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A LEWY–STAMPACCHIA INEQUALITY IN VARIABLE SOBOLEV SPACES FOR PSEUDOMONOTONE OPERATORS

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Abstract. In this paper, we are interested in proving the Lewy-Stampacchia inequality in the general framework of an obstacle problem for a nonlinear pseudomonotone elliptic operator in $W_0^{1,p(\cdot)}(\Omega)$ where $p(\cdot)$ is a log-Hölder continuous exponent. Our aim is to adapt to the context of variable exponent Sobolev spaces a previous work of the first author, based on a penalization method.

1. Introduction

The inequality of Lewy-Stampacchia has been initially proved by H. Lewy and G. Stampacchia [11] in the context of the superharmonic problem. Then, a huge amount of literature has ben devoted lately to this kind of problems; let us cite for example the two monographs: J. F. Rodrigues [20] and G. M. Troianiello [25] (for general results on variational inequalities, obstacle problems and their applications) and the references therein. Since the last two decades, the study of Lewy-Stampacchia inequalities found a renewed interest, either to prove the inequality by itself, or to help to clarify the regularity of solutions to some obstacle problems, or for applications to concrete problems. Let us quote below some recent works on this topic.

In [2], A. Azevedo, J. F. Rodrigues and L. Santos considered a model based on a N-system for linear second-order elliptic equations with sequential constraints. In [4], L. Boccardo uses a Lewy-Stampacchia inequality in studying G-convergence in unilateral problems. In [5], S. Challal, A. Lyaghfouri and J. F. Rodrigues consider a divergence operator of type $-\mathrm{div}[\frac{a(Du)}{|Du|}Du]$ in an associated functional framework based on Orlicz Lebesgue and Sobolev spaces. They consider entropy solutions to an obstacle problem for L^1-data, derive Lewy-Stampacchia inequalities and show convergence and stability results for such operators.

In this general formulation, they extend or complete the results of J.F. Rodrigues for p-Laplace operators [21], J.F. Rodrigues, M. Sanchón and J.-M. Urbano for a $p(\cdot)$-Laplacian with variable exponent [22] and J. F. Rodrigues and R. Teymurazyan for more general Orlicz space [23]. In [18], S. Ouaro and S. Traore also consider entropy

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solutions to similar problem for $p(\cdot)$-Laplace operator type and L^1 data, with general assumptions on the exponent $p(\cdot)$. In [10], C. Leone is interested in obstacle problems for a strongly monotone and Lipschitz operator A, with measure data. M. Matzeu and R. Servadei in [12] and A. Mokrane and F. Murat in [17], consider the case of semilinear variational inequality with a lower order nonlinear term, and, via Lewy-Stampacchia's estimates, they study the Hölder regularity of the solution of the problem. Then, M. C. Palmeri [19] considers the evolution parabolic case. Let us also quote works by R. Servadei and E. Valdinoci [24] for non-local operators like the fractional Laplacian, and integro-differential operators in general.

Then, we invite the reader interested in older papers to consult the references cited in the above cited ones.

In all these papers, the main operator is assumed to be strictly monotone. The Lewy-Stampacchia inequality is a part of the papers and it is used to derive additional information, like the regularity of the solution of the obstacle problem under consideration, when the operator allows it.

An other kind of questions is the proof of the Lewy-Stampacchia inequality for a solution (not unique *a priori*) of obstacle problems when the operator belongs to a large class of Leray-Lions operators. The technique needs to be adapted and it is what was envisaged by A. Mokrane and F. Murat in the papers [13], [14], [16] and [15]; and this is what we propose to adapt to the context of variable exponent Sobolev spaces. This allows to consider singular/degenerate pseudomonotone operators, depending on given sub-domains.

In this paper, we are interested in proving the Lewy-Stampacchia inequality, namely, in the sense of the Radon measures

$$0 \leqslant \mu = A(u) - f \leqslant (f - A(\psi))^-, \tag{1.1}$$

in the general framework for a nonlinear pseudomonotone elliptic problem with obstacles of the type

$$\begin{cases} u \in K(\psi) := \{w \in W_0^{1,p(\cdot)}(\Omega) : \; w \geqslant \psi \text{ a.e. in } \Omega\}, \\ \langle A(u), v-u \rangle \geqslant \langle f, v-u \rangle, \qquad \forall v \in K(\psi), \end{cases} \tag{1.2}$$

where the data are:
- a Leray-Lions pseudomonotone operator $A(v) = -\operatorname{div}(a(x,v,Dv))$, which acts from $W_0^{1,p(\cdot)}(\Omega)$ into $W^{-1,p'(\cdot)}(\Omega)$ (the definitions of such spaces are given in the next section),
- the obstacle ψ, which belongs to $W^{1,p(\cdot)}(\Omega)$ with $\psi \leqslant 0$ on $\partial\Omega$,
- the right hand side f, which is assumed to be such that $g = f - A(\psi)$ belongs to the order dual $V^*_{p'(\cdot)} = (W^{-1,p'(\cdot)}(\Omega))^+ - (W^{-1,p'(\cdot)}(\Omega))^+$.

After this first section, we propose a second one where we recall the framework of Lebesgue and Sobolev spaces with variable exponents. For the convenience of the reader, we recall properties used in the sequel, present the assumptions on the data, derive some technical lemmata and state the main result. The last section is devoted to the proof of the result, following the method: setting $g = f - A(\psi) \in V^*_{p'(\cdot)}$, we first consider the case where

$$g^- \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega),\ \psi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \text{ with } \psi \leqslant 0 \text{ on } \partial\Omega.$$

In such a case, consider the penalized problem

$$A(u_\varepsilon) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = f$$

where $f = g^+ - g^- + A(\psi)$. Setting

$$z_\varepsilon = g^- - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-,$$

we prove that $z_\varepsilon^- \to 0$ in $L^1(\Omega)$. Then, defining $\mu_\varepsilon = \dfrac{1}{\varepsilon}(u_\varepsilon - \psi)^-$ and passing to the limit in $g^- - \mu_\varepsilon = z_\varepsilon = z_\varepsilon^+ - z_\varepsilon^- \geqslant -z_\varepsilon^-$, we obtain the Lewy-Stampacchia inequality $g^- - \mu \geqslant 0$.

The general case is then obtained by passing to the limit in the sequence of the solutions associated with the approximation of g^- by $\hat{g}_n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, $\hat{g}_n \geqslant 0$ (the existence of such approximations is also proved by a penalization method) and ψ by $\psi_n \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $\psi_n \leqslant 0$ on $\partial\Omega$.

The main technical problems are:
- one has to consider different techniques depending on the sets $\Omega_1 = \{x \in \Omega : p(x) < 2\}$ and $\Omega_2 = \{x \in \Omega : p(x) \geqslant 2\}$,
- one is not able to use Poincaré inequality with the integrals (the modulus), but just with the norms.

2. Statement of the main result

In the sequel, we consider a natural number d and a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$.

2.1. Variable Lebesgue and Sobolev spaces

For convenience, let us recall some well known properties of the spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ which can be found for instance in S. N. Antontsev and S. Shmarev [1], L. Diening, P Harjulehto, P Hästö and M Ruzicka [6], X. Fan and D. Zhao [7] or O. Kováčik and J. Rákosník [9].

In the sequel, we call exponent any measurable function $p : \Omega \mapsto [1, +\infty[$ and we set $p^- = \operatorname{ess\,inf}_\Omega p$ and $p^+ = \operatorname{ess\,sup}_\Omega p$.

For any exponent $p(\cdot)$ and any measurable function f, we set

$$\rho_{p(\cdot)} : f \mapsto \int_\Omega |f(x)|^{p(x)} \mathrm{d}x,$$

the Luxemburg norm $\|f\|_{p(\cdot)} = \inf\{\lambda > 0/\,\rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\}$, and define

$$\begin{aligned} L^{p(\cdot)}(\Omega) &= \{u : \Omega \to \mathbb{R},\ \text{measurable}/\, x \mapsto |u(x)|^{p(x)} \in L^1(\Omega)\}, \\ W^{1,p(\cdot)}(\Omega) &= \{u \in L^{p(\cdot)}(\Omega)\ /\ \partial_{x_i} u \in L^{p(\cdot)}(\Omega),\, i = 1,\dots,d\}, \\ W_0^{1,p(\cdot)}(\Omega) &= W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega). \end{aligned}$$

1. Endowed with the Luxembourg norm, $L^{p(\cdot)}(\Omega)$ is a Banach space, separable if $p^+ < +\infty$. In that case,

$$\left[L^{p(\cdot)}(\Omega)\right]' = L^{p'(\cdot)}(\Omega)$$

where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{and} \quad L^{p(\cdot)}(\Omega) \text{ is uniformly convex if } 1 < p^- \leqslant p^+ < +\infty.$$

Note also (J. Giacomoni and G. Vallet [8]) that if $(f_n) \subset L^{p(\cdot)}(\Omega)$ converges to f weakly in $L^{p(\cdot)}(\Omega)$, then, $\rho_p(f_n)$ converges to $\rho_p(f)$ implies that f_n converges to f in $L^{p(\cdot)}(\Omega)$.

