2} \cap \mathbf{L}^\infty$ solutions decay as $t^{-1-\frac{1}{\rho+\sigma-1}}$ in $\mathbf{L}^\infty$ and the main term of the asymptotic depend on the first moment of the solution.

In this paper we present a further development of the theory of subcritical nonlinear pseudodifferential equations on a half-line with a nonanalytic symbol, considering the case of rational symbol

$$K(p) = (-1)^{n+1} p^{2n} \prod_{j=1}^{n} (p^2 - a_j^2)^{-1}.$$

We prove global in time existence of solutions to the initial-boundary value problem and find the asymptotic behavior of solutions for large time. The main difficulty in the study of equation (1.1) on a half-line is that its symbol $K(p)$ is nonanalytic, therefore we can not apply the methods of book [6] directly. To prove the well-posedness of problem (1.1) we use the integral representation for sectionally analytic function and a theory of singular integro-differential equations with Hilbert kernel. Thus we show that we do not need any boundary data in problem (1.1), whereas if we rewrite equation (1.1) in the pseudoparabolic form, then n boundary data should be posed in the problem for its correct solvability (see [6]). For example, to prove the well-posedness for equation (1.2) we need to put one boundary datum, whereas no boundary value is necessary in the case of equation (1.1). Since very often model equations of mathematical physics are derived only by using the corresponding dispersion relation (see, e.g. [16]) and have a pseudodifferential form, we address a very interesting question to Physicist: which problem is more adequate for describing the physical phenomena?

Now we state the main result of this paper. By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denote by $\mathbf{L}^p$, $1 \leqslant p \leqslant \infty$, the weighted Lebesgue space $\mathbf{L}^{1,a}$ is defined by

$$\mathbf{L}^{p,a} = \left\{ \phi \in \mathbf{L}^p(\mathbb{R}^+); \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle x\rangle^a \phi\|_{\mathbf{L}^p} < \infty \right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $a \geqslant 0$. Denote

$$\theta = \left| \int_{\mathbb{R}^+} x u_0(x) dx \right|,$$

$$\widetilde{G}(x) = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} e^{(-1)^{n+1} p^{2n}} e^{px} p dp$$

and

$$\eta = \operatorname{Re} \lambda \int_{\mathbb{R}^+} x \mathscr{N}(\widetilde{G}(x)) dx > 0.$$

THEOREM 1. *Let* $\eta > 0$. *Assume that the initial data* $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}$, $a \in (0,1)$ *are sufficiently small* $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,1+a}} \leqslant \varepsilon$, *and* $n(1-\varepsilon^{n+1}\eta) < \sigma < n$. *Then the initial-boundary value problem (1.1) has a unique global solution* $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}\right)$. *Furthermore there exist some constant* A , *a real constant* w *and a function* $V \in \mathbf{L}^{1,1+a} \cap \mathbf{L}^\infty$ *such that*

$$\left\| \left(u(t) - At^{-\frac{\mu}{\sigma} - \frac{1}{n}} e^{i\omega \log t} V\left(\cdot t^{-\frac{1}{2n}}\right)\right)\right\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{\mu}{\sigma} - \frac{1}{n} - \gamma}, \tag{1.5}$$

where $\mu = 1 - \frac{\sigma}{n}$, $\gamma = \frac{1}{2}\min(a, n-\sigma)$.

We organize the rest of our paper as follows. In Section 2 we obtain some preliminary estimates for the Green operator and prove the well-posedness of the linearized problem (1.1). Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary lemmas

Let $\phi(q)$ be a complex function, which obeys the Hölder condition for all finite q and tends to a definite limit $\phi(\infty)$ as $q \to \infty$. Then Cauchy type integral

$$F(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(q)}{q-z} dq$$

constitutes a function analytic in the left and right semi-planes. Here and below this functions will be denoted $F^+(z)$ and $F^-(z)$, respectively. This functions have the limiting values $F^+(p)$ and $F^-(p)$ at all points of imaginary axis Re $p=0$, on approaching the contour from the left and from the right, respectively. These limiting values are expressed by Sokhotzki-Plemelj formula

$$F^+(p) - F^-(p) = \phi(p). \tag{2.1}$$

All the integrals are understood in the sense of the principal values. Now we consider the linear initial boundary-value problem on a half-line

$$\begin{cases} \partial_t u + \mathbb{K}u = f(t,x),\ x \in \mathbb{R}^+,\ t \in \mathbb{R}^+, \\ u(0,x) = u_0(x), x \in \mathbb{R}^+. \end{cases} \tag{2.2}$$

Denote by $\mathscr{G}(t)$

$$\mathscr{G}(t)\phi = \mathscr{L}^{-1}\left\{ e^{-K(p)t}\left(\widehat{\phi}(p) - \widehat{\phi}(-p)\right)\right\}. \tag{2.3}$$

THEOREM 2. *Let*

$$u_0 \in \mathbf{L}^1\left(\mathbb{R}^+\right), f(t,x) \in \mathbf{C}^0(\mathbb{R}^+, \mathbf{L}^1 \cap \mathbf{C}).$$

Then solution of (2.2) has the following form

$$u(t,x) = \mathscr{G}(t)u_0 + \int_0^t \mathscr{G}(t-\tau) f(\tau) d\tau.$$

Proof. To derive an integral representation for the solutions of the problem (2.2) we suppose that there exists a solution $u(t,x)$ of problem (2.2), which is continued by zero outside of $x>0$:

$$u(t,x)=0 \quad \text{for all} \quad x<0.$$

Let $\phi(p)$ be a function of the complex variable p, which obeys the Hölder condition for all finite p and tends to 0 as $p\to\pm i\infty$. We define the operator

$$\mathbb{P}\phi(z)=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\phi(q)dq.$$

We have for the Laplace transform

$$\mathscr{L}\{\mathbb{K}\mathrm{u}\}=\mathbb{P}\{K(p)\mathscr{L}\{u\}\}.$$

Since $\mathscr{L}\{u\}$ is analytic for all Re $q>0$ we have

$$\widehat{u}(t,p)=\mathscr{L}\{u\}=\mathbb{P}\widehat{u}(t,p). \tag{2.4}$$

Taking Laplace transform with respect to the space variable we get

$$\begin{cases}\mathbb{P}\left(\partial_t\widehat{u}(t,q)+K(q)\widehat{u}(t,q)-\widehat{f}(t,q)\right)dq=0,\ x\in\mathbb{R}^+,t>0,\\ \widehat{u}(0,p)=\widehat{u}_0(p),\ x\in\mathbb{R}^+.\end{cases} \tag{2.5}$$

We rewrite the equation (2.5) in the form

$$\partial_t\widehat{u}(t,p)+K(p)\widehat{u}(t,p)-\widehat{f}(t,p)=\Psi(t,p), \tag{2.6}$$

with some function $\Psi(t,p)=O(\langle p\rangle^{-1})$ such that

$$\mathbb{P}^-\{\Psi(t,p)\}=0. \tag{2.7}$$

Applying the Laplace transformation with respect to time variable to problem (2.6) we find for Re $p>0$

$$\widehat{\widehat{u}}(\xi,p)=\frac{1}{K(p)+\xi}\left(\widehat{u}_0(p)+\widehat{\widehat{f}}(\xi,p)+\widehat{\Psi}(\xi,p)\right). \tag{2.8}$$

Here the functions $\widehat{\widehat{u}}(\xi,p),\widehat{\Psi}(\xi,p)$ and $\widehat{\widehat{f}}(\xi,p)$ are the Laplace transforms for $\widehat{u}(t,p)$, $\Psi(t,p)$ and $\widehat{f}(t,p)$ with respect to time, respectively. We will find the function $\widehat{\Psi}(\xi,p)$ using the analytic properties of function $\widehat{\widehat{u}}$ in the right-half complex planes Re $p>0$ and Re $\xi>0$. We have for Re $p=0$ the sufficient condition

$$\widehat{\widehat{u}}(\xi,p)=-\frac{1}{\pi i}VP\int_{-i\infty}^{i\infty}\frac{1}{q-p}\widehat{\widehat{u}}(\xi,q)dq. \tag{2.9}$$

Here and below PV means the Cauchy principal value of the singular integral. Taking into account the assumed condition (2.7) and making use of Sokhotzki-Plemelj formula (2.1) we perform the condition (2.9) in the form of nonhomogeneous Riemann problem

$$\Omega^{+}(\xi,p)=\frac{K(p)+\xi}{\xi}\Omega^{-}(\xi,p)-K(p)\Lambda^{+}(\xi,p), \tag{2.10}$$

where the sectionally analytic functions functions $\Omega(z,\xi)$ and $\Lambda(z,\xi)$ given by formulas

$$\Omega(\xi,z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{K(q)}{K(q)+\xi}\widehat{\Psi}(\xi,p)dq, \tag{2.11}$$

$$\Lambda(\xi,z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{K(q)+\xi}\left(\widehat{u}_0(q)+\widehat{\widehat{f}}(\xi,p)\right)dq. \tag{2.12}$$