2. There exists a constant c such that if $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then the following Hölder inequality holds

$$\int_\Omega |f(x)g(x)|\, \mathrm{d}x \leqslant c\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.1}$$

3. If p^+ is finite, then, $\lim_{n\to\infty} u_n = 0$ in $L^{p(\cdot)}(\Omega)$ if and only if $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = 0$.

4. $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ if and only if $q(\cdot) \leqslant p(\cdot)$. Moreover, the norm of the embedding operator does not exceed $|\Omega| + 1$.

5. *Nemitsky operator in* $L^{p(\cdot)}(\Omega)$: If $h : (x,t) \in \Omega \times \mathbb{R} \mapsto h(x,t) \in \mathbb{R}$ is a Carathéodory function such that

$$\exists g \in L^{q(\cdot)}(\Omega),\ \exists a \geqslant 0,\ |h(x,t)| \leqslant g(x) + a|t|^{\frac{p(x)}{q(x)}},$$

and if $p(\cdot)$ and $q(\cdot)$ are bounded exponents, then the Nemitsky operator associated to h is bounded and continuous from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$.

DEFINITION 1. We say that a bounded exponent $p(\cdot)$ is log-Hölder continuous on Ω if there exists $c_1 > 0$ such that $|p(x) - p(y)| \leqslant c_1 \log(e + 1/|x-y|)$ for all $x, y \in \Omega$.

6. Endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}} = \|Du\|_{p(\cdot)},$$

$W_0^{1,p(\cdot)}(\Omega)$ is a Banach space (resp. $\|u\|_{W^{1,p(\cdot)}} = \|u\|_{p(\cdot)} + \|Du\|_{p(\cdot)}$ for $W^{1,p(\cdot)}(\Omega)$), separable if p^+ is bounded and reflexive if $1 < p^- \leqslant p^+ < +\infty$. Moreover, if $p(\cdot)$ is log-Hölder continuous with $p^- > 1$, for any $F \in [W_0^{1,p(\cdot)}(\Omega)]'$, there exists $(f_0, \vec{f}) \in [L^{p'(\cdot)}(\Omega)]^{d+1}$ such that, for any $u \in W_0^{1,p(\cdot)}(\Omega)$,

$$\langle F, u \rangle = \int_\Omega \{f_0 u + \vec{f} \cdot Du\} dx.$$

In other words $[W_0^{1,p(\cdot)}(\Omega)]' = W^{-1,p'(\cdot)}(\Omega)$.

7. If $p(\cdot)$ is log-Hölder continuous, then $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\!\!\!\rightarrow L^{p(\cdot)}(\Omega)$. If moreover, $q(\cdot)$ is an exponent with $q^+ < +\infty$, then $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ if $q(\cdot) \leqslant p^\star(\cdot) := \frac{dp(\cdot)}{d-p(\cdot)}$, where the embedding constant depends only on $|\Omega|$, d, $c\log(p)$ and q^+.

8. *Poincaré inequality*: If $p(\cdot)$ is log-Hölder continuous, there exists a constant c depending only on d and $c_{\log}(p)$ such that, for any $u \in W_0^{1,p(\cdot)}(\Omega)$,

$$\|u\|_{p(\cdot)} \leqslant c\,\mathrm{diam}(\Omega)\|u\|_{W_0^{1,p(\cdot)}}.$$

9. *Nemitsky operator in $W_0^{1,p(\cdot)}(\Omega)$*: Let

$$h : (x, \xi_0, \xi_1, .., \xi_d) \in \Omega \times \mathbb{R}^{d+1} \mapsto h(x, \xi_0, \xi_1, .., \xi_d) \in \mathbb{R}$$

be a Carathéodory function such that there exist $g \in L^{p'(\cdot)}(\Omega)$ and $C > 0$ with

$$|h(x, \xi_0, \xi_1, .., \xi_d)| \leqslant g(x) + C \sum_{k=0}^{d} |\xi_k|^{p(x)-1},$$

for any $(\xi_0, \xi_1, .., \xi_d) \in \mathbb{R}^{d+1}$ and a.e. $x \in \Omega$. Let $\alpha \in \mathbb{Z}_+^d$ with $|\alpha| \leqslant 1$, then the operator

$$\begin{aligned} N_\alpha : W_0^{1,p(\cdot)}(\Omega) &\to W^{-1,p'(\cdot)}(\Omega) \\ u &\mapsto \left\{ v \in W_0^{1,p(\cdot)}(\Omega) \mapsto \int_\Omega h(x, u(x), Du(x)) D^\alpha v(x) dx \right\} \end{aligned}$$

is continuous and bounded.

In what follows, we assume that $p(\cdot)$ is a log-Hölder continuous exponent such that $1 < p^- \leqslant p^+ < +\infty$.

Let us set a technical lemma used in the sequel.

LEMMA 1. 1) *Assume that $u \in W_0^{1,p(\cdot)}(\Omega)$ and that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous function such that $f(0) = 0$. Then,*

$$f(u) \in W_0^{1,p(\cdot)}(\Omega) \quad \textit{and} \quad Df(u) = f'(u)Du \textit{ a.e.}.$$

Moreover, $u \in W_0^{1,p(\cdot)}(\Omega) \mapsto f(u) \in W_0^{1,p(\cdot)}(\Omega)$ is a continuous (resp. weakly continuous) mapping.

2) Assume that $u \in W_0^{1,p(\cdot)}(\Omega)$ and that $f, f_n : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous functions such that $f(0) = f_n(0) = 0$ and $\|f_n'\| \leqslant M$ for all n. Then, if f_n' converges a.e. to f', then $f_n(u)$ converges to $f(u)$ in $W_0^{1,p(\cdot)}(\Omega)$.

Proof. 1) By definition, if $u \in W_0^{1,p(\cdot)}(\Omega)$, then $u \in W_0^{1,1}(\Omega)$. Thus, the classical chain rule asserts that $f(u) \in W_0^{1,1}(\Omega)$ with $Df(u) = f'(u)Du$ a.e.. Then, one concludes the first part of the assertion by noticing that

$$\int_\Omega |f(u)|^{p(x)}dx \leqslant \int_\Omega \|f'\|_\infty^{p(x)}|u|^{p(x)}dx \leqslant \max\left[\|f'\|_\infty^{p-}, \|f'\|_\infty^{p^+}\right]\int_\Omega |u|^{p(x)}dx,$$

$$\int_\Omega |Df(u)|^{p(x)}dx \leqslant \max\left[\|f'\|_\infty^{p-}, \|f'\|_\infty^{p^+}\right]\int_\Omega |Du|^{p(x)}dx.$$

If (u_n) converges to u in $L^{p(\cdot)}(\Omega)$, then the properties of the Nemitsky operator $u \mapsto f(u)$ in $L^{p(\cdot)}(\Omega)$ yield the convergence of $f(u_n)$ to $f(u)$ in $L^{p(\cdot)}(\Omega)$.

If (u_n) converges weakly to u in $W_0^{1,p(\cdot)}(\Omega)$, then, it converges to u in $L^{p(\cdot)}(\Omega)$ and the above convergence holds. Moreover, the above inequalities ensure that $f(u_n)$ is a bounded sequence in $W_0^{1,p(\cdot)}(\Omega)$. Up to a subsequence, $f(u_{n_k})$ converges weakly to a limit-point v in $W_0^{1,p(\cdot)}(\Omega)$. Since $f(u_n)$ converges to $f(u)$ in $L^{p(\cdot)}(\Omega)$, one gets that $v = f(u)$ and the result since the whole sequence $f(u_n)$ will converge weakly to $f(u)$ in $W_0^{1,p(\cdot)}(\Omega)$.

If (u_n) converges to u in $W_0^{1,p(\cdot)}(\Omega)$, it converges weakly and $f(u_n)$ converges weakly to $f(u)$ in $W_0^{1,p(\cdot)}(\Omega)$ and strongly in $L^{p(\cdot)}(\Omega)$.

Using again the classical chain rule result in $W_0^{1,1}(\Omega)$, $f'(u_n)Du_n$ converge to $f'(u)Du$ in $L^1(\Omega)^d$, thus a.e. for a subsequence denoted similarly.

Since (u_n) converges to u in $W_0^{1,p(\cdot)}(\Omega)$, Du_n converges to Du in $[L^{p(\cdot)}(\Omega)]^d$ and $|Du_n(x) - Du(x)|^{p(x)}$ converges to 0 in $L^1(\Omega)$. Thus, up to a second subsequence indexed again by n, there exists $h \in L^1(\Omega)$ such that $|Du_n(x) - Du(x)|^{p(x)} \leqslant h(x)$ a.e. Then,

$$\begin{aligned}|Df(u_n) - Df(u)|^{p(\cdot)} &= |f'(u_n)Du_n - f'(u)Du|^{p(\cdot)} \\ &\leqslant 2^{p^+-1}[|f'(u_n)[Du_n - Du]|^{p(\cdot)} + |[f'(u_n) - f'(u)]Du|^{p(\cdot)}] \\ &\leqslant C(p^-, p^+, \|f'\|_\infty)[h(\cdot) + |Du|^{p(\cdot)}] = k(\cdot) \in L^1(\Omega).\end{aligned}$$

It is therefore possible to apply the convergence theorem of Lebesgue to conclude that $Df(u_n)$ converges to $Df(u)$ in $[L^{p(\cdot)}(\Omega)]^d$, for the subsequence first, then for the whole sequence since any subsequence of $f(u_n)$ has to converge to $f(u)$.

2) By assumption, there exists a subset Λ of full-measure in $\mathbb{R}$ such that for any $s \in \Lambda$, $f_n'(s)$ converges to $f'(s)$. Thus, since $Du = 0$ a.e. in $\{x \in \Omega, u(x) \in \mathbb{R}\backslash\Lambda\}$, one has that $f_n'(u)Du$ converges a.e. to $f'(u)Du$. Since $|f_n'(u)Du| \leqslant M|Du|$, the dominated convergence theorem yields the conclusion.

2.2. Assumptions on the operator

Let A be the nonlinear operator of Leray-Lions type acting from $W_0^{1,p(\cdot)}(\Omega)$ into its dual $W^{-1,p'(\cdot)}(\Omega)$, which is defined by

$$A(u) = -\text{div}(a(x,u,Du)).$$

The function $a:\Omega\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ is assumed to be a strictly monotone Carathéodory function

$$i.e.\begin{cases}\forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^d, & x\mapsto a(x,s,\xi)\quad\text{is measurable,}\\ \text{for a.e. } x\in\Omega, & (s,\xi)\mapsto a(x,s,\xi)\quad\text{is continuous,}\\ \forall\xi,\eta\in\mathbb{R}^d,\xi\neq\eta,\forall x\in\Omega\text{ a.e.}, & [a(x,s,\xi)-a(x,s,\eta)][\xi-\eta]>0.\end{cases}\tag{2.2}$$

We also assume that there exist three constants $\bar{\alpha}>0$, $\bar{\beta}>0$, $\bar{\gamma}\geqslant 0$, a function $\bar{h}$ in $L^1(\Omega)$ and a function $\bar{k}$ in $L^{p(\cdot)}(\Omega)$ and two exponents q,r such that, for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$ and for all $\xi\in\mathbb{R}^d$ one has $1\leqslant q(x),r(x)\leqslant q^+<p^-$ and

$$a(x,s,\xi)\xi\geqslant\bar{\alpha}\mid\xi\mid^{p(x)}-\left[\bar{\gamma}|s|^{q(x)}+|\bar{h}(x)|\right],\tag{2.3}$$

$$\mid a(x,s,\xi)\mid\leqslant\bar{\beta}\left[|\bar{k}(x)|+\mid s\mid^{\frac{r(x)}{p(x)}}+\mid\xi\mid\right]^{p(x)-1}.\tag{2.4}$$

We propose to derive from these assumptions some technical lemmata used in the sequel. They are somehow classical results, but we propose to give the details of the proofs in this particular framework: variable exponents and dependence on u for A.

LEMMA 2. *There exists a constant $\bar{C}>0$ such that, for any $u\in W_0^{1,p(\cdot)}(\Omega)$,*

$$\int_\Omega a(x,u,Du)Du\,dx\geqslant\frac{\bar{\alpha}}{2}\min\left[\|u\|^{p^+}_{W_0^{1,p(\cdot)}},\|u\|^{p^-}_{W_0^{1,p(\cdot)}}\right]-\bar{C}.$$

Proof. For any $u\in W_0^{1,p(\cdot)}(\Omega)$, since $\int_\Omega|u|^{q(x)}dx\leqslant 1+\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)}$,

$$\begin{aligned}\int_\Omega a(x,u,Du)Du\,dx&\geqslant\bar{\alpha}\int_\Omega|Du|^{p(x)}dx-\bar{\gamma}\int_\Omega|u|^{q(x)}dx-\|\bar{h}\|_{L^1(\Omega)}\\&\geqslant\bar{\alpha}\int_\Omega|Du|^{p(x)}dx-\bar{\gamma}\left[1+\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)}\right]-\|\bar{h}\|_{L^1(\Omega)}\\&\geqslant\bar{\alpha}\int_\Omega|Du|^{p(x)}dx-C\|u\|^{q^+}_{W_0^{1,p(\cdot)}}-\bar{\gamma}-\|\bar{h}\|_{L^1(\Omega)}\end{aligned}\tag{2.5}$$

where C is related to the embeddings $W_0^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$.

Assume on the one hand that $\|u\|_{W_0^{1,p(\cdot)}}\geqslant 1$. If for any positive δ,

$$C(\delta)=\delta^{\frac{-q^+}{p^--q^+}},$$

then (2.5) and Young inequality yield

$$\int_\Omega a(x,u,Du)Du\,dx\geqslant\bar{\alpha}\|u\|^{p^-}_{W_0^{1,p(\cdot)}}-C\delta\|u\|^{p^-}_{W_0^{1,p(\cdot)}}-C.C(\delta)-\bar{\gamma}-\|\bar{h}\|_{L^1(\Omega)}.$$

Choosing $\delta=\frac{\bar{\alpha}}{2C}$, one gets the required inequality.

Assume on the other hand that $\|u\|_{W_0^{1,p(\cdot)}} < 1$. If for any positive δ,

$$C(\delta) = \delta^{\frac{-q^+}{p^+-q^+}},$$

then (2.5) and Young inequality yield

$$\int_\Omega a(x,u,Du)Dudx \geqslant \bar{\alpha}\|u\|^{p^+}_{W_0^{1,p(\cdot)}} - C\delta\|u\|^{p^+}_{W_0^{1,p(\cdot)}} - C.C(\delta) - \bar{\gamma} - \|\bar{h}\|_{L^1(\Omega)}.$$

Choosing $\delta = \frac{\bar{\alpha}}{2C}$, one gets the required inequality.

LEMMA 3. *For any $\bar{\delta} > 0$, there exist constants $\bar{C}, \bar{C}(\bar{\delta}) > 0$ such that, for any $u, v \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\int_\Omega a(x,u,Du)Dvdx \leqslant \bar{\delta}\|u\|^{p^+}_{W_0^{1,p(\cdot)}} + \bar{\delta}^2\|u\|^{p^-}_{W_0^{1,p(\cdot)}} + \bar{C}\bar{\delta} + \bar{C}(\bar{\delta})\int_\Omega |Dv|^{p(x)}dx.$$

Proof.

$$\int_\Omega a(x,u,Du)Dvdx \leqslant \bar{\beta}\int_\Omega \left[|\bar{k}(x)| + |u(x)|^{\frac{r(x)}{p(x)}} + |Du(x)|\right]^{p(x)-1}|Dv|dx.$$

For any $\delta > 0$, Young inequality yields

$$\begin{aligned}
&\int_\Omega a(x,u,Du)Dvdx \\
&\qquad \leqslant \bar{\beta}\delta\int_\Omega \left[|\bar{k}(x)| + |u(x)|^{\frac{r(x)}{p(x)}} + |Du(x)|\right]^{p(x)}dx + \int_\Omega \frac{|Dv|^{p(x)}}{\delta^{p(x)-1}}dx \\
&\qquad \leqslant 3^{p^+-1}\bar{\beta}\delta\int_\Omega \left[|\bar{k}(x)|^{p(x)} + |u(x)|^{r(x)} + |Du(x)|^{p(x)}\right]dx \\
&\qquad\qquad + \max[\frac{1}{\delta^{p^+-1}}, \frac{1}{\delta^{p^--1}}]\int_\Omega |Dv|^{p(x)}dx,
\end{aligned}$$

and, if $\delta = \frac{\bar{\delta}}{\bar{\beta}3^{p^+-1}}$, there exists possibly different constants $C, C(\bar{\delta})$ such that

$$\begin{aligned}
&\int_\Omega a(x,u,Du)Dvdx \\
&\quad \leqslant \bar{\delta}\int_\Omega |Du(x)|^{p(x)}dx + \bar{\delta}\int_\Omega |u(x)|^{r(x)}dx + C\bar{\delta} + C(\bar{\delta})\int_\Omega |Dv|^{p(x)}dx \\
&\quad \leqslant \bar{\delta}\int_\Omega |Du(x)|^{p(x)}dx + \bar{\delta}\|u\|^{p^-}_{W_0^{1,p(\cdot)}} + C\bar{\delta} + C(\bar{\delta})\int_\Omega |Dv|^{p(x)}dx \\
&\quad \leqslant \bar{\delta}\|u\|^{p^+}_{W_0^{1,p(\cdot)}} + \bar{\delta}\|u\|^{p^-}_{W_0^{1,p(\cdot)}} + C\bar{\delta} + C(\bar{\delta})\int_\Omega |Dv|^{p(x)}dx, \qquad (2.6)
\end{aligned}$$

by using Young inequality and $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$. Then, the lemma is proved.