It is required to find two functions for some fixed point ξ, Re $\xi>0$: $\Omega^{+}(z,\xi)$, analytic in Re $z<0$ and $\Omega^{-}(z,\xi)$, analytic in Re $z>0$, which satisfy on the contour Re $p=0$ the relation (2.10).Note that bearing in mind formula (2.11) we can find unknown function $\widehat{\Phi}(p,\xi)$ which involved in the formula (2.8) by the relation

$$\widehat{\Psi}(\xi,p)=\frac{K(p)+\xi}{K(p)}\left(\Omega^{+}(\xi,p)-\Omega^{-}(\xi,p)\right). \tag{2.13}$$

We introduce the function

$$W=K(p)+\xi.$$

There are exist n roots $\phi_j(\xi)$ of the equation $K(p)=-\xi$, such that $\operatorname{Re}\phi_j(\xi)>0$ for all $\operatorname{Re}\xi>0$, and $\lim_{\xi\to\infty}\phi_j(\xi)=a_j$. Therefore the function $W(p,\xi)$ can be represented as the ratio of the functions $Y^{+}(p)$ and $Y^{-}(p)$ constituting the boundary values of functions, $Y^{+}(z)$ and $Y^{-}(z)$, analytic in the left and right complex semi-plane and having in these domains no zero

$$W(\xi,p)=\frac{Y^{+}(\xi,p)}{Y^{-}(\xi,p)}. \tag{2.14}$$

These functions are given by formula

$$Y^{+}(\xi,p)=\prod_{j=1}^{n}\frac{(\xi+1)(p-\phi_j(\xi))}{(p-a_j)},$$

$$Y^{-}(\xi,p)=\prod_{j=1}^{n}\frac{(p+a_j)}{(\xi+1)(p+\phi_j(\xi))}$$

Replacing in equation (2.10) the coefficient of the Riemann problem $W(\xi,p)$ by (2.14) we reduce the nonhomogeneous Riemann problem (2.10) in the form

$$\frac{\Omega^{+}(\xi,p)}{Y^{+}(\xi,p)}=\frac{\Omega^{-}(\xi,p)}{Y^{-}(\xi,p)}-\frac{1}{Y^{+}}K(p)\Lambda^{+}(\xi,p). \tag{2.15}$$

Since $\frac{1}{Y^+}K(p)\Lambda^+(\xi,p)$ satisfies on Re $p=0$ the Hölder condition, it can be uniquely represented in the form of the difference of the functions $U^+(\xi,p)$ and $U^-(\xi,p)$, constituting the boundary values of the analytic function $U(\xi,z)$, given by formula

$$U(\xi,z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-z}\frac{1}{Y^+}K(p)\Lambda^+(\xi,p)dq. \tag{2.16}$$

Therefore the problem (2.15) takes the form

$$\frac{\Omega_1^+(\xi,p)}{Y^+(\xi,p)}+U^+(\xi,p)=\frac{\Omega_1^-(\xi,p)}{Y^-(\xi,p)}+U^-(\xi,p).$$

The last relation indicates that the function $\frac{\Omega_1^+}{Y^+}+U^+$, analytic in Re $z<0$, and the function $\frac{\Omega_1^-}{Y^-}+U^-$, analytic in Re$z>0$, constitute the analytic continuation of each other through the contour Re $z=0$. Consequently, they are branches of unique analytic function in the entire plane,which has zero in the point $z=\infty$ .According to Liouville theorem this function is zero. Thus we get

$$\Omega^+=-Y^+U^+, \tag{2.17}$$
$$\Omega^-=-\xi Y^-U^-.$$

From (2.17) under (2.14) the difference limiting values of solution of (2.10) are given by formula

$$\Omega^+-\Omega^-=-\frac{K(p)}{K(p)+\xi}Y^+U^+ \tag{2.18}$$

Replacing the difference $\Omega^+-\Omega^-$ in the relation (2.13) by formula (2.18) we get

$$\widehat{\Psi}(\xi,p)=-Y^+U^+.$$

It is easily to observe that $\widehat{\Psi}(\xi,p)$ is boundary value of the function analytic in the left complex semi-plane and therefore satisfies our basic assumption (2.7). Having determined the function $\widehat{\Psi}(\xi,p)$from (2.8) we determine required function $\widehat{\widehat{u}}$

$$\widehat{\widehat{u}}=\frac{1}{K(p)+\xi}\left(\widehat{u}_0(p)+\widehat{\widehat{f}}(\xi,p)-Y^+U^+\right). \tag{2.19}$$

Via (2.1) we rewrite last formula in the form

$$\widehat{\widehat{u}}=-Y^-\widetilde{U}^-, \tag{2.20}$$

where

$$\widetilde{U}^-(\xi,z)=\frac{1}{2\pi i}\int\frac{1}{q-z}\frac{1}{Y^+}(\widehat{u}_0(q)+\widehat{\widehat{f}}(\xi,q)).$$

Note that the function $\widehat{\widehat{u}}$ is the limiting value of an analytic function in Re $z>0$. Note the fundamental importance of the proven fact, that the solution $\widehat{\widehat{u}}$ constitutes an analytic function in $\operatorname{Re} z > 0$ and, as a consequence, its inverse Laplace transform vanish for all $x < 0$. Taking inverse Laplace transform of (2.20) with respect to time and inverse Fourier transform with respect to space variables we obtain

$$u(t,x) = \int_0^\infty G(t,x,y)u_0(y)dy + \\ + \int_0^t d\tau \int_0^\infty G(t-\tau,x,y)f(\tau,y)d\tau, \tag{2.21}$$

where

$$G(t,x,y) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}Y^-(\xi,p)I^-(\xi,p,y),$$

$$I(\xi,z,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}\frac{1}{Y^+(\xi,q)}dq.$$

Using Sokhotzki-Plemelj formula and Cauchy Theorem we get

$$G(t,x,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}e^{-py}dp + F(t,x,y). \tag{2.22}$$

where

$$F(t,x,y) = -\frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}Y^-(p,\xi)I^+(p,\xi,y)dp.$$

Since by definition $Y^+(\xi,z)$ is analytic in $\operatorname{Re} z < 0$ via Cauchy Theorem we have

$$J^+(\xi,p,y) = \frac{1}{2\pi i}\lim_{z\to p,\operatorname{Re} z<0}\int_{-i\infty}^{i\infty}\frac{e^{qy}}{q-z}\frac{1}{Y^+(\xi,q)}dq \\ = \frac{e^y}{Y^+(\xi,p)}. \tag{2.23}$$

In another hand making the change of variable $q \to -q$ and using obvious identity $\frac{1}{Y^+(-q,\xi)} = Y^-(q,\xi)$ we obtain

$$\int_{-i\infty}^{i\infty}\frac{e^{qy}}{q-z}\frac{1}{Y^+(\xi,q)}dq \\ = -\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q+z}\frac{1}{Y^+(\xi,-q)}dq \\ = -\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q+z}Y^-(\xi,q)dq. \tag{2.24}$$

Via estimates (2.23) and (2.24) we rewrite the function $F(t,x,y)$ in the form

$$F(t,x,y) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x+y)-K(p)t}e^{-py}dp + \widetilde{F}(t,x,y), \tag{2.25}$$

where

$$\widetilde{F}(t,x,y) = \frac{1}{2\pi i}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\int_{-i\infty}^{i\infty} e^{px}Y^-(\xi,p)\widetilde{I}^+(\xi,p,y)dp,$$

$$\begin{aligned}\widetilde{I}(z,\xi,y) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{-qy}}{q-z}\frac{1}{Y^+(\xi,q)}\left(1-Y^-(\xi,q)Y^+(\xi,q)\frac{q-z}{q+z}\right)dq\\ &= O(z^{-1}).\end{aligned}$$

Now we prove that $\widetilde{F}(t,x,y)$ is identically zero. Indeed, taking residue in point $p = -\phi_j(\xi)$ we get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}Y^-(\xi,p)\widetilde{I}^+(\xi,p,y)dp \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}Y^+(\xi,p)\frac{1}{K(p)+\xi}\widetilde{I}^+(\xi,p,y)dp\\ &= \sum_{j=1}^{n} e^{-\phi_j(\xi)x}Y^+(\xi,-\phi_j(\xi))\frac{1}{K'(-\phi_j)}\widetilde{I}^+(\xi,-\phi_j,y).\end{aligned} \tag{2.26}$$