LEMMA 4. *There exist positive constants $\bar{C}_1, \bar{C}_2$ such that,*

$$\int_\Omega a(x,u,Du)D[u-v]dx \geqslant \frac{\bar{\alpha}}{2}\min[\|u\|^{p^+}_{W_0^{1,p(\cdot)}}, \|u\|^{p^-}_{W_0^{1,p(\cdot)}}] - C_1\int_\Omega |Dv|^{p(x)}dx - C_2,$$

for any $u, v \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. Thanks to (2.5) and Young inequality, for any $\bar{\delta} > 0$, there exists $C_{\bar{\delta}} > 0$, independent on u, such that

$$\int_\Omega a(x,u,Du)Dudx \geqslant \bar{\alpha}\int_\Omega |Du|^{p(x)}dx - \bar{\delta}\|u\|^{q^+}_{W_0^{1,p(\cdot)}} - C_{\bar{\delta}}.$$

Thanks to the first of the three inequalities denoted by (2.6), for any $\bar{\delta} > 0$, there exist C", $C'_{\bar{\delta}} > 0$, independent on u and v, such that

$$\int_\Omega a(x,u,Du)Dvdx \leqslant \bar{\delta}\int_\Omega |Du(x)|^{p(x)}dx + \bar{\delta}\|u\|^{r^+}_{W_0^{1,p(\cdot)}} + C'_{\bar{\delta}}\int_\Omega |Dv|^{p(x)}dx + C''\bar{\delta},$$

and,

$$\int_\Omega a(x,u,Du)D[u-v]dx + C_{\bar{\delta}} + C''\bar{\delta}$$
$$\geqslant [\bar{\alpha} - \bar{\delta}]\int_\Omega |Du|^{p(x)}dx - \bar{\delta}\left[\|u\|^{q^+}_{W_0^{1,p(\cdot)}} + \|u\|^{r^+}_{W_0^{1,p(\cdot)}}\right] - C'_{\bar{\delta}}\int_\Omega |Dv|^{p(x)}dx.$$

Then, for a suitable choice of $\bar{\delta}$, and arguing as in Lemma 2 concerning the comparison of the norm of u in $W_0^{1,p(\cdot)}(\Omega)$ to 1, one gets the result.

We finally assume that for any $m \in \mathbb{R}^+$ there exist four positive constants $\alpha_m, \beta_m, \gamma_m$ and δ_m and three functions h_m, k_m, l_m in $L^{p(\cdot)}(\Omega)$ such that, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, $t \in \mathbb{R}$ with $|s| \leqslant m$, $|t| \leqslant m$, and $|s-t| \leqslant \varepsilon_m$ for some small ε_m, and for all $\xi \in \mathbb{R}^d$, $\eta \in \mathbb{R}^d$ one has

$$[a(x,s,\xi) - a(x,s,\eta)][\xi - \eta] \geqslant \alpha_m \frac{|\xi - \eta|^{\max(2,p(x))}}{(|h_m(x)| + |\xi| + |\eta|)^{2-\min(p(x),2)}}, \tag{2.7}$$

$$|a(x,s,\xi) - a(x,s,\eta)| \leqslant \beta_m\left[|k_m(x)| + |\xi| + |\eta|\right]^{(p(x)-2)^+}|\xi - \eta|^{\min(p(x)-1,1)}, \tag{2.8}$$

$$|a(x,s,\xi) - a(x,t,\xi)| \leqslant \gamma_m|s-t|^{\frac{1+\delta_m}{\min(p'(x),2)}}\left[|l_m(x)| + |\xi|\right]^{p(x)-1}. \tag{2.9}$$

REMARK 1.
1) An example of an operator which satisfies all the above assumptions is the so-called $p(\cdot)$-Laplacian, defined for $1 < p^- \leqslant p(\cdot) \leqslant p^+ < \infty$ by

$$-\Delta_{p(\cdot)}(u) = -\mathrm{div}(|Du|^{p(\cdot)-2}Du).$$

A more general example is the case where

$$A(u) = -\mathrm{div}(b(\cdot,u)|Du|^{p(\cdot)-2}Du), \qquad i.e.\ a(x,s,\xi) = b(x,s)|\xi|^{p(x)-2}\xi;$$

if b is a Carathéodory function which satisfies $\alpha \leqslant b(x,s) \leqslant \beta$, for some $0 < \alpha \leqslant \beta < \infty$, then hypotheses (2.2), (2.3), (2.4), (2.7) and (2.8) are satisfied; hypothesis (2.9) is satisfied if $b(x,\cdot)$ is locally Hölder continuous with an Hölder exponent which is strictly greater than $\frac{1}{p'(x)}$ when $p(x) \geqslant 2$ and than $1/2$ when $1 < p(x) \leqslant 2$.

Let us mentioned that one can also consider the case of an operator A with an additional vector field $\vec{B}(\cdot,u)$ inside the divergence operator.

2) As a consequence of the Carathéorory assumption in (2.2), the growth condition (2.4), and the properties of the Nemitsky operators in $W^{1,p(\cdot)}(\Omega)$, one gets the well-definedness, the continuity and the boundedness of the operator

$$A : w \in W_0^{1,p(\cdot)}(\Omega) \mapsto -\mathrm{div}(a(x,w,Dw)) \in W^{-1,p'(\cdot)}(\Omega).$$

Then, the compact embedding of $W_0^{1,p(\cdot)}(\Omega)$ in $L^{p(\cdot)}(\Omega)$, the properties of Nemitsky operators in $L^{p(\cdot)}(\Omega)$, (2.2) and (2.4) allow to prove, by classic arguments, that A is pseudomonotone.
Moreover, A is coercive. Indeed, if $(u_k) \subset W_0^{1,p(\cdot)}(\Omega)$ is a sequence of functions whose norms in $W_0^{1,p(\cdot)}(\Omega)$ go to infinity, thanks to Lemma 2, for sufficiently large k we have

$$\int_\Omega \frac{a(x,u_k,Du_k)Du_k}{\|u_k\|_{W_0^{1,p(\cdot)}}}dx \geqslant \frac{\bar{\alpha}}{2}\|u_k\|^{p^- -1}_{W_0^{1,p(\cdot)}(\Omega)} - \frac{\bar{C}}{\|u_k\|_{W_0^{1,p(\cdot)}}},$$

and A is coercive since $p^- > 1$.

2.3. The main result

Let $\psi : \Omega \to \overline{\mathbb{R}}$ be a given function (the obstacle) which is assumed to satisfy

$$\psi \in W^{1,p(\cdot)}(\Omega), \qquad \text{with} \qquad \psi \leqslant 0 \qquad \text{on } \partial\Omega. \tag{2.10}$$

Let $K(\psi)$ be defined by $K(\psi) = \{v \in W_0^{1,p(\cdot)}(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega\}$ (note that $K(\psi)$ is non empty, since ψ^+ belongs to $K(\psi)$).

Finally, we fix $f \in W^{-1,p'(\cdot)}(\Omega)$, define

$$g = f - A(\psi) = f + \mathrm{div}(a(x,\psi,D\psi)) \tag{2.11}$$

and assume that

$$g \in V^*_{p'(\cdot)}, \tag{2.12}$$

where $V^*_{p'(\cdot)}$ is the order dual space of $W_0^{1,p(\cdot)}(\Omega)$, which is defined as the set of those elements g of $W^{-1,p'(\cdot)}(\Omega)$ which are also elements of the space of Radon measures $\mathscr{M}(\Omega)$ and are such that g^+ and g^- belong to $W^{-1,p'(\cdot)}(\Omega)$, or equivalently as the set of those elements g of $W^{-1,p'(\cdot)}(\Omega)$ which are such that there exist g^p and g^n (where the superscripts p and n stand for "positive" and "negative") such that

$$\begin{cases} g = g^p - g^n, \\ g^p \in W^{-1,p'(\cdot)}(\Omega), \quad g^p \geqslant 0, \quad g^n \in W^{-1,p'(\cdot)}(\Omega), \quad g^n \geqslant 0. \end{cases}$$

Our goal is to prove the following theorem:

THEOREM 1. *Under the above assumptions (2.2)-(2.12), there exists at least one function u, which is a solution of the variational inequality*

$$\begin{cases} \displaystyle\int_\Omega a(x,u,Du)D(v-u)dx \geqslant \langle f, v-u\rangle, \quad \forall v \in K(\psi), \\ u \in K(\psi), \end{cases} \tag{2.13}$$

and which is such that the distribution μ defined by

$$\mu = -\mathrm{div}(a(x,u,Du)) - f \tag{2.14}$$

satisfies the Lewy-Stampacchia inequality

$$\mu \leqslant g^- = (f + \mathrm{div}[a(x,\psi,D\psi)])^-. \tag{2.15}$$

REMARK 2. Theorem 1 states in particular that μ is a nonnegative Radon measure that belongs to $W^{-1,p'(\cdot)}(\Omega) \cap \mathscr{M}(\Omega)$. Then, since according to [6, Chapter 10-11] any function of $W^{1,p(\cdot)}(\Omega)$ is $p(\cdot)$-quasicontinuous and thus measurable for the nonnegative measure μ which belongs to $W^{-1,p'(\cdot)}(\Omega)$, one gets that $(u-\psi) = 0$ μ-a.e. in Ω.

3. Proof of Theorem 1

We will perform the proof of Theorem 1, mainly without distinguishing between the sets $\Omega_1 = \{x \in \Omega : \ p(x) < 2\}$ and $\Omega_2 = \{x \in \Omega : \ p(x) \geqslant 2\}$, except for the second part of the proof of the convergence of z_ε^- in $L^1(\Omega)$, where some different technicalities appear in Ω_1 and Ω_2.

3.1. Some preliminary results

Here we give two existence results and a density one. Since we propose to adapt to the context of variable exponents what was proposed by A. Mokrane and F. Murat in [13, Sections 5-6], we will just focus our attention on the operator A and we invite the reader interested in the detail of the proofs to consult the above mentioned reference.

Concerning the first two theorems, we recall that since we do not suppose that

$$p_- \geqslant \frac{2d}{d+2},$$

it is not clear whether $(u_\varepsilon - \psi)^- v$ belongs to $L^1(\Omega)$ when v only belongs to $W_0^{1,p(\cdot)}(\Omega)$.

THEOREM 2. *Let $f \in W^{-1,p'(\cdot)}(\Omega)$, and let a be a Carathéodory function which satisfies (2.2), (2.3) and (2.4). Finally let $\psi : \Omega \longrightarrow \overline{\mathbb{R}}$ be a measurable function such that*

$$\exists v^\star \in W_0^{1,p(\cdot)}(\Omega) \quad \text{with } v^\star \geqslant \psi \quad \text{a.e. in } \Omega. \tag{3.1}$$

Then for each $\varepsilon > 0$ there exists at least one u_ε such that

$$\begin{cases} u_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), \quad (u_\varepsilon - \psi)^- \in L^2(\Omega), \\ \displaystyle\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)Dv dx - \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - \psi)^- v dx = \langle f, v\rangle, \\ \forall v \in W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega). \end{cases} \tag{3.2}$$

Moreover,

$$\begin{cases} (u_\varepsilon - \psi)^- u_\varepsilon \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)Du_\varepsilon dx - \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - \psi)^- u_\varepsilon dx = \langle f, u_\varepsilon\rangle. \end{cases} \tag{3.3}$$

Proof. To prove this result, one has just to adapt to the case of variable exponents the proof of A. Mokrane and F. Murat [13, Th. 6.1] and [13, Prop. 6.1]. Concerning A. Mokrane and F. Murat [13, Th. 6.1], one just notes that from Remark 1, we know that A is a coercive pseudomonotone operator. Then, denoting by T_n the truncation at height n, the operator

$$B : w \mapsto -\mathrm{div}(a(x,w,Dw)) - \frac{1}{\varepsilon}T_n(w-\psi)^-$$

is well defined from $W_0^{1,p(\cdot)}(\Omega)$ in $W^{-1,p'(\cdot)}(\Omega)$, and is a strongly continuous perturbation of A.

Concerning A. Mokrane and F. Murat [13, Prop. 6.1], one just needs to verify that the function $v = T_k(u_\varepsilon)^+$ (resp. $T_k(u_\varepsilon)^-$) belongs to $W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$, and that $T_k(u_\varepsilon)^+$ tends strongly to u_ε^+ in $W_0^{1,p(\cdot)}(\Omega)$ as $k \to +\infty$. Since this is a consequence of Lemma 1, the result holds.

A result whose proof is similar to the one of Theorem 2 and A. Mokrane and F. Murat [13, Th. 6.2] is the following one:

THEOREM 3. *Let* $f \in W^{-1,p'(\cdot)}(\Omega)$, $v \in W_0^{1,p(\cdot)}(\Omega)$, *and let* a *be a Carathéodory function which satisfies (2.2), (2.3), (2.4). Then for each* $\varepsilon > 0$ *there exists at least one* v_ε *such that*

$$\begin{cases} v_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), \quad v_\varepsilon - v \in L^2(\Omega), \\ \displaystyle\int_\Omega a(x, v_\varepsilon, Dv_\varepsilon) Dw\, dx + \frac{1}{\varepsilon}\int (v_\varepsilon - v) w\, dx = \langle f, w\rangle, \\ \forall w \in W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega). \end{cases} \tag{3.4}$$

Now we give a density lemma for the nonnegative cone of $W^{-1,p'(\cdot)}(\Omega)$, extending to the variable exponent framework the result of A. Mokrane and F. Murat in [13, Section 5].

LEMMA 5. *The nonnegative cone of* $W_0^{1,p(\cdot)}(\Omega)$ *is dense in the nonnegative cone of* $W^{-1,p'(\cdot)}(\Omega)$.

Proof. For $f \in W^{-1,p'(\cdot)}(\Omega)$, $f \geqslant 0$, we define v, v_ε and $\hat{f}_\varepsilon$ by

$$v \in W_0^{1,p(\cdot)}(\Omega), -\mathrm{div}(|Dv|^{p(\cdot)-2} Dv) = f \quad \text{in} \quad \mathscr{D}'(\Omega),$$

$$\begin{cases} v_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), \quad v_\varepsilon - v \in L^2(\Omega), \quad \forall w \in W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega), \\ \displaystyle\int_\Omega |Dv_\varepsilon|^{p(\cdot)-2} Dv_\varepsilon Dw\, dx + \frac{1}{\varepsilon}\int_\Omega (v_\varepsilon - v) w\, dx = 0, \end{cases} \tag{3.5}$$

$$\hat{f}_\varepsilon = -\frac{1}{\varepsilon}(v_\varepsilon - v) = -\mathrm{div}(|Dv_\varepsilon|^{p(\cdot)-2} Dv_\varepsilon).$$

Let us note first that the existence of v_ε is ensured by Theorem 3. Then, following the proof of [13, Section 5], one gets that $v_\varepsilon - v \leqslant 0$ a.e. in Ω and $0 \leqslant \hat{f}_\varepsilon = -\frac{1}{\varepsilon}(v_\varepsilon - v) \in W_0^{1,p(\cdot)}(\Omega)$.

It remains to prove that $\hat{f}_\varepsilon$ strongly converges to f in $W^{-1,p'(\cdot)}(\Omega)$. Using in (3.5) the test function $(v_\varepsilon - v)$ we obtain

$$\begin{aligned} \int_\Omega |Dv_\varepsilon|^{p(\cdot)-2} Dv_\varepsilon Dv_\varepsilon dx + \frac{1}{\varepsilon}\int_\Omega |v_\varepsilon - v|^2 dx & \\ &= \int_\Omega |Dv_\varepsilon|^{p(\cdot)-2} Dv_\varepsilon Dv\, dx \\ &\leqslant \int_\Omega \frac{1}{p'(x)}|Dv_\varepsilon|^{p(x)} + \frac{1}{p(x)}|Dv|^{p(x)}\, dx, \end{aligned} \tag{3.6}$$

which implies that

$$\frac{1}{p^+}\min\left[\|v^\varepsilon\|^{p^+}_{W_0^{1,p(\cdot)}(\Omega)}, \|v^\varepsilon\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega)}\right] + \frac{1}{\varepsilon}\|v^\varepsilon - v\|^2_{L^2(\Omega)} \leqslant C.$$

Therefore

$$v^\varepsilon \rightharpoonup v \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega). \tag{3.7}$$

Rewriting now (3.6) as

$$\begin{cases} \int_\Omega \left[|Dv_\varepsilon|^{p(\cdot)-2}Dv_\varepsilon - |Dv|^{p(\cdot)-2}Dv\right] D[v_\varepsilon - v]dx + \frac{1}{\varepsilon}\int_\Omega |v_\varepsilon - v|^2 dx \\ = -\int_\Omega |Dv|^{p(\cdot)-2}DvD[v_\varepsilon - v]dx, \end{cases}$$

we deduce from (3.7) that

$$\int_\Omega [|Dv_\varepsilon|^{p(\cdot)-2}Dv_\varepsilon - |Dv|^{p(\cdot)-2}Dv]D[v_\varepsilon - v]dx \longrightarrow 0. \tag{3.8}$$

Since $J : W_0^{1,p(\cdot)}(\Omega) \to \mathbb{R}$, $u \mapsto \int_\Omega \frac{1}{p(x)}|Du(x)|^{p(x)}dx$ is a proper, convex, continuous and Gâteaux-differentiable mapping, its Gâteaux-derivative is the single-valued maximal monotone operator, from $W_0^{1,p(\cdot)}(\Omega)$ to its dual space, defined by

$$\langle DJ(u), v\rangle_{W_0^{1,p(\cdot)}(\Omega)} = \int_\Omega |Du|^{p(x)-2}DuDvdx$$

for any (u,v) in $W_0^{1,p(\cdot)}(\Omega)$.