Since by the definition $K(-\phi_l) = K(\phi_l) = -\xi$ we get

$$\frac{1}{K'(-\phi_l)} = -\phi_l'(\xi).$$

Substituting formula (2.26) into definition of $\widetilde{F}(t,x,y)$ and changing variable $-\phi_j(\xi) = p$ we obtain

$$\begin{aligned}\widetilde{F}(t,x,y) &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{-i\infty}^{i\infty} d\xi\,\phi_j'(\xi)e^{\xi t}e^{-\phi_j(\xi)x}Y^+(\xi,-\phi_j(\xi))\widetilde{I}^+(\xi,-\phi_j,y)\\ &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\Gamma_j} dp e^{-K(p)t}e^{px}Y^+(-K(p),p)\widetilde{I}^+(-K(p),p,y),\end{aligned}$$

where for $\xi \in (-i\infty, i\infty)$

$$\Gamma_j = \left\{z = -\phi_j(\xi) \in \mathbb{C},\ \operatorname{Re} z < 0, \operatorname{Re} K(z) = 0\right\}.$$

Note that by definition

$$Y^+(-K(p),p) = 0$$

and therefore

$$\widetilde{F}(t,x,y) = 0. \tag{2.27}$$

Thus from (2.21) via (2.22) and (2.25) we have for solution $u(t,x)$

$$u(t,x) = \mathscr{G}(t)u_0 + \int_0^t \mathscr{G}(t-\tau)f(\tau)d\tau, \tag{2.28}$$

where the Green operator $\mathcal{G}(t)$ is given by (2.3). Proposition is proved.

We introduce operator $\mathcal{G}_0(t)$ is given by

$$\mathcal{G}_0(t)\phi = \int_0^{+\infty} G_1(t,x,y)\phi(y)\,dy,$$

where the kernel

$$G_1(t,x,y) = t^{-\frac{1}{2n}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{(-1)^{n+1}z^{2n}} (e^{z(x-y)t^{-\frac{1}{2n}}} - e^{z(x+y)t^{-\frac{1}{2n}}})dz.$$

Denote

$$G_0(t,x) = \partial_y G_1(t,x,y)\big|_{y=0} = t^{-\frac{1}{n}} \widetilde{G}(xt^{-\frac{1}{2n}}), \tag{2.29}$$

where

$$\widetilde{G}(x) = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} e^{(-1)^{n+1}z^{2n} + zxt^{-\frac{1}{2n}}} z\,dz.$$

From [7] we easily get the following

LEMMA 1. *Let* $\phi \in \mathbf{L}^r$

$$\|\mathcal{G}_0(t)\phi\|_{\mathbf{L}^q} \leqslant C\langle t\rangle^{\frac{1}{2n}\left(\frac{1}{q}-\frac{1}{r}\right)} \|\phi\|_{\mathbf{L}^r},$$

is true for all $t > 0,\ 1 \leqslant q \leqslant \infty, 1 \leqslant r \leqslant \infty$. *Furthermore we assume that* $\phi \in \mathbf{L}^{1,1+a}$, *then the estimate*

$$\left\|(\cdot)^b (\mathcal{G}_0(t)\phi - \vartheta G_0(t))\right\|_{\mathbf{L}^q} \leqslant Ct^{-\frac{1}{n}+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,1+a}}$$

is valid for all $t > 0$, *where* $1 \leqslant q \leqslant \infty,\ b \in [0, 1+a]$ *and*

$$\vartheta = \int_0^{+\infty} x\phi(x)\,dx.$$

We now collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ defined by (2.3), in the norms $\|\phi\|_{\mathbf{L}^r}$ and $\|\phi\|_{\mathbf{L}^{1,1+w}}$, where $w \in (0,1),\ 1 \leqslant r \leqslant \infty$.

LEMMA 2. *Suppose that the function* $\phi \in \mathbf{L}^\infty(\mathbb{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbb{R}^+)$, *where* $a \in (0,1)$. *Then the estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^r} \leqslant C\langle t\rangle^{-\frac{1}{2n}\left(\frac{1}{r_1}-\frac{1}{r}\right)} \|\phi\|_{\mathbf{L}^{r_1}} + e^{-t}\|\phi\|_{\mathbf{L}^r},$$

$$\|\mathcal{G}(t)\phi - \vartheta G_0(t)\|_{\mathbf{L}^\infty} \leqslant Ct^{-\frac{1}{n}-\frac{a}{2n}} \|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-t}\|\phi\|_{\mathbf{L}^\infty},$$

and

$$\left\|(\cdot)^b (\mathcal{G}(t)\phi - \vartheta G_0)\right\|_{\mathbf{L}^1} \leqslant Ct^{-\frac{1}{2n}+\frac{b-a}{2n}} \|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-t}\|\phi\|_{\mathbf{L}^1}$$

are valid for all $t > 0$, *where* $1 \leqslant r \leqslant r_1 \leqslant \infty, 0 < b \leqslant a$.

Proof. Note that the Green operator $\mathscr{G}(t)$ can be represented as

$$\mathscr{G}(t)\phi = \mathscr{G}_0(t)\phi + e^{-t}\phi + \mathscr{R}_1(t)\phi, \tag{2.30}$$

where the remainder

$$\mathscr{R}_1(t)\phi = \int_0^{+\infty}(R_1(t,x-y) - R_1(t,x+y))\phi(y)\,dy$$

with a kernel

$$R_1(t,x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}\widehat{R}_1(t,p)\,dp,$$

where

$$\widehat{R}_1(t,p) = e^{-K(p)t} - e^{Ctp^{2n}} - e^{-t},$$

where $C = (-1)^{n+1}$. From Lemma 1 the operator $\mathscr{G}_0(t)$ satisfies the estimates of the Lemma .

Now we estimate the remainder $\mathscr{R}_1(t)$. We represent

$$\widehat{R}_1(t,p) = e^{-K(p)t}\left(1 - e^{-tCp^{2n}+K(p)t}\right) - e^{-t}$$

for all $|p| \leqslant 1$, and

$$\widehat{R}_1(t,p) = -e^{-Ctp^{2n}} + e^{-t}\left(e^{(1-K(p))t} - 1\right)$$

for all $|p| \geqslant 1$, then we see that

$$\left|\partial_p^j \widehat{R}_1(t,p)\right| \leqslant C\langle t\rangle^{\frac{j}{2}-1} e^{\frac{1}{2}Ctp^{2n}} + C\langle t\rangle^{1+\frac{1}{n}} e^{-t}\left(1-p^{2n}\right)^{-3}$$

for all $\operatorname{Re} p = 0,\ t > 0,\ 0 \leqslant j \leqslant 4.$ Therefore we have

$$\begin{aligned}|R_1(t,x)| &\leqslant C\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-4}\langle t\rangle^{-\frac{1}{2n}-1} + C\langle x\rangle^{-4}\langle t\rangle^{1+\frac{1}{n}} e^{-t} \\ &\leqslant C\left\langle x\langle t\rangle^{-\frac{1}{2n}}\right\rangle^{-4}\langle t\rangle^{-\frac{1}{2n}-1}\end{aligned}$$

for all $x \in \mathbb{R},\ t > 0.$ Applying this estimate by the Young inequality we find

$$\|\mathscr{R}_1(t)\phi\|_{\mathbf{L}^r} \leqslant C\langle t\rangle^{-\frac{1}{2n}\left(\frac{1}{q}-\frac{1}{r}\right)-1}\|\phi\|_{\mathbf{L}^q}$$

for all $1 \leqslant q \leqslant r \leqslant \infty$ and

$$\|\mathscr{R}_1(t)\phi\|_{\mathbf{L}^{1,w}} \leqslant C\langle t\rangle^{-\frac{1}{n}}\left(\langle t\rangle^{\frac{w}{2n}}\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,w}}\right)$$

for all $t > 0.$ Now by representation (2.30) the estimates of the lemma follow. Lemma 2 is proved.

3. Proof of Theorem 1

We rewrite the initial-boundary value problem (1.1) as the following integral equation

$$u(t) = \mathscr{G}(t)\,u_0 - \int_0^t \mathscr{G}(t-\tau)\,\mathscr{N}(u(\tau))\,d\tau, \tag{3.1}$$

where the Green operator $\mathscr{G}$ of the corresponding linear problem and $\mathscr{N}(u) = \lambda\,|u|^{\sigma}\,u$.