In terms of DJ, limit relation (3.8) reads as

$$\lim_\varepsilon \langle DJ(v_\varepsilon) - DJ(v), v_\varepsilon - v\rangle_{W_0^{1,p(\cdot)}(\Omega)} = 0$$

and the properties of maximal monotone operators in reflexive Banach spaces (*cf. e.g.* V. Barbu [3]) ensure that $J(v_\varepsilon) \to J(v)$ and

$$\langle DJ(v_\varepsilon), v_\varepsilon\rangle_{W_0^{1,p(\cdot)}(\Omega)} \to \langle DJ(v), v\rangle_{W_0^{1,p(\cdot)}(\Omega)}.$$

Then, the remark, saying that the weak convergence of a sequence in $L^{p(\cdot)}(\Omega)$ along with the convergence of the modulus implies the strong convergence (J. Giacomoni and G. Vallet [8, Appendix]), leads to the convergence of Dv_ε to Dv in $L^{p(\cdot)}(\Omega)$, *i.e.* of v_ε to v in $W_0^{1,p(\cdot)}(\Omega)$. Therefore

$$\hat{f}_\varepsilon = -\text{div}(|Dv_\varepsilon|^{p(\cdot)-2}Dv_\varepsilon) \longrightarrow -\text{div}(|Dv|^{p(\cdot)-2}Dv) = f \text{ in } W^{-1,p'(\cdot)}(\Omega)$$

as required. This completes the proof of Lemma 5.

We will now turn to the proof of Theorem 1.

3.2. The case where $g^- \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\psi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$

Hypotheses (2.12) on g and (2.10) on ψ assert that

$$g = f - A(\psi) \in V^*_{p'(\cdot)}, \; \psi \in W^{1,p(\cdot)}(\Omega) \text{ with } \psi \leqslant 0 \text{ on } \partial\Omega.$$

In this subsection, we further assume that $g = g^p - g^n$ where

$$g^p \in W^{-1,p'(\cdot)}(\Omega),\ g^p \geqslant 0, \quad g^n \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega),\ g^n \geqslant 0, \tag{3.9}$$

and that

$$\psi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \quad \text{with} \quad \psi \leqslant 0 \quad \text{on } \partial\Omega. \tag{3.10}$$

For convenience, we set $v^\star = \psi^+$. Thus,

$$v^\star \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \text{ with } v^\star \geqslant \psi \text{ a.e. in } \Omega. \tag{3.11}$$

3.2.1. Penalization and *a priori* estimates

Under hypotheses (2.2), (2.3), (2.4), (3.11) and (2.11), consider u_ε, a solution of problem (3.2) given by Theorem 2. Using (3.3) and the test function $v^\star$ in (3.2), we obtain

$$\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)D(u_\varepsilon - v^\star)dx - \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - \psi)^-(u_\varepsilon - v^\star)dx = \langle f, u_\varepsilon - v^\star\rangle. \tag{3.12}$$

Then, by Lemma 4 and the definition of $v^\star$, the following estimate holds:

$$\begin{aligned}\int_\Omega a(x,u_\varepsilon,Du_\varepsilon)D[u_\varepsilon - v^\star]dx - \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - \psi)^-(u_\varepsilon - v^\star)dx \\ \geqslant \frac{\bar\alpha}{2}\min\Big[\|u_\varepsilon\|^{p^+}_{W_0^{1,p(\cdot)}}, \|u_\varepsilon\|^{p^-}_{W_0^{1,p(\cdot)}}\Big] - C_1\int_\Omega |Dv^\star|^{p(x)}dx \\ -C_2 + \frac{1}{\varepsilon}\int_\Omega |(u_\varepsilon - \psi)^-|^2 dx.\end{aligned}$$

Here, estimating the second term in the left-hand side of equality (3.12), we used the fact that $v^\star \geqslant \psi$ a.e. in Ω.

Finally the right hand side of (3.12) is estimated by

$$|\langle f, u_\varepsilon - v^\star\rangle| \leqslant \|f\|_{W^{-1,p'(\cdot)}} \| u_\varepsilon \|_{W_0^{1,p(\cdot)}(\Omega)} - \langle f, v^\star\rangle \leqslant \frac{\overline{\alpha}}{4}\|Du_\varepsilon\|^{p^-}_{L^p(\Omega)} + C,$$

and from the above computation we deduce that

$$\| u_\varepsilon \|_{W_0^{1,p(\cdot)}(\Omega)} \leqslant C \quad \text{and} \quad \| (u_\varepsilon - \psi)^- \|^2_{L^2(\Omega)} \leqslant C\varepsilon. \tag{3.13}$$

3.2.2. Proof of the existence result

Define $\mu_\varepsilon = \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-$. From equation (3.2) we deduce that

$$-\operatorname{div}(a(x,u_\varepsilon,Du_\varepsilon)) - \mu_\varepsilon = f \quad \text{in } \mathscr{D}'(\Omega), \tag{3.14}$$

which in view of the growth condition (2.4) on a, the Hölder and Poincaré inequalities and of the $W_0^{1,p(\cdot)}(\Omega)$ estimate (3.13) on u_ε imply that μ_ε is bounded in $W^{-1,p'(\cdot)}(\Omega)$.

We can thus extract a subsequence (still denoted by ε) such that $u_\varepsilon \rightharpoonup u$ weakly in $W_0^{1,p(\cdot)}(\Omega)$, and a.e., and $\mu_\varepsilon \rightharpoonup \mu$ weakly in $W^{-1,p'(\cdot)}(\Omega)$. Then, using Lemma 1 for the truncations T_n, the sequel of the proof of A. Mokrane and F. Murat [13, Section 4.1.2] holds similarly and we obtain that u solves the variational inequality (2.13) and that (2.14) holds.

3.2.3. Strong convergence of z_ε^- in $L^1(\Omega)$

First step. Let us define $z_\varepsilon = g^n - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-$ and remark that z_ε belongs to $W_0^{1,p(\cdot)}(\Omega)$ in view of (3.9) and (2.10).

Let us fix $k>0$ and set $E_\varepsilon = \{x\in\Omega : -k < z_\varepsilon(x) < 0\}$. Then, following [13, Section 4.1.3], we have

$$|u_\varepsilon - \psi| \leqslant (\|g^n\|_{L^\infty(\Omega)} + k)\varepsilon \quad \text{ on } E_\varepsilon. \tag{3.15}$$

Setting $m = \|\psi\|_{L^\infty(\Omega)} + \|g^n\|_{L^\infty(\Omega)} + k$, supposing $\varepsilon \leqslant 1$ and taking into account (3.15), we get

$$|\psi| \leqslant m \text{ a.e. in } \Omega \text{ and } |u_\varepsilon| \leqslant m \text{ a.e on } E_\varepsilon. \tag{3.16}$$

By definitions

$$g = f + \operatorname{div}(a(x,\psi,D\psi)),\ z_\varepsilon = g^n - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \quad \text{and} \quad g = g^p - g^n.$$

Then, (3.2) yields that for any $v \in W_0^{1,p(\cdot)}(\Omega)\cap L^2(\Omega)$,

$$\int_\Omega [a(x,u_\varepsilon,Du_\varepsilon) - a(x,\psi,D\psi)]Dv\,dx + \int_\Omega z_\varepsilon v\,dx = \langle g^p, v\rangle. \tag{3.17}$$

Moreover, since $g^n \geqslant 0$, by definition of z_ε, $E_\varepsilon \subset \{x\in\Omega : u_\varepsilon(x) - \psi(x) < 0\}$ and a.e. in Ω

$$D[-T_k(z_\varepsilon^-)] = Dz_\varepsilon 1_{E_\varepsilon} = [Dg^n + \frac{1}{\varepsilon}(Du_\varepsilon - D\psi)]1_{E_\varepsilon}.$$

Thus, by using $-T_k(z_\varepsilon^-)$ as a test function in (3.17), and since $g^p \geqslant 0$, we obtain

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{E_\varepsilon} [a(x,u_\varepsilon,Du_\varepsilon) - a(x,u_\varepsilon,D\psi)]D[u_\varepsilon - \psi]dx + \int_\Omega z_\varepsilon^- T_k(z_\varepsilon^-)dx \\ \leqslant -\displaystyle\int_{E_\varepsilon} [a(x,u_\varepsilon,Du_\varepsilon) - a(x,u_\varepsilon,D\psi)]Dg^n dx \\ \qquad -\displaystyle\int_{E_\varepsilon} [a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)]Dg^n dx \\ \qquad -\dfrac{1}{\varepsilon}\displaystyle\int_{E_\varepsilon} [a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)][Du_\varepsilon - D\psi]dx. \end{cases} \tag{3.18}$$