By standard method we can prove the local existence of weak solutions to the initial boundary-value problem (1.1) (see, for example, [7])

THEOREM 3. *Let $u_0 \in \mathbf{L}^{1,1+a}(\mathbb{R}^+) \cap \mathbf{L}^{\infty}(\mathbb{R}^+)$, $a \geqslant 0$. Then for some $T > 0$ there exists an unique solution $u \in \mathbf{C}\left([0,T];\mathbf{L}^{1,1+a}(\mathbb{R}^+) \cap \mathbf{L}^{\infty}(\mathbb{R}^+)\right)$ to the problem (1.1).*

By Theorem 3, it follows that the global solution (if it exists) is unique. Indeed, suppose that there exist two global solutions with the same initial data, which are different at some time $t > 0$. By virtue of the continuity of solutions with respect to time, we can find a maximal time segment $[0,T]$, where the solutions are equal, but for $t > T$ they are different. Now we apply the local existence theorem taking the initial time T and obtain that these solutions coincide on some interval $[T,T_1]$, which gives us a contradiction with the fact that T is a maximal time until which the solutions coincide. So our main purpose in the proof of Theorem 1 is to show the global in time existence of solutions. Denote

$$\|g\|_{\mathbf{Z}} = \left(\|g(t)\|_{\mathbf{L}^{\infty}} + \|g(t)\|_{\mathbf{L}^{1,1+a}}\right),$$

and

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(\langle t\rangle^{\frac{1}{n}}\,\|\phi(t)\|_{\mathbf{L}^{\infty}} + \langle t\rangle^{-\frac{a}{2n}}\,\|\phi(t)\|_{\mathbf{L}^{1,1+a}}\right),$$

where $a \in (0,1)$. Note that the $\mathbf{L}^1$ - norm is estimated by the norm $\mathbf{X}$

$$\begin{aligned}\|\phi(t)\|_{\mathbf{L}^1} &= \int_0^{\langle t\rangle^{\frac{1}{n}}} |\phi(t,x)|\,dx + \int_{\langle t\rangle^{\frac{1}{n}}}^{+\infty} |1+x|^{-1-\alpha}\,|x|^{1+\alpha}\,|\phi(t,x)|\,dx \\ &\leqslant C\langle t\rangle^{\frac{1}{n}}\,\|\phi(t)\|_{\mathbf{L}^{\infty}} + C\langle t\rangle^{-\frac{a}{2n}}\,\|\phi(t)\|_{\mathbf{L}^{1,1+a}} \leqslant C\,\|\phi\|_{\mathbf{X}}.\end{aligned} \tag{3.2}$$

In the next lemma we estimate the Green operator in our basic norm X.

LEMMA 3. *Let the function $f(t,x)$ have a zero first moment $\int_0^{+\infty} x f(t,x)\,dx = 0$. Then the following inequality*

$$\left\|\int_0^t \mathscr{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{X}} \leqslant C\,\|\langle t\rangle\,f\|_{\mathbf{X}}$$

is valid , provided that the right-hand side is finite.

Proof. In view of Lemma 2 we get

$$\left\|\int_0^t \mathscr{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^{\infty}} + \left\|\int_0^t \mathscr{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^{1,1+a}}$$

$$\leqslant C \|\langle t \rangle f\|_{\mathbf{X}} \leqslant C \|\langle t \rangle f\|_{\mathbf{X}}$$

for all $0 \leqslant t \leqslant 1$. We now consider $t > 1$. By virtue of Lemma 2 we obtain

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} \leqslant C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{n}-\frac{a}{2n}} \left(\|f(\tau)\|_{\mathbf{L}^\infty} + \|f(\tau)\|_{\mathbf{L}^{1,1+a}} \right) d\tau + C \int_{\frac{t}{2}}^t \|f(\tau)\|_{\mathbf{L}^\infty}\, d\tau,$$

hence using the definition of the norm $\mathbf{X}$ we get

$$\begin{aligned} \left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} &\leqslant C \|\langle t \rangle f\|_{\mathbf{X}} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{n}-\frac{a}{2n}} \langle \tau \rangle^{\frac{a}{2n}-1}\, d\tau \\ &\quad + C \|\langle t \rangle f\|_{\mathbf{X}} \int_{\frac{t}{2}}^t \langle \tau \rangle^{-\frac{1}{n}-1}\, d\tau \\ &\leqslant C t^{-\frac{1}{n}} \|\langle t \rangle f\|_{\mathbf{X}} \end{aligned}$$

and similarly

$$\begin{aligned} &\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^{1,1+a}} \\ &\quad \leqslant C \int_0^t \|f(\tau)\|_{\mathbf{L}^{1,1+a}}\, d\tau \leqslant C \|\langle t \rangle f\|_{\mathbf{X}} \int_0^t \tau^{\frac{a}{2n}-1} d\tau \\ &\quad \leqslant C t^{\frac{a}{2n}} \|\langle t \rangle f\|_{\mathbf{X}} \end{aligned}$$

for all $t > 4$. Hence the result of the lemma follows. Lemma 3 is proved.

From Lemmas 1 and 2 we easily get the following estimates

$$\left\| G_0(\cdot t^{-\frac{1}{2}}) \right\|_{\mathbf{X}} \leqslant C, \tag{3.3}$$

$$\begin{aligned} &\left\| \langle t \rangle^{\frac{a}{2}} \left(\mathcal{G}_0(t)\phi - G_0(\cdot, t) \int_{R^+} x\phi(x) dx \right) \right\|_{\mathbf{X}} \\ &+ \left\| \langle t \rangle^{\frac{a}{2}} \left(\mathcal{G}(t) - \mathcal{G}_0(t) \right) \phi \right\|_{\mathbf{X}} \leqslant C \|\phi\|_{\mathbf{Z}}. \end{aligned} \tag{3.4}$$

Also by direct calculation we have for $\mu = 1 - \sigma \frac{1}{n}$

$$\left\| t^{1-\mu} x (\mathcal{N}(u_1) - \mathcal{N}(u_2)) \right\|_{\mathbf{X}} \leqslant C \|(u_1 - u_2)\|_{\mathbf{X}} \left(\|u_1\|_{\mathbf{X}}^\sigma + \|u_2\|_{\mathbf{X}}^\sigma \right). \tag{3.5}$$

We make a change of the dependent variable $u(t,x) = v(t,x) e^{-\varphi(t) + i\psi(t)}$ in equation (1.1). Then for the new function $v(t,x)$ we get the following equation

$$\partial_t v + e^{\varphi(t) - i\psi(t)} \mathcal{N}(v e^{-\varphi(t) + i\psi(t)}) + \mathbb{K}(v) - (\varphi' - i\psi')v = 0.$$

Since for any $z \in \mathbb{C}$

$$e^{z}\mathcal{N}(ve^{-z}) = e^{-\sigma \operatorname{Re} z}\mathcal{N}(v),$$

we have

$$e^{\varphi(t)-i\psi(t)}\mathcal{N}(ve^{-\varphi(t)+i\psi(t)}) = e^{-\sigma\varphi}\mathcal{N}(v).$$

We assume that

$$\int_{\mathbb{R}^+} x\left(e^{-\sigma\varphi}\mathcal{N}(v) - (\varphi' - i\psi')v\right)dx = 0.$$

Then since

$$\int_{\mathbb{R}^+} x\mathbb{K}v dx = \partial_p (K(p)\widehat{v}(t,p))\big|_{p=0} = 0$$

the first moment value of new function $v(t,x)$ satisfies a conservation law:

$$\frac{d}{dt}\int_{\mathbb{R}^+} xv(t,x)\,dx = 0,$$

hence $\int_{\mathbb{R}^+} xv(t,x)\,dx = \int_{\mathbb{R}^+} xv_0(t,x)\,dx$ for all $t > 0$. We can choose $\varphi(0) = 0$ and $\psi(0) = \arg\hat{u}_0(0)$ such that

$$\int_{\mathbb{R}^+} xv_0(t,x)\,dx = \left|\int_{\mathbb{R}^+} xu_0(t,x)\,dx\right| = \theta > 0.$$

Thus we consider the problem for the new dependent variables $(v(t,x), \varphi(t))$

$$\begin{cases} \partial_t v + \mathbb{K}v = -e^{-\sigma\varphi}\left(\mathcal{N}(v) - \frac{v}{\theta}\int_{\mathbb{R}^+} x\mathcal{N}(v)dx\right) \\ \partial_t \varphi(t) = \frac{1}{\theta}e^{-\sigma\varphi}\operatorname{Re}\int_{\mathbb{R}^+} x\mathcal{N}(v)dx, \\ v(0,x) = v_0(x),\ \varphi(0) = 0, \end{cases} \tag{3.6}$$

where

$$v_0(x) = u_0(x)\exp(-i\arg\hat{u}_0(0))$$

and

$$\psi(t) = \arg\hat{u}_0(0) - \frac{1}{\theta}\int_0^t e^{-\sigma\varphi}\operatorname{Im}\int_{\mathbb{R}^+} x\mathcal{N}(v)dx. \tag{3.7}$$

We denote $h(t) = e^{\sigma\varphi}$ and write (3.6) as

$$\begin{cases} \partial_t v + \mathbb{K}v = f(v,h),\ v(0,x) = v_0(x), \\ \partial_t h = \frac{\sigma}{\theta}\operatorname{Re}\int_{\mathbb{R}^+} x\mathcal{N}(v)dx,\ h(0) = 1, \end{cases} \tag{3.8}$$

where

$$f(v,h) = -h^{-1}(t)\left(\mathcal{N}(v) - \frac{v}{\theta}\int_{\mathbb{R}^+} x\mathcal{N}(v)dx\right).$$

We note that the first moment value of the nonlinearity $f(v,h) = 0$ for all $t > 0$. We now prove the existence of the solution $(v(t,x), h(t))$ for the problem (3.8) by the successive approximations $(v_m(t,x), h_m(t))$, $m = 1,2,...,$ defined as follows

$$\begin{cases} \partial_t v_m + \mathbb{K}v_m = f(v_{m-1}, h_{m-1}),\ v_m(0,x) = v_0(x), \\ h_m = 1 + \frac{\sigma}{\theta}\int_0^t d\tau \operatorname{Re}\int_{\mathbb{R}^+} x\mathcal{N}(v_m)dx \end{cases} \tag{3.9}$$

for all $m \geqslant 2$, where

$$v_1 = \mathscr{G}(t)\, v_0.$$

Denote by

$$\delta = \frac{\sigma\theta^{\sigma}}{\mu}\eta, \tag{3.10}$$

where

$$\eta = \operatorname{Re}\lambda \int_{\mathbb{R}^+} x\mathscr{N}(\widetilde{G}(x))dx > 0, \mu = 1 - \frac{\sigma}{n}.$$

Here $\widetilde{G}(x)$ is defined as

$$\widetilde{G}(x) = \frac{1}{\pi i}\int_{-i\infty}^{i\infty} e^{(-1)^{n+1}z^{2n}+zx} z dz.$$

Note that by condition of Theorem

$$n(1-\varepsilon^{n+1}\eta) < \sigma < n$$

and therefore

$$\delta > \varepsilon^{-1}. \tag{3.11}$$

We now prove by induction the following estimates for all $m \geqslant 1, t > 0$

$$\begin{aligned} &\|v_m\|_{\mathbf{X}} \leqslant C\varepsilon,\ \|v_m(t) - \mathscr{G}(t)\, v_0\|_{\mathbf{X}} \leqslant C\varepsilon^{\sigma+1}, \\ &|h_m(t)| > C\delta t^{\mu} \end{aligned} \tag{3.12}$$

and

$$\int_{\mathbb{R}^+} x v_m dx = \theta, \int_{\mathbb{R}^+} x f_m dx = 0. \tag{3.13}$$

Via (3.3) and (3.4) , we have

$$\begin{aligned} &\|\mathscr{G}(t)\, v_0\|_{\mathbf{X}} \leqslant C\varepsilon, \\ &\|v_1 - \mathscr{G}(t)\, v_0\|_{\mathbf{X}} \leqslant C\varepsilon^{\sigma+1}. \end{aligned}$$

Also using (3.10) we get

$$\begin{aligned} h_1(t) &= 1 + \sigma\theta^{\sigma}\int_0^t d\tau \operatorname{Re}\int_{\mathbb{R}^+} x\mathscr{N}(G_0(x,t))dx \\ &= 1 + \sigma\theta^{\sigma}\eta\int_0^t \tau^{-\frac{\sigma}{n}} d\tau \\ &= 1 + t^{\mu}\delta. \end{aligned}$$

Therefore estimates (3.12) are valid for $m = 1$. Also since

$$\int_{\mathbb{R}^+} x\mathscr{G} v_0 dx = \int_{\mathbb{R}^+} x v_0 dx$$

we easily obtain (3.13) is valid for $m=1$. Now we assume that estimates (3.12)-(3.13) are true with m replaced by $m-1$. The integral equation associated with (3.9) is written as

$$\begin{cases} v_m(t) = \mathcal{G}(t)v_0 + \int_0^t \mathcal{G}(t-\tau) f(v_{m-1}(\tau), h_{m-1}(\tau))\, d\tau, \\ h_m(t) = 1 + \frac{\sigma}{\theta}\int_0^t d\tau \operatorname{Re}\int_{\mathbb{R}^+} x\mathcal{N}(v_m)dx. \end{cases} \tag{3.14}$$

We have

$$\left\| t^{1-\mu}\mathcal{N}(u)\right\|_{\mathbf{X}} \leqslant C\|u\|_{\mathbf{X}}^{\sigma+1}$$

and

$$\left\| t^{1-\mu} u \int_{\mathbb{R}^+} x\mathcal{N}(u)dx \right\|_{\mathbf{X}} \leqslant \left\| t^{1-\mu} x\mathcal{N}(u)\right\|_{\mathbf{L}^1} \|u\|_{\mathbf{X}} \leqslant C\|u\|_{\mathbf{X}}^{\sigma+2}$$

Therefore due to (3.12) and (3.11) we get

$$\begin{aligned} &\|tf(v_{m-1}(\tau), h_{m-1}(\tau))\|_{\mathbf{X}} \\ &\leqslant \frac{1}{|h_{m-1}|} t^{\mu}\left(\varepsilon^{\sigma+1} + \frac{1}{\theta}\varepsilon^{\sigma+2}\right) \leqslant \frac{\varepsilon}{t^{\mu}} t^{\mu}\left(\varepsilon^{\sigma+1} + \frac{1}{\theta}\varepsilon^{\sigma+2}\right) \leqslant \varepsilon^{\sigma+1} \end{aligned}$$

Since $f(v_{m-1}(\tau), h_{m-1}(\tau))$ have the zero the first moment value, from Lemma 3 we get

$$\begin{aligned} \left\| \int_0^t \mathcal{G}(t-\tau) f(v_{m-1}(\tau), h_{m-1}(\tau))\, d\tau \right\|_{\mathbf{X}} &\leqslant \|tf(v_{m-1}(\tau), h_{m-1}(\tau))\|_{\mathbf{X}} \leqslant C\varepsilon^{\sigma+1} \\ &\leqslant C\varepsilon. \end{aligned}$$

It follows that

$$\|v_m\|_{\mathbf{X}} \leqslant C\varepsilon, \|v_m(t) - \mathcal{G}(t)v_0\|_{\mathbf{X}} \leqslant C\varepsilon^{\sigma+1} \tag{3.15}$$

We have

$$h_m(t) = \delta t^{\mu} + R_m,$$

where

$$\begin{aligned} R_m = \frac{\sigma}{\theta}\int_0^t d\tau x(\mathcal{N}(v_m) - \mathcal{N}(\mathcal{G}(\tau)v_0)) \\ + \frac{\sigma}{\theta}\int_0^t d\tau x(\mathcal{N}(\mathcal{G}(\tau)v_0) - \mathcal{N}(\mathcal{G}_0(\tau)v_0)) \\ + \frac{\sigma}{\theta}\int_0^t d\tau x(\mathcal{N}(\mathcal{G}(\tau)v_0) - \mathcal{N}(\theta G_0(\tau,x))). \end{aligned}$$

Via (3.2) and (3.5) we obtain

$$\begin{aligned} &\left| \int_0^t d\tau \int_{\mathbb{R}^+} x(\mathcal{N}(u_1) - \mathcal{N}(u_2))\, dx \right| \\ &\leqslant \left\| \langle t\rangle^{\gamma}(u_1-u_2)\right\|_{\mathbf{X}} (\|u_1\|_{\mathbf{X}}^{\sigma} + \|u_2\|_{\mathbf{X}}^{\sigma}) \int_0^t \tau^{-1+\mu}\langle\tau\rangle^{-\gamma} \\ &\leqslant \left\| \langle t\rangle^{\gamma}(u_1-u_2)\right\|_{\mathbf{X}} (\|u_1\|_{\mathbf{X}}^{\sigma} + \|u_2\|_{\mathbf{X}}^{\sigma}) \frac{1}{\mu-\gamma} t^{\mu-\gamma}. \end{aligned} \tag{3.16}$$