Let us set

$$\Omega_1 = \{x\in\Omega :\ p(x) < 2\},\ \ E_\varepsilon^1 = E_\varepsilon \cap \Omega_1,$$

$$\Omega_2 = \{x \in \Omega : \ p(x) \geqslant 2\}, \ \ E_\varepsilon^2 = E_\varepsilon \cap \Omega_2.$$

Second step. For $x \in \Omega_2$, we have $p(x) \geqslant 2$, and we deduce from strong monotonicity condition (2.7) and from estimate (3.16) on u_ε that

$$\frac{1}{\varepsilon}\int_{E_\varepsilon^2}[a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,D\psi)]D[u_\varepsilon-\psi]dx \geqslant \frac{\alpha_m}{\varepsilon}\int_{E_\varepsilon^2}|Du_\varepsilon - D\psi|^{p(\cdot)}dx. \quad (3.19)$$

Similarly, the $L^\infty(\Omega)$ estimates (3.16), the local Lipschitz continuity condition (2.8), and Young inequality together with the fact that $(p(\cdot)-2)p'(\cdot)+p'(\cdot)=p(\cdot)$ and the $W_0^{1,p(\cdot)}(\Omega)$ estimate (3.13) on u_ε, yield [1], since ε is small,

$$\begin{aligned}
&\left|-\int_{E_\varepsilon^2}[a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,D\psi)]Dg^n dx\right| \\
&\quad \leqslant \beta_m \int_{E_\varepsilon^2}[|k_m(x)|+|Du_\varepsilon|+|D\psi|]^{p(x)-2}|Du_\varepsilon - D\psi||Dg^n|dx \\
&\quad \leqslant \frac{\alpha_m}{2\varepsilon}\int_{E_\varepsilon^2}|Du_\varepsilon - D\psi|^{p(x)}dx \\
&\qquad\quad + \beta_m \int_{E_\varepsilon^2}\left[\frac{2\beta_m}{\alpha_m}\varepsilon\right]^{\frac{1}{p(x)-1}}[|k_m(x)|+|Du_\varepsilon|+|D\psi|]^{\frac{(p(x)-2)p(x)}{p(x)-1}}|Dg^n|^{\frac{p(x)}{p(x)-1}}dx \\
&\quad \leqslant \frac{\alpha_m}{2\varepsilon}\int_{E_\varepsilon^2}|Du_\varepsilon - D\psi|^{p(x)}dx \\
&\qquad\quad + C_m\varepsilon^{\frac{1}{p^+-1}}\int_{E_\varepsilon^2}[|k_m(x)|+|Du_\varepsilon|+|D\psi|]^{\frac{(p(x)-2)p(x)}{p(x)-1}}|Dg^n|^{\frac{p(x)}{p(x)-1}}dx \\
&\quad \leqslant \frac{\alpha_m}{2\varepsilon}\int_{E_\varepsilon^2}|Du_\varepsilon - D\psi|^{p(x)}dx + C_m\varepsilon^{\frac{1}{p^+-1}}.
\end{aligned} \quad (3.20)$$

We observe that producing (3.20), with the use of the first inequality of (3.13) we estimated

$$\begin{aligned}
&\int_{E_\varepsilon^2}[|k_m(x)|+|Du_\varepsilon|+|D\psi|]^{\frac{(p(x)-2)p(x)}{p(x)-1}}|Dg^n|^{\frac{p(x)}{p(x)-1}}dx \\
&\qquad\qquad \leqslant \int_{E_\varepsilon^2}[|k_m(x)|+|Du_\varepsilon|+|D\psi|+|Dg^n|]^{p(x)}dx \leqslant C_m.
\end{aligned}$$

Moreover, using (2.9), (3.16) and the second inequality of (3.13), we get

$$\left|-\int_{E_\varepsilon^2}[a(x,u_\varepsilon,D\psi)-a(x,\psi,D\psi)]Dg^n dx\right| \to 0 \text{ as } \varepsilon \to 0. \quad (3.21)$$

Finally using Young inequality, (2.9), (3.16) and (3.15), for sufficiently small ε we have

$$\left|-\frac{1}{\varepsilon}\int_{E_\varepsilon^2}[a(x,u_\varepsilon,D\psi)-a(x,\psi,D\psi)]D[u_\varepsilon-\psi]dx\right| - \frac{\alpha_m}{2\varepsilon}\int_{E_\varepsilon^2}|Du_\varepsilon - D\psi|^{p(x)}dx$$

[1] Here and in what follows, C_m denotes nonnegative constants which do not depend on ε, but can depend on m and vary from line to line.
If $|k_m(x)|+|Du_\varepsilon|+|D\psi|=0$, add 1 to $|k_m|$ to avoid this situation.

$$
\begin{aligned}
&\leqslant \frac{C_m}{\varepsilon}\int_{E_\varepsilon^2}|a(x,u_\varepsilon,D\psi)-a(x,\psi,D\psi)|^{p'(x)}dx\\
&\leqslant \frac{C_m}{\varepsilon}\int_{E_\varepsilon^2}\gamma_m^{p'(x)}|u_\varepsilon-\psi|^{1+\delta_m}[l_m(x)+|D\psi|]^{(p(x)-1)p'(x)}dx\\
&\leqslant \frac{C_m}{\varepsilon}[(\|g^n\|_{L^\infty(\Omega)}+k)\varepsilon]^{1+\delta_m}\int_{E_\varepsilon^2}\gamma_m^{p'(x)}[l_m(x)+|D\psi|]^{p(x)}dx\\
&\leqslant C_m\varepsilon^{\delta_m}.
\end{aligned}
\tag{3.22}
$$

Third step. Define the function F_ε and the set Z_ε by

$$
F_\varepsilon=|h_m(x)|+|Du_\varepsilon|+|D\psi|,\quad Z_\varepsilon=\{x\in\Omega : F_\varepsilon(x)\neq 0\}.
$$

Note that $Du_\varepsilon(x)=D\psi(x)=0$ in $\Omega\setminus Z_\varepsilon$. Therefore, in particular, we have

$$
Du_\varepsilon(x)-D\psi(x)=0 \quad \text{in} \quad \Omega\setminus Z_\varepsilon. \tag{3.23}
$$

From (3.23), (2.7) and the $L^\infty(\Omega)$ estimate (3.16) on u_ε we deduce that

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{E_\varepsilon^1}[a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,D\psi)]D[u_\varepsilon-\psi]dx\\
&\qquad=\frac{1}{\varepsilon}\int_{E_\varepsilon^1\cap Z_\varepsilon}[a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,D\psi)]D[u_\varepsilon-\psi]dx\\
&\qquad\geqslant\frac{\alpha_m}{\varepsilon}\int_{E_\varepsilon^1\cap Z_\varepsilon}\frac{|Du_\varepsilon-D\psi|^2}{(|h_m(x)|+|Du_\varepsilon|+|D\psi|)^{2-p(x)}}dx\\
&\qquad=\frac{\alpha_m}{\varepsilon}\int_{E_\varepsilon^1\cap Z_\varepsilon}\frac{|Du_\varepsilon-D\psi|^2}{|F_\varepsilon|^{2-p(x)}}dx.
\end{aligned}
\tag{3.24}
$$

Since because of condition (2.4) and of the inclusions $u_\varepsilon,\psi\in W^{1,p(\cdot)}(\Omega)$ the first integral in (3.24) is finite, the last integral in (3.24) is also finite.

Further, condition (2.8), estimates (3.16), property (3.23), Young inequality with the exponents $\bar q(x)=2/(p(x)-1)$ and $\bar q(x)/(\bar q(x)-1)$ for $x\in E_\varepsilon^1$, the equality

$$
\frac{2-p(x)+\bar q(x)}{\bar q(x)-1}=p(x),\quad x\in E_\varepsilon^1,
$$

and the boundedness of $\{F_\varepsilon\}$ in $L^{p(\cdot)}(\Omega)$ yield

$$
\begin{aligned}
&\left|-\int_{E_\varepsilon^1}[a(x,u_\varepsilon,Du_\varepsilon)-a(x,u_\varepsilon,D\psi)]Dg^n dx\right|\\
&\qquad\leqslant \beta_m\int_{E_\varepsilon^1}|Du_\varepsilon-D\psi|^{p(x)-1}|Dg^n|dx\\
&\qquad=\beta_m\int_{E_\varepsilon^1\cap Z_\varepsilon}|Du_\varepsilon-D\psi|^{p(x)-1}|Dg^n|dx
\end{aligned}
$$

$$\leqslant \beta_m \delta \int_{E_\varepsilon^1 \cap Z_\varepsilon} \left[\frac{|Du_\varepsilon - D\psi|^{p(x)-1}}{|F_\varepsilon|^{\frac{2-p(x)}{\overline{q}(x)}}} \right]^{\overline{q}(x)} dx$$

$$+ \beta_m \int_{E_\varepsilon^1 \cap Z_\varepsilon} \delta^{\frac{-1}{\overline{q}(x)-1}} \left[|F_\varepsilon|^{\frac{2-p(x)}{\overline{q}(x)}} |Dg^n| \right]^{\frac{\overline{q}(x)}{\overline{q}(x)-1}} dx$$

$$\leqslant \frac{\alpha_m}{2\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} \frac{|Du_\varepsilon - D\psi|^{(p(x)-1)\overline{q}(x)}}{|F_\varepsilon|^{2-p(x)}} dx$$

$$+ C_m \varepsilon^{\frac{p^- - 1}{3-p^-}} \int_{E_\varepsilon^1 \cap Z_\varepsilon} (|F_\varepsilon| + |Dg^n|)^{p(x)} dx$$

$$\leqslant \frac{\alpha_m}{2\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} \frac{|Du_\varepsilon - D\psi|^2}{|F_\varepsilon|^{2-p(x)}} dx + C_m \varepsilon^{\frac{p^- - 1}{3-p^-}}, \tag{3.25}$$

by setting $\delta = \frac{\alpha_m}{2\varepsilon\beta_m}$.