Therefore using (3.15) and $\left\|\langle t\rangle^{\gamma_1}\left(\mathscr{G}(\tau)v_0-\mathscr{G}_0(\tau)v_0\right)\right\|_{\mathbf{X}}\leqslant C\varepsilon$ we get from (3.16)

$$|R_m|<C\varepsilon^{\sigma+1}\langle t\rangle^{\mu}.$$

Hence

$$|h_m(t)|>C\delta t^{\mu}$$

for all $t>1$. Also integrating $v_m=\mathscr{M}(v_{m-1})$ with respect to $x\in\mathbb{R}^+$ we get

$$\begin{aligned}\int_{\mathbb{R}^+}xv_m(t,x)\,dx&=\int_{\mathbb{R}^+}x\mathscr{G}(t)v_0dx-\int_{\mathbb{R}^+}xdx\int_0^t\mathscr{G}(t-\tau)f_{m-1}(\tau)\,d\tau\\&=\theta.\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^+}xf_m(t,x)\,dx=\int_{\mathbb{R}^+}\left(x\mathscr{N}(v_m)-\frac{v_m(t,x)}{\theta}\int_{\mathbb{R}^+}x\mathscr{N}(v_m)dx\right)dx=0.$$

Thus by induction we see that estimates (3.12)-(3.13) are valid for all $m\geqslant 1$. In the same way by induction we can prove that

$$\|v_m-v_{m-1}\|_{\mathbf{X}}\leqslant\frac{1}{4}\|v_{m-1}-v_{m-2}\|_{\mathbf{X}},$$

and

$$\begin{aligned}\sup_{t>0}\delta t^{\mu}|h_m(t)-h_{m-1}(t)|\leqslant\frac{1}{4}\|v_{m-1}-v_{m-2}\|_{\mathbf{X}}\\+\frac{1}{4}\sup_{t>0}\delta t^{\mu}|h_{m-1}(t)-h_{m-2}(t)|\end{aligned}$$

for all $m>2$. Therefore taking the limit $m\to\infty$, we obtain a unique solution

$$\begin{aligned}&\lim_{m\to\infty}v_m(t,x)=v(t,x)\in\mathbf{X},\\&\lim_{m\to\infty}h_m(t)=h(t)=e^{(\sigma-1)\varphi(t)}\in\mathbf{C}(0,\infty)\end{aligned}$$

satisfying equations

$$\begin{cases}v(t)=\mathscr{G}(t)v_0+\int_0^t\mathscr{G}(t-\tau)f(v(\tau),h(\tau))\,d\tau,\\ h(t)=1+\frac{\sigma}{\theta}\int_0^t d\tau\,\mathrm{Re}\int_{\mathbb{R}^+}x\mathscr{N}(v)dx,\end{cases}$$

and estimates for $t>1$

$$\begin{aligned}\|v(t)-\mathscr{G}(t)v_0\|_{\mathbf{X}}\leqslant C\varepsilon^{\sigma+1},\|v(t)\|_{\mathbf{X}}\leqslant C\varepsilon\\|h(t)|>C\delta t^{\mu},\delta>\varepsilon^{-1}.\end{aligned}\qquad(3.17)$$

We now compute the asymptotics of the solution. First we show the existence of solutions to the integral equation

$$V(\xi) = V_0(\xi) - \frac{1}{\beta}\int_0^1 \frac{dz}{z^{1+\frac{1}{2n}}(1-z)^{\frac{1}{n}}}\int_{\mathbb{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2n}}}, \frac{yz^{\frac{1}{2n}}}{(1-z)^{\frac{1}{2n}}}\right) F(y)\,dy, \quad (3.18)$$

where $V_0(\xi) = G_0(\xi)$, and

$$F(y) = \mathscr{N}(V(y)) - V(y)\int_{\mathbb{R}^+} \xi \mathscr{N}(V(\xi))d\xi,$$
$$\beta = \frac{\sigma}{\mu}\int_{\mathbb{R}^+} \xi \mathscr{N}(V(\xi))d\xi.$$

We define successive approximations $V_{k+1} = \mathscr{R}(V_k)$ for $k = 0, 1, 2, \ldots$, where

$$\mathscr{R}(V_k)(\xi) = V_0(\xi) - \frac{1}{\beta_k}\int_0^1 \frac{dz}{z^{1+\frac{1}{2n}}(1-z)^{\frac{1}{n}}}\int_{\mathbb{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2n}}}, \frac{yz^{\frac{1}{2n}}}{(1-z)^{\frac{1}{2n}}}\right) F_k(y)\,dy,$$

$$F_k(y) = \mathscr{N}(V_k(y)) - V_k(y)\int_{\mathbb{R}^+} \xi \mathscr{N}(V_k(\xi))d\xi \text{ and}$$
$$\beta_k = \frac{\sigma}{\mu}\operatorname{Re}\int_{\mathbb{R}^+} \xi \mathscr{N}(V_k(\xi))d\xi\ .$$

By induction we prove the estimates

$$\|V_k - V_0\|_{\mathbf{Z}} \leqslant C\varepsilon,\ \|V_k\|_{\mathbf{Z}} \leqslant C,\ \beta_k \geqslant C\varepsilon^{-1},$$

and

$$\|V_{k+1} - V_k\|_{\mathbf{Z}} \leqslant \frac{1}{2}\|V_k - V_{k-1}\|_{\mathbf{Z}} \quad (3.19)$$

for all $k \geqslant 1$. Firstly we have to show that

$$\int_{\mathbb{R}^+} yF_k(y)\,dy = 0 \text{ and } \int_{\mathbb{R}^+} yV_k(y)\,dy = 1. \quad (3.20)$$

Since $\int_{\mathbb{R}^+} yV_0(y)\,dy = 1$ by the definition of $F_k(y)$, we see that (3.20) is true for $k = 0$. We assume that (3.20) holds for some k. Then we have

$$\int_{\mathbb{R}^+} \xi V_{k+1}(\xi)\,d\xi = 1 - \frac{1}{\beta_k}\int_0^1 \frac{dz}{z^{1+\frac{1}{2n}}(1-z)^{\frac{1}{2n}}}\int_{\mathbb{R}^+} \xi\,d\xi$$
$$\times \int_{\mathbb{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2n}}}, \frac{yz^{\frac{1}{2n}}}{(1-z)^{\frac{1}{2n}}}\right) F_k(y)\,dy = 0$$

hence it follows that $\int_{\mathbb{R}^+} F_{k+1}(y)\,dy = 0$. Thus we get (3.20) for any k. Estimates (3.19)-(3.20) are valid for $k = 0$. Changing $\tau = tz$ and $y_1 = \tau^{\frac{1}{2n}} y$ and using

$$\left\| t^{-1} V((\cdot)t^{\frac{1}{2n}}) \right\|_{\mathbf{X}} = \|V(\cdot)\|_{\mathbf{Z}}$$

we get

$$\begin{aligned}
&\|V_{k+1}-V_0\|_{\mathbf{Z}} \\
&\quad = \frac{C}{\beta_k}\left\|\int_0^1 \frac{dz}{z^{1+\frac{1}{2n}}(1-z)^{\frac{1}{n}}}\int_{\mathbb{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2n}}},\frac{yz^{\frac{1}{2n}}}{(1-z)^{\frac{1}{2n}}}\right)F_k(y)\,dy\right\|_{\mathbf{Z}} \\
&\quad \leqslant \frac{C}{\beta_k}\left\|\int_0^t \tau^{-\mu}\mathscr{G}_0(t-\tau)\,\tau^{-\frac{\sigma}{n}}F_k\left(\cdot\tau^{-\frac{1}{2n}}\right)d\tau\right\|_{\mathbf{X}}.
\end{aligned}$$

Since

$$\left\|\int_0^t \tau^{-\mu}\mathscr{G}_0(t-\tau)\,\tau^{-\frac{\sigma}{n}}F_k\left(\cdot\tau^{-\frac{1}{2n}}\right)d\tau\right\|_{\mathbf{X}} \leqslant C\left\|t^{1-\mu-\frac{\sigma}{n}}F_k(\cdot t^{-\frac{1}{2n}})\right\|_{\mathbf{X}}$$

and

$$\begin{aligned}
\left\|t^{1-\mu-\frac{\sigma}{n}}F_k(\cdot t^{-\frac{1}{2n}})\right\|_{\mathbf{X}} &\leqslant C\left\|t^{-\frac{1}{n}}V_k(\cdot t^{-\frac{1}{2n}})\right\|_{\mathbf{X}}^{\sigma+1} \\
&\leqslant C\,\|V_k(\cdot)\|_{\mathbf{Z}}^{\sigma+1}
\end{aligned}$$

we get

$$\|V_{k+1}-V_0\|_{\mathbf{Z}} \leqslant C\varepsilon.$$

By condition of Theorem $n(1-\varepsilon^{n+1}\eta)<\sigma<n$ and therefore $\mu=(1-\frac{\sigma}{n})>\varepsilon^{\sigma+1}$. Thus we have

$$\begin{aligned}
\beta_k &= \frac{\sigma}{\mu}\operatorname{Re}\int_{\mathbb{R}^+}\xi\mathscr{N}(V_k(\xi))d\xi \\
&= \frac{\sigma}{\mu}\operatorname{Re}\int_{\mathbb{R}^+}\xi\mathscr{N}(G_0(\xi))d\xi+\frac{\sigma}{\mu}O(\varepsilon^{\sigma})>\varepsilon^{-1}.
\end{aligned}$$