A similar argument as in the previous case yields

$$\left| - \int_{E_\varepsilon^1} [a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)] Dg^n dx \right| \to 0 \text{ as } \varepsilon \to 0. \tag{3.26}$$

As far as the last term of (3.18) is concerned, using (3.23) and Young inequality, we obtain

$$\left| -\frac{1}{\varepsilon} \int_{E_\varepsilon^1} [a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)] D[u_\varepsilon - \psi] dx \right|$$

$$= \left| -\frac{1}{\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} [a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)] D[u_\varepsilon - \psi] dx \right|$$

$$\leqslant \frac{1}{\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} |a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)| \frac{|Du_\varepsilon - D\psi|}{|F_\varepsilon|^{\frac{2-p(x)}{2}}} |F_\varepsilon|^{\frac{2-p(x)}{2}} dx$$

$$\leqslant \frac{\alpha_m}{2\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} \frac{|Du_\varepsilon - D\psi|^2}{|F_\varepsilon|^{2-p(x)}} dx$$

$$+ \frac{2}{\alpha_m \varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} |a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)|^2 |F_\varepsilon|^{2-p(x)} dx. \tag{3.27}$$

In order to estimate the last term of the right hand side of (3.27), we use again (2.9), (3.15), (3.16) and the facts that $\{F_\varepsilon\}$ is bounded in $L^{p(\cdot)}(\Omega)$ and $2(p(\cdot)-1)+2-p(\cdot) = p(\cdot)$. Then for sufficiently small ε we obtain

$$\frac{1}{\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} |a(x,u_\varepsilon,D\psi) - a(x,\psi,D\psi)|^2 |F_\varepsilon|^{2-p(\cdot)} dx$$

$$\leqslant \frac{1}{\varepsilon} \int_{E_\varepsilon^1 \cap Z_\varepsilon} \gamma_m^2 |u_\varepsilon - \psi|^{1+\delta_m} [|l_m(x)| + |D\psi|]^{2(p(\cdot)-1)} |F_\varepsilon|^{2-p(\cdot)} dx$$

$$\leqslant \frac{1}{\varepsilon} [(\|g^n\|_{L^\infty(\Omega)} + k)\varepsilon]^{1+\delta_m} \int_{E_\varepsilon^1 \cap Z_\varepsilon} \gamma_m^2 [|l_m(x)| + |D\psi|]^{2(p(\cdot)-1)} |F_\varepsilon|^{2-p(\cdot)} dx$$

$$\leqslant C_m \varepsilon^{\delta_m}. \tag{3.28}$$

Summing up (3.18)-(3.22) and (3.24)-(3.28), we obtain for an arbitrary $k > 0$,

$$\int_\Omega z_\varepsilon^- T_k(z_\varepsilon^-)dx \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

and therefore, $z_\varepsilon^- \to 0$ strongly in $L^1(\Omega)$.

3.2.4. Proof of the Lewy-Stampacchia inequality

Coming back to the definitions of z_ε and μ_ε, we have

$$z_\varepsilon = z_\varepsilon^+ - z_\varepsilon^- = g^n - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \quad \text{and} \quad g^n + z_\varepsilon^- = z_\varepsilon^+ + \mu_\varepsilon \geqslant \mu_\varepsilon.$$

Passing to the limit in the above inequality thanks to the strong convergence of z_ε^- to zero in $L^1(\Omega)$ and the weak convergence in $W^{-1,p'(\cdot)}(\Omega)$ of $\{\mu_\varepsilon\}$ to μ, we deduce that

$$g^n \geqslant \mu. \tag{3.29}$$

This is the Lewy-Stampacchia inequality that completes the proof of Theorem 1 in the case of this section, *i.e.* where $g = g^p - g^n$ and assumptions (3.9) and (3.10) hold.

3.3. The general case: approximation of g and ψ and passing to the limit

Let us consider now general data as assumed in hypotheses (2.10) and (2.12). By Lemma 5, there exists $\hat{g}_n$ such that

$$\hat{g}_n \in W_0^{1,p(\cdot)}(\Omega), \quad \hat{g}_n \geqslant 0, \quad \hat{g}_n \to g^- \text{ in } W^{-1,p'(\cdot)}(\Omega). \tag{3.30}$$

Using Lemma 1 with the truncation T_n at height n, we can assume by a further approximation that each of those functions $\hat{g}_n$ also belongs to $L^\infty(\Omega)$. Moreover, setting for every $n \in \mathbb{N}$, $\psi_n = T_n(\psi)$, we obtain

$$\psi_n \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad \psi_n \leqslant 0 \text{ on } \partial\Omega, \quad \psi_n \to \psi \text{ in } W^{1,p(\cdot)}(\Omega). \tag{3.31}$$

Define now $f_n = g^+ - \hat{g}_n + A(\psi_n)$, and let u_n be a solution given by Theorem 1 where f, ψ and g are replaced by f_n, ψ_n and $g^+ - \hat{g}_n$ respectively. Therefore,

$$\forall v \in K(\psi_n), \ \int_\Omega a(x,u_n,Du_n)D(v-u_n)dx \geqslant \langle f_n, v-u_n\rangle, \tag{3.32}$$

$$u_n \in K(\psi_n), \ \mu_n = -\mathrm{div}(a(x,u_n,Du_n)) - f_n, \quad \mu_n \leqslant \hat{g}_n. \tag{3.33}$$

Then, following the proof proposed by A. Mokrane and F. Murat in [13, Section 4.2], one proves that $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. This and condition (2.4) imply that $\{A(u_n)\}$ is bounded in $W^{-1,p'(\cdot)}(\Omega)$. Then we obtain, up to a subsequence still indexed by n, that u_n converges weakly to a given u in $W_0^{1,p(\cdot)}(\Omega)$ and $A(u_n)$ and μ_n converge weakly to given φ and μ in $W^{-1,p'(\cdot)}(\Omega)$. Moreover, since for every n, $u_n \in K(\psi_n)$, taking into account the mentioned convergence of u_n to u and convergence of ψ_n to ψ in Ω, we get $u \in K(\psi)$. This and the definition of ψ_n imply that for every n, $T_n(u) \in K(\psi_n)$. Then, by virtue of (3.32), for every n we have

$$\langle A(u_n), u_n - T_n(u_n)\rangle \leqslant \langle f_n, u_n - T_n(u_n)\rangle.$$

Hence, using the convergence of $\{T_n(u)\}$ to u in $W_0^{1,p(\cdot)}(\Omega)$, the convergence of $\{f_n\}$ to f in $W^{-1,p'(\cdot)}(\Omega)$, the weak convergence of $\{u_n\}$ to u in $W_0^{1,p(\cdot)}(\Omega)$ and the weak convergence of $\{A(u_n)\}$ to φ in $W^{-1,p'(\cdot)}(\Omega)$, we obtain

$$\limsup_n \langle A(u_n), u_n\rangle \leqslant \langle \varphi, u\rangle.$$

Since the operator A is pseudomonotone, the latter inequality along with the weak convergence of $\{A(u_n)\}$ to φ in $W^{-1,p'(\cdot)}(\Omega)$ implies that $\varphi = A(u)$ and $\langle A(u_n), u_n\rangle \to \langle A(u), u\rangle$. Then, fixing an arbitrary $v \in K(\psi)$, using $T_n(v)$ as a test function in (3.32) and passing to the limit in (3.32) and (3.33), we obtain that $\langle A(u), v-u\rangle \geqslant \langle f, v-u\rangle$, $\mu = A(u) - f$ and $\mu \leqslant g^-$. This completes the proof of Theorem 1 in the general case.

REMARK 3. Let us mention that techniques of the same kind can be applied to the bilateral problem, for example by adapting [16], but one needs *ad hoc* assumptions on the operator.

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