Therefore (3.19)-(3.20) are true for any k. Using (3.5) we have

$$\begin{aligned}
&\left\|t^{1-\mu}x(\mathscr{N}(u_1)-\mathscr{N}(u_2))\right\|_{\mathbf{X}} \\
&\qquad+\left\|t^{1-\mu}\left(u_1\int_{\mathbb{R}^+}x\mathscr{N}(u_1)dx-u_2\int_{\mathbb{R}^+}x\mathscr{N}(u_2)dx\right)\right\|_{\mathbf{X}} \\
&\qquad\qquad\leqslant C\,\|(u_1-u_2)\|_{\mathbf{X}}\,(\|u_1\|_{\mathbf{X}}^{\sigma}+\|u_2\|_{\mathbf{X}}^{\sigma}).
\end{aligned} \tag{3.21}$$

Thus in the same manner we obtain

$$\|V_{k+1}-V_k\|_{\mathbf{Z}} \leqslant \frac{1}{2}\,\|V_k(\cdot)-V_{k-1}(\cdot)\|_{\mathbf{Z}}$$

and therefore estimate (3.19) is valid for any $k\geqslant 1$. Hence $\mathscr{R}$ is a contraction mapping and there exists a unique solution $V(\xi)$ to integral equation (3.18).

We are now in a position to prove asymptotics of solutions v. We prove by induction

$$\left\|\langle t\rangle^{\gamma}\left(v_k(t)-t^{-\frac{1}{n}}\theta V_k\left((\cdot)t^{-\frac{1}{2n}}\right)\right)\right\|_{\mathbf{X}}<C\varepsilon, \tag{3.22}$$

where $\gamma > 0$ is small. The estimate (3.22) is true for $k = 0$ since

$$\left\| \langle t \rangle^{\gamma} \left(v_0 (t) - t^{-\frac{1}{n}} \theta V_0 \left((\cdot) t^{-\frac{1}{2n}} \right) \right) \right\|_{\mathbf{X}} = \left\| \langle t \rangle^{\gamma} (\mathscr{G}(t) v_0 - \theta G_0 (t)) \right\|_{\mathbf{X}} \leqslant C\varepsilon. \tag{3.23}$$

We assume that (3.22) is valid for some k. Due to (3.16) we have

$$\left| \int_0^t d\tau \int_{\mathbb{R}^+} x (\mathscr{N}(u_1) - \mathscr{N}(u_2))\, dx \right| \leqslant \frac{C}{\mu} t^{\mu} \left\| (u_1 - u_2) \right\|_{\mathbf{X}} (\|u_1\|_{\mathbf{X}}^{\sigma} + \|u_2\|_{\mathbf{X}}^{\sigma})$$

and therefore for $t > 0, \mu > \varepsilon^{\sigma+1}$

$$\begin{aligned} &\int_0^t d\tau \int_{\mathbb{R}^+} x \left| \mathscr{N}(v_k) - \mathscr{N}\left(t^{-\frac{1}{n}} \theta V_k \left(x t^{-\frac{1}{2n}} \right) \right) \right|_{\mathbf{L}^1} \\ &\leqslant \frac{C}{\mu} t^{\mu-\gamma} \left(\|v_k\|_{\mathbf{X}} + \theta \tau^{-\frac{1}{n}} \|V_k\|_{\mathbf{X}} \right)^{\sigma} \left\| \langle t \rangle^{\gamma} \left(v_k (\tau, \cdot) - \theta \tau^{-\frac{1}{n}} V_k \left(\cdot \tau^{-\frac{1}{2n}} \right) \right) \right\|_{\mathbf{X}} \\ &\leqslant C\varepsilon^{\sigma+1} t^{\mu-\gamma}. \end{aligned}$$

Then it follows that for $t > 1$

$$\begin{aligned} |h_k (t) - \theta^{\sigma} \beta_k t^{\mu}| &= \left| 1 + \frac{\sigma}{\theta} \operatorname{Re} \int_0^t \int_{\mathbb{R}^+} x \mathscr{N}(v_k) dx d\tau \right. \\ &\qquad \left. - t^{\mu} \frac{\sigma \theta^{\sigma+1}}{\mu} \operatorname{Re} \int_{\mathbb{R}^+} \xi \mathscr{N}(V_k (\xi)) d\xi \right| \\ &= \left| 1 + \frac{\sigma}{\theta} \operatorname{Re} \int_0^t \Big(\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} x \mathscr{N}(v_k) dx d\tau \right. \\ &\qquad \left. - \theta^{\sigma+1} \tau^{-\frac{1}{n}+\mu} \int_{\mathbb{R}^+} \xi \mathscr{N}(V_k(\xi)) d\xi \Big) d\tau \right| \\ &\leqslant 1 + \frac{C}{\theta} \int_0^t \int_{\mathbb{R}^n} x \left| \mathscr{N}(v_k) - \mathscr{N}\left(\theta \tau^{-\frac{1}{n}} V_k \left(x t^{-\frac{1}{2n}} \right) \right) \right| dx d\tau \\ &\leqslant 1 + C\varepsilon^{\sigma} t^{\mu-\gamma}. \end{aligned} \tag{3.24}$$

Changing variables such that $\tau = zt$ and $\xi \tau^{-\frac{1}{2n}} = y$ we have

$$\begin{aligned} &\frac{1}{\beta_k} \int_0^t \tau^{\sigma-1} \mathscr{G}_0 (t - \tau) \tau^{-(\sigma-1)} F_k \left(\cdot \tau^{-\frac{1}{2n}} \right) d\tau \\ &= \frac{t^{-\frac{1}{n}}}{\beta_k} \int_0^1 \frac{dz}{z^{1+\frac{1}{2n}} (1-z)^{\frac{1}{2n}}} \int_{\mathbb{R}^+} dy G_1 \left(\frac{(\cdot)}{(1-z)^{\frac{1}{2n}}}, \frac{y z^{\frac{1}{2n}}}{(1-z)^{\frac{1}{2n}}} \right) F_k (y) dy \\ &= t^{-\frac{1}{n}} \left(V_0 \left(x t^{-\frac{1}{2n}} \right) - V_{k+1} \left(x t^{-\frac{1}{2n}} \right) \right). \end{aligned}$$

Therefore we obtain

$$\left\| \langle t \rangle^{\gamma} \left(\theta t^{-\frac{1}{n}} V_{k+1} \left(\cdot t^{-\frac{1}{2n}} \right) - v_{k+1} (t) \right) \right\|_{\mathbf{X}}$$

$$
\begin{aligned}
&\leqslant C\left\|\langle t\rangle^{\gamma}\left(\theta t^{-\frac{1}{n}}V_{k+1}\left(\cdot t^{-\frac{1}{2n}}\right)-\mathscr{G}_0(t)v_0+\int_0^t h_k^{-1}(\tau)\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right)\right\|_{\mathbf{X}}\\
&\quad+C\left\|\langle t\rangle^{\gamma}\int_0^t h_k^{-1}(\tau)\left(\mathscr{G}(t-\tau)-\mathscr{G}_0(t-\tau)\right)f_k(\tau)d\tau\right\|_{\mathbf{X}}\\
&\quad+C\left\|\langle t\rangle^{\gamma}\left(\mathscr{G}(t)v_0-\mathscr{G}_0(t)v_0\right)\right\|_{\mathbf{X}}\\
&\leqslant C\left\|\langle t\rangle^{\gamma}\left(\theta t^{-\frac{1}{n}}V_0\left(\cdot t^{-\frac{1}{2n}}\right)-\mathscr{G}_0(t)v_0\right)\right\|_{\mathbf{X}}\\
&\quad+C\left\|\langle t\rangle^{\gamma}\int_0^t\left(h_k^{-1}(\tau)-\frac{\theta^{-\sigma}}{\beta_k}\tau^{-\mu}\right)\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right\|_{\mathbf{X}}\\
&\quad+\frac{C}{\beta_k\theta^{\sigma-1}}\left\|\langle t\rangle^{\gamma}\int_0^t\mathscr{G}_0(t-\tau)\left(f_k(\tau)-\frac{\theta^{\sigma+1}}{\tau^{(\sigma+1)}}F_k\left(\cdot\tau^{-\frac{1}{2n}}\right)\right)\tau^{-\mu}d\tau\right\|_{\mathbf{X}}\\
&\quad+C\left\|\langle t\rangle^{\gamma}\left(\mathscr{G}(t)v_0-\mathscr{G}_0(t)v_0\right)\right\|_{\mathbf{X}}\\
&\quad+C\left\|\langle t\rangle^{\gamma}\int_0^t h_k^{-1}(\tau)\left(\mathscr{G}(t-\tau)-\mathscr{G}_0(t-\tau)\right)f_k(\tau)d\tau\right\|_{\mathbf{X}}\\
&\equiv I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

From (3.23) we have $I_1\leqslant C\varepsilon$. We rewrite

$$
\begin{aligned}
I_2&=C\left\|\langle t\rangle^{\gamma}\int_0^t\left(h_k^{-1}(\tau)-\frac{\theta^{-\sigma+1}}{\beta_k}\tau^{-\mu}\right)\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right\|_{\mathbf{X}}\\
&\leqslant C\left\|\langle t\rangle^{\gamma}\int_0^t\tau^{-\mu}\left|\tau^{\mu}-\frac{\theta^{-\sigma+1}}{\beta_k}h_k(\tau)\right|h_k^{-1}(\tau)\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right\|_{\mathbf{X}}.
\end{aligned}
$$

Using (3.24) we get

$$
\frac{\theta^{-\sigma+1}}{\beta_k}h_k(\tau)=t^{\mu}+\frac{\theta^{-\sigma+1}}{\beta_k}O(1+C\varepsilon^{\sigma-1}t^{\mu-\gamma})
$$

and therefore we obtain

$$
\begin{aligned}
I_2&\leqslant\left\|\langle t\rangle^{\gamma}\int_0^t\tau^{-\mu}\left(\frac{\theta^{-\sigma}}{\beta_k}+\varepsilon\tau^{\mu-\gamma}\right)\right.\\
&\qquad\left.\times h_k^{-1}(\tau)\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right\|_{\mathbf{X}}\\
&\leqslant C\varepsilon^{-\sigma+2}\left\|\langle t\rangle^{\gamma}\int_0^t\tau^{-\gamma}h_k^{-1}(\tau)\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right\|_{\mathbf{X}}.
\end{aligned}
$$

Since $\int_{\mathbb{R}^+}xf_k(\tau)dx=0$ from Lemma 3 we have for $\gamma\geqslant 0$

$$
\left\|\langle t\rangle^{\gamma}\int_0^t\tau^{-\gamma}\langle\tau\rangle^{-\mu}\mathscr{G}_0(t-\tau)f_k(\tau)d\tau\right\|_{\mathbf{X}}\leqslant C\left\|t^{1-\mu}f_k(t,x)\right\|_{\mathbf{X}}
$$

and using $\left\|t^{1-\mu}f_k(t,x)\right\|_{\mathbf{X}}\leqslant C\varepsilon^{\sigma}$ we obtain

$$
I_2\leqslant C\varepsilon.
$$

By condition (3.21) and the estimate (3.22) we have

$$\left\|t^{1-\mu}\langle t\rangle^{-\gamma}\left(f_k(t)-\frac{\theta^{\sigma+1}}{\tau^{(\sigma+1)}}F_k\left(\cdot t^{-\frac{1}{2n}}\right)\right)\right\|_{\mathbf{X}}\leqslant C\varepsilon^{\sigma}$$

and

$$\int_{\mathbb{R}^+}\left(f_k(\tau,x)-\frac{\theta^{\sigma+1}}{\tau^{(\sigma+1)}}F_k\left(\cdot t^{-\frac{1}{2n}}\right)\right)dx=0.$$

Therefore by analogy with the estimate for I_2 we get for $\beta_k>\varepsilon^{-1}$

$$\begin{aligned}
I_3&=\frac{C}{\beta_k\theta^{\sigma-1}}\left\|\langle t\rangle^{\gamma}\int_0^t\mathscr{G}_0(t-\tau)\left(f_k(\tau)-\frac{\theta^{\sigma}}{\tau^{\frac{1}{2n}\sigma}}F_k\left(\cdot\tau^{-\frac{1}{2n}}\right)\right)\frac{\tau^{\frac{1}{2n}(\sigma-1)}d\tau}{\tau}\right\|_{\mathbf{X}}\\
&\leqslant\frac{C}{\beta_k\theta^{\sigma-1}}\left\|\langle t\rangle^{\gamma}\int_0^t\mathscr{G}_0(t-\tau)\left(f_k(\tau)-\frac{\theta^{\sigma+1}}{\tau^{(\sigma+1)}}F_k\left(\cdot\tau^{-\frac{1}{2n}}\right)\right)\tau^{-\mu}d\tau\right\|_{\mathbf{X}}\\
&\leqslant\frac{C}{\beta_k\theta^{\sigma}}\left\|t^{1-\mu}\left(f_k(t)-\frac{\theta^{\sigma+1}}{\tau^{(\sigma+1)}}F_k\left(\cdot t^{-\frac{1}{2n}}\right)\right)\right\|_{\mathbf{X}}\\
&\leqslant C\varepsilon.
\end{aligned}$$

In the same way we easily get

$$I_4+I_5\leqslant\varepsilon.$$

Hence by induction (3.22) is true for any $k\geqslant 0$ uniformly with respect to k. Taking a limit $k\to\infty$ in (3.22) we get

$$\left\|\langle t\rangle^{\gamma}\left(v(t)-t^{-\frac{1}{n}}\theta V\left(\cdot t^{-\frac{1}{2n}}\right)\right)\right\|_{\mathbf{X}}\leqslant C\varepsilon\,. \tag{3.25}$$

Via (3.7) we also get

$$\begin{aligned}
\psi(t)&=\arg\hat{u}_0(0)-\frac{1}{\theta}\int_0^t d\tau h^{-1}\operatorname{Im}\int_{\mathbb{R}^+}x\mathscr{N}(v)dx\\
&=\psi(0)-\int_0^t\beta^{-1}(\tau+1)^{-1}d\tau\operatorname{Im}\int_{\mathbb{R}^+}y\mathscr{N}(V)dy\\
&\quad-\int_0^t\left(\theta^{\sigma}h^{-1}(\tau)(\tau+1)^{(\sigma-1)}-\beta^{-1}(\tau+1)^{-1}\right)d\tau\\
&\quad\times\operatorname{Im}\int_{\mathbb{R}^+}y\mathscr{N}(V)dy-\frac{1}{\theta}\operatorname{Im}\int_0^t h^{-1}(\tau)\left(\int_{\mathbb{R}^+}x\mathscr{N}(v)dx\right.\\
&\left.-\frac{\theta^{\sigma+1}}{(\tau+1)^{1-\sigma}}\int_{\mathbb{R}^+}y\mathscr{N}(V)dy\right)d\tau.
\end{aligned}$$

Therefore

$$\psi(t)=\omega\log t+\Psi+O\left(t^{-\gamma}\right), \tag{3.26}$$

where $\omega=-\frac{1}{\beta}\operatorname{Im}\int_{\mathbb{R}^+}y\mathscr{N}(V)dy$ and

$$\Psi \equiv \psi(0) - \int_{\mathbb{R}^+} \left(\theta^{\sigma-1} h^{-1}(\tau)(\tau+1)^{-(\sigma-1)} - \beta^{-1}(\tau+1)^{-1} \right) d\tau \operatorname{Im} \int_{\mathbb{R}^+} y \mathcal{N}(V) dy - \operatorname{Im} \int_0^\infty \theta^{\sigma-1} h^{-1}(\tau) \left(\theta^{-\sigma} \int_{\mathbb{R}^+} x \mathcal{N}(v) dx - (\tau+1)^{-(\sigma-1)} \int_{\mathbb{R}^+} y \mathcal{N}(V) dy \right) d\tau.$$

Also from (3.24) we obtain

$$|h(t) - \theta^\sigma \beta t^\mu| \leqslant 1 + C\theta^\sigma \beta t^{\mu-\gamma}, \tag{3.27}$$

where

$$\beta = \frac{\sigma}{\mu} \operatorname{Re} \int_{\mathbb{R}^+} \xi \mathcal{N}(V(\xi)) d\xi \ .$$

Therefore via the formula

$$u(t,x) = e^{-\varphi(t)+i\psi(t)} v(t,x) = e^{i\psi(t)} h^{-\frac{1}{\sigma}} v(t,x)$$

and (3.25), (3.26) and (3.27) we obtain the asymptotics of the solution

$$\left\| \left(u(t) - At^{-\frac{\mu}{\sigma}-\frac{1}{n}} e^{i\omega \log t} V\left(\cdot t^{-\frac{1}{2n}} \right) \right) \right\|_{L^\infty} \leqslant C\varepsilon \langle t \rangle^{-\gamma-\frac{\mu}{\sigma}-\frac{1}{n}}$$

with a constant $A = \beta^{-\frac{1}{\sigma-1}} e^{i\Psi}$. This completes the proof of Theorem 1.

